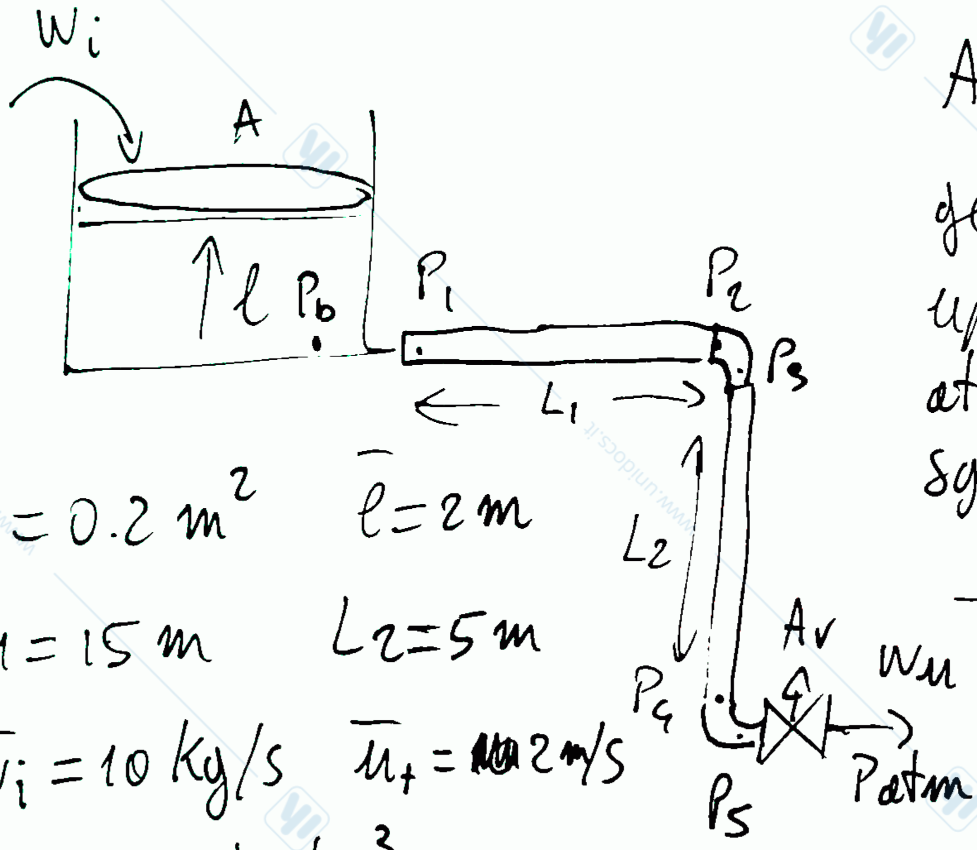
LEVEL CONTROL IN A TANK



$A = 0.2\ m^2$ $\bar{l} = 2m$

$L1 = 15\ m$ $L2 = 5m$

$\bar{w}_i = 10\ kg/s$ $\bar{u}_t = 2\ m/s$

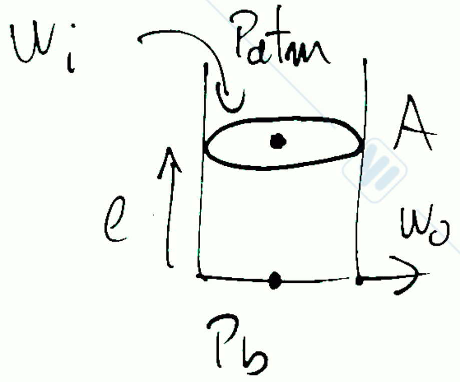
$\rho = 1000\ kg/m^3$

A cylindrical tank of cross-section A gets a variable flow rate w_i from some upstream process and discharges at atmospheric pressure throug a piping system and a control valve

-> Objective of control: keep $l \sim$ const despite the variations in w_i

-> Identify what are the control-relevant dynamics and design a level controller

Tank model



$\rho = 1000\ Kg/m^3 = const$

- Mass balance
- Momentum balance ↕ direction

$$\frac{dM}{dt} = \dot{w}_i - w_o \qquad M = \rho V = \rho l A = \underbrace{\rho A}_{const} l$$

$\Delta P \Leftrightarrow V$

$\Delta W \Leftrightarrow I$

$$\begin{cases} \rho A \frac{dl}{dt} = w_i - w_o \\ P_b - P_{atm} = \rho g l \end{cases}$$

w_i, w_o, l, P_b

$\Delta \cancel{l}$ state → ΔP_b state

$$\frac{d\Delta P_b}{dt} = \rho g \frac{d\Delta l}{dt}$$

$$\frac{d\Delta l}{dt} = \frac{1}{\rho g} \frac{d\Delta P_b}{dt}$$

$$\begin{cases} \rho A \frac{d\Delta l}{dt} = \Delta w_i - \Delta w_o \\ \Delta P_b = \rho g \Delta l \end{cases}$$

$\bar{w}_i = \bar{w}_o \qquad \bar{P}_b - P_{atm} = \rho g \bar{l}$

$$\frac{\rho A}{\rho g}\frac{d\Delta P_b}{dt}=\Delta W_i-\Delta W_0$$

steady-state

$$\bar{W}_1-\bar{W}_0=0$$

$$\bar{P}_b-P_{atm}=\rho g e$$

$$\bar{P}_1-\bar{P}_2=\frac{cf}{2}\frac{\omega L_1}{\rho A_t^3}\bar{W}_u^2$$

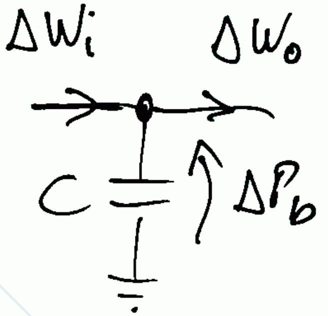
$$\bar{P}_3-\bar{P}_4=-\rho g L_2+\frac{cf}{2}\frac{\omega L_2}{\rho A_t^3}\bar{W}_u^2$$

$$\bar{P}_b-\bar{P}_1=\rho\frac{\bar{u}^2}{2}$$

$$\bar{P}_2-\bar{P}_3=0.6\,\rho\frac{\bar{u}^2}{2}$$

$$\bar{P}_4-\bar{P}_5=0.6\,\rho\frac{u^2}{2}$$

$$\bar{u}=\frac{\bar{w}}{\rho A_t}\qquad \bar{W}_u=\bar{A}_V\sqrt{\rho(P_5-P_{atm})}$$



$$C=\frac{A}{g}$$

$$A_t=\frac{\bar{w}}{\rho\bar{u}}=\frac{10}{1000\cdot 2}=5\cdot 10^{-3}\ m^2$$

circular pipe

$$A_t=\pi r^2$$
$$\omega=2\pi r$$

$$\omega=2\sqrt{\pi A_t}=0.25\ m$$

$$cf=0.005$$

$\bar{w}_0 = \bar{w}_i = 10$ Kg/s

$\bar{P}_b - \bar{P}_1 = \rho \dfrac{\bar{u}^2}{2} = 2000$ Pa

$\bar{P}_1 - \bar{P}_2 = 7500$ Pa

$R = 2 \dfrac{\bar{P}_1 - \bar{P}_0}{\bar{w}}$ | duct

$\bar{P}_2 - \bar{P}_3 = 1200$ Pa

$\bar{P}_3 - \bar{P}_4 = -49050 + 2500$

$\bar{P}_4 - \bar{P}_5 = 1200$ Pa

~~$\bar{P}_b - \bar{P}_{atm} =$~~ $\bar{P}_b - \bar{P}_{atm} = 19620$ Pa

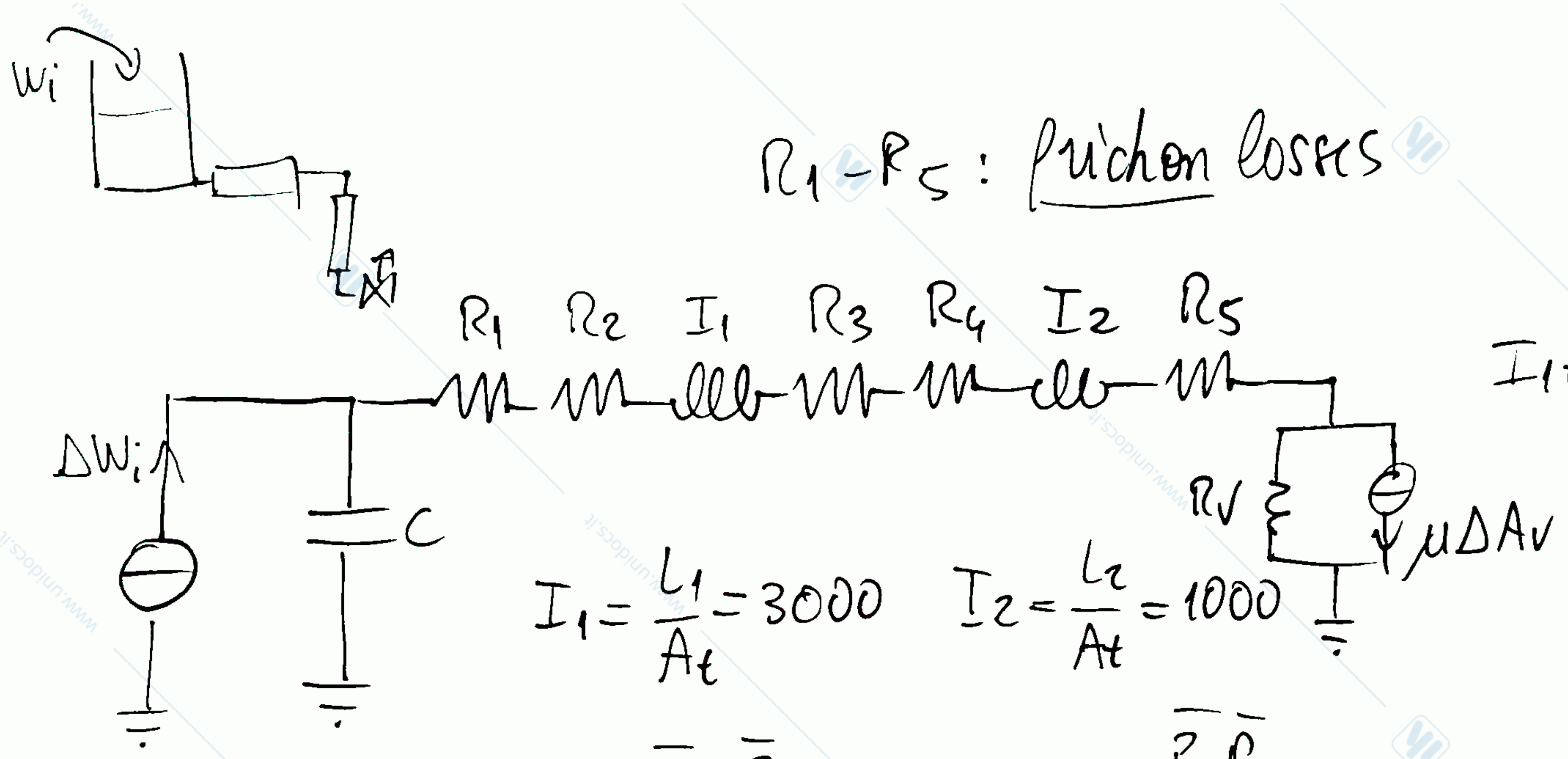
$\bar{P}_5 - \bar{P}_{atm} = \cancel{P_b} - P_{atm} + (\cancel{P_1} - \cancel{P_b}) + (\cancel{P_2} - \cancel{P_1}) + (\cancel{P_3} - \cancel{P_2}) + (\cancel{P_4} - \cancel{P_3}) + (P_5 - \cancel{P_4})$

$= 19620 - 2000 - 7500 - 1200 + 49050 - 2500 - 1200 =$

$= 68670 - (14400) = 54270$ → consistent w/ design principle ΔPvalve >> ΔPduction

$\bar{W}_u = \bar{A}_v \sqrt{\rho(P_5 - P_{atm})} \rightarrow \bar{A}_v = \dfrac{w_{11}}{\sqrt{\rho(P_5 - P_{atm})}} = 1.35 \cdot 10^{-3} m^3$

$R_1 - R_5$: friction losses



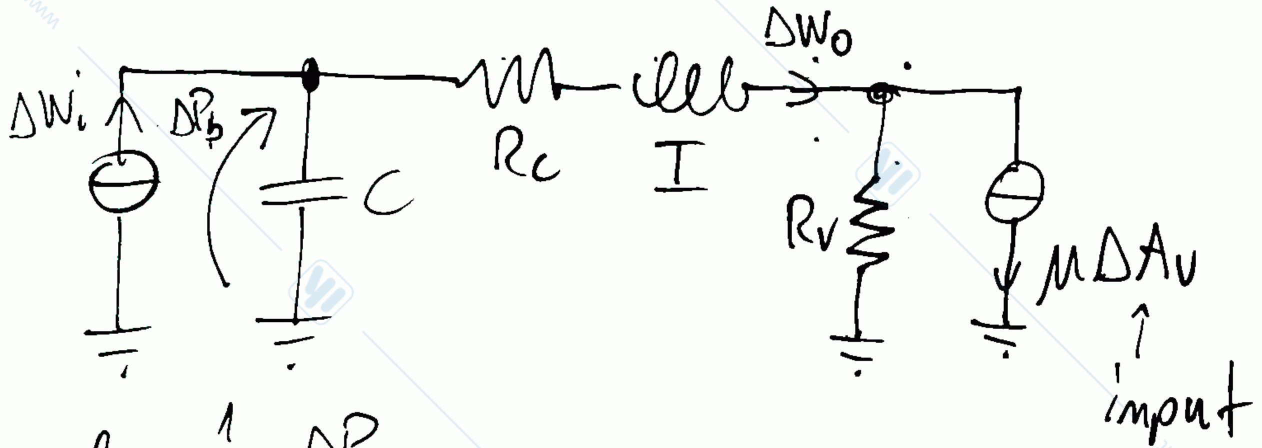
$I_1 + I_2 = I = 4000$

$I_1 = \frac{L_1}{A_t} = 3000$ $I_2 = \frac{L_2}{A_t} = 1000$

$C = \frac{A}{g} = 0.02$ $R_1 = 2\frac{\overline{P_b} - \overline{P_1}}{\overline{W_u}} = 400$ $R_2 = 2\frac{\overline{P_1} - \overline{P_2}}{\overline{W_u}} = 1500$ $R_3 = 2\frac{\overline{P_2} - \overline{P_3}}{\overline{W_u}} = 240$

$R_4 = 2\frac{\overline{P_3} - \overline{P_4}/P_{uc}}{\overline{W_u}} = 500$ $R_5 = R_3 = 240$ $R_V = 2\frac{\overline{P_5} - P_{atm}}{\overline{W_u}} = 10850$

$\mu = \frac{\overline{W_u}}{A_V} = 7400$ $R_1 + R_2 + R_3 + R_4 + R_5 = R_c = 2880$



$R_c = 2880$

$R_V = 10850$

$\mu = 7400$

$c = 0.02$

$I = 4000$

$$\Delta l = \frac{1}{\rho g} \Delta P_b$$

$$\Delta W_o = \mu \Delta A_V \cdot \frac{R_V}{R_V + sI + R_c + \frac{1}{sC}}$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta P_b = -\Delta W_o \cdot \frac{1}{sC} = \mu \Delta A_V \frac{R_V}{1 + s(R_V + R_c)C + s^2 IC}$$

approximate T_1, T_2 $\underline{Hp}$ 2 real poles, $\boxed{T_1 \gg T_2}$

$$(1 + sT_1)(1 + sT_2) = 1 + s(T_1 + T_2) + s^2 T_1 T_2 = 1 + as + bs^2$$

$$T_1 \sim a \qquad T_2 = \frac{b}{T_1} \simeq \frac{b}{a}$$

$T_1 \simeq (R_V + R_c) C = 275\ s$

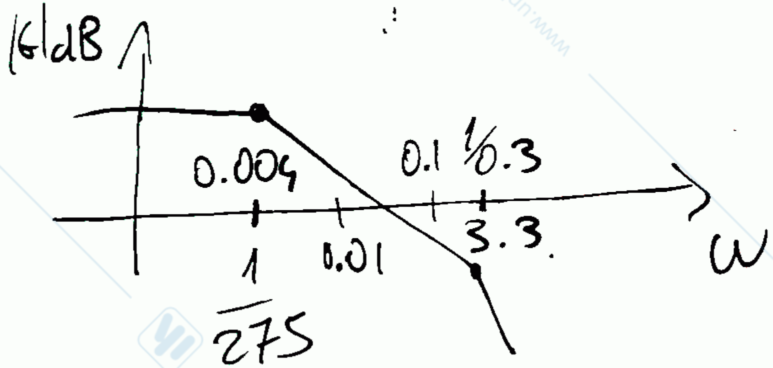
$T_2 \simeq \frac{I\gamma}{(R_V+R_c)\gamma} = 0.3\ s$ → it is true $T_1 >> T_2$

$\boxed{\omega_c < 0.1\ rad/s}$

$G(s) = \frac{\Delta \ell}{\Delta A_V} = -\frac{1}{\rho g} \mu R_V \frac{1}{(1+sT_1)(1+sT_2)}$

$\omega_c > 0.01$

$\boxed{\bullet\ T_2 \text{ can be neglected}}$



$G(j\omega)$ well approx $\omega \sim \omega_c$

- R_c could be neglected
- neglect friction 100%

↳ $\bar{P}_s - P_{atm} =$

$\leftarrow\ = \rho g (\bar{\ell} + L_2)$

$G(s) \simeq -\frac{1}{\rho g} \mu R_V \frac{1}{1+sT_1}$

$R_c = 0$

$\boxed{R_V = 2 \frac{\rho g (\bar{\ell} + L_2)}{\bar{W}_u}}$

$T_1 \simeq R_V C \qquad \mu = \frac{\bar{W}_u}{\bar{A}_V}$

$$G(s) = -\frac{1}{\cancel{\rho g}} \frac{\cancel{\bar{A}_v}\sqrt{\rho\,\rho g(\bar{l}+L_2)}}{\cancel{\bar{A}_v}}\, 2\, \frac{\cancel{\rho} g(\bar{l}+L_2)}{\bar{w}_u} \frac{1}{1+sT_1}$$

$$= -\frac{2\rho\sqrt{g(\bar{l}+L_2)}\,(\bar{l}+L_2)}{\bar{w}_u} \frac{1}{1+sT_1}$$

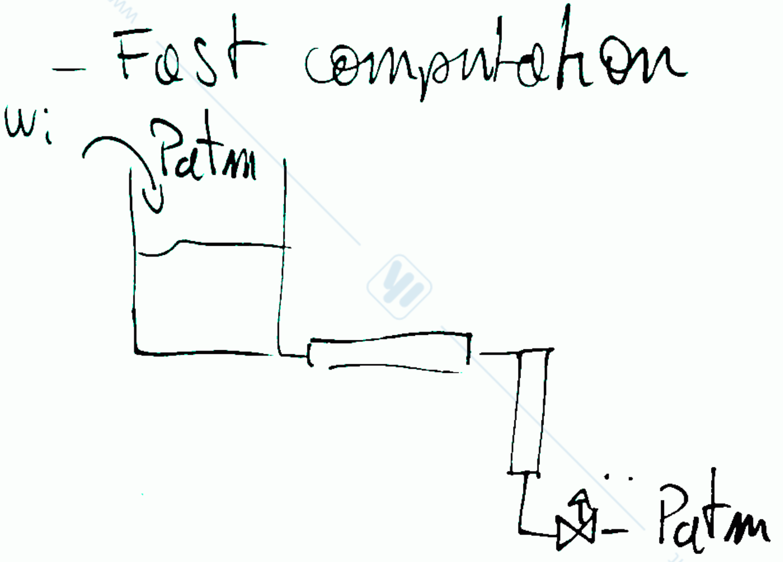
$$T_1 \sim R_v C = 2\frac{\rho \cancel{g}(\bar{l}+L_2)}{\bar{w}_u} \cdot \frac{A}{\cancel{g}} = \frac{2\cancel{\rho}A(\bar{l}+L_2)}{\bar{w}_u}$$

$$G(s) = -\frac{\mu}{1+sT_1} \qquad \mu = \frac{2\rho\sqrt{g(\bar{l}+L_2)}\,(\bar{l}+L_2)}{\bar{w}_u}$$

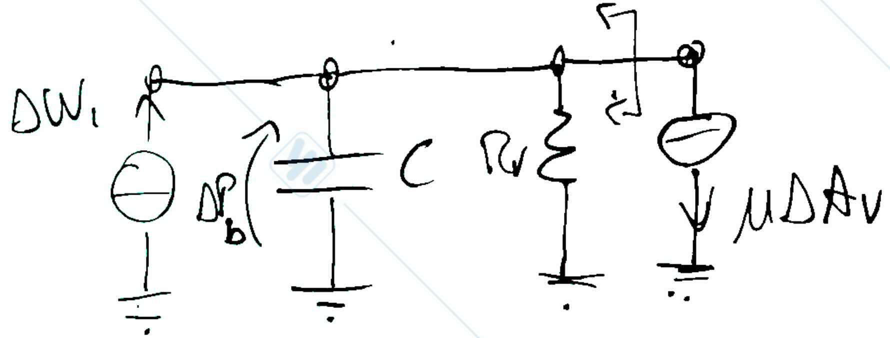
$$T_1 = \frac{2A(\bar{l}+L_2)}{\bar{w}_u}$$

$\bar{l} \pm 1m$ $\qquad \frac{\mu}{T_1}$ changes 10% $\qquad \mu \propto \sqrt{\bar{l}+L_2}$

- Fast computation



$c = \frac{A}{g}$.



$C = \frac{A}{g}$

Simplifying assumptions

- $\rho = \text{const}$
- good-design principle → Neglect friction
- $T \simeq \frac{\rho \bar{u} L}{2(P_1 - P_0)} = \frac{1000 \cdot 2 \cdot 20}{2 \cdot 70'000} = \frac{40000}{140000} \simeq 0.35\,s$

~~inertia~~

small wrt $\frac{1}{\omega_c}$

no inductor (inertance)

$$R_v = 2\frac{P_{1v} - P_{0v}}{\bar{w}_u} = 2\frac{\rho g(\bar{l} + L_2)}{\bar{w}_u}$$

$$u = \frac{\bar{w}_u}{A_v} = \frac{\cancel{A_v}\sqrt{\rho\rho g(\bar{l}+L_2)}}{\cancel{A_v}}$$

$$\Delta l = \frac{1}{\rho g}\Delta P_b = -\frac{1}{\rho g} u\Delta A_v R_v // \frac{1}{sC} = \frac{-1}{\rho g} \cdot u\Delta A_v \frac{R_v/sC}{R_v + \frac{1}{sC}}$$

$$\Delta l = -\frac{M}{\rho g} R_V \frac{1}{1+sT_1} \qquad T_1 = C \cdot R_V$$

$$\Delta l = \frac{1}{\rho g} \Delta P_b = -\frac{1}{\rho g} \frac{M \Delta A_V R_V}{1+sT_1} \qquad T_1 = R_V \cdot C$$

$$= -\frac{M}{\rho g} R_V \frac{1}{1+sT_1}$$

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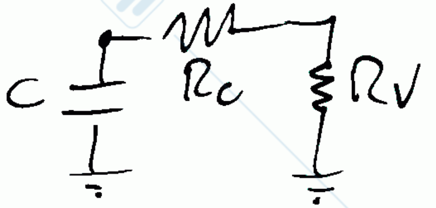
2 Time constants $T_1 >> T_2$ slow $I = 0$

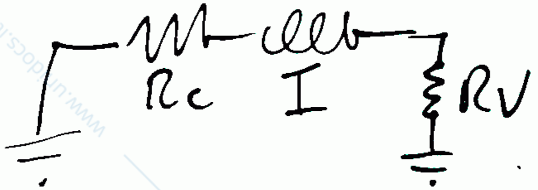
slow $\Leftrightarrow$ tank $\quad T_1 = (R_c + R_V)C = 275\ s$ slow $\updownarrow$ C



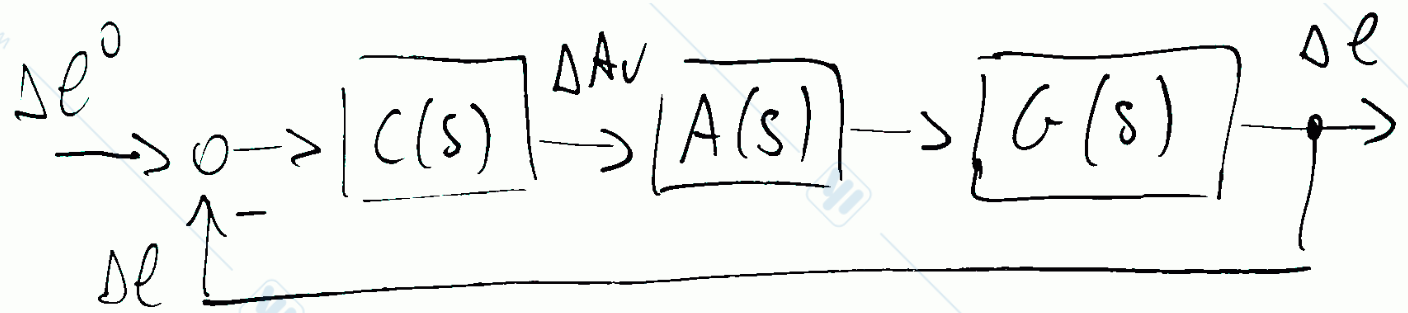
fast $\Leftrightarrow$ pipes

$$T_2 = \frac{I}{R_c + R_V} = 0.3\ s$$

fast capacitor voltage fixed=0



⑩



$$A(s) = \frac{1}{1+sT_a}$$

$$C(s) = K_p \frac{1+sT_i}{sT_i}$$

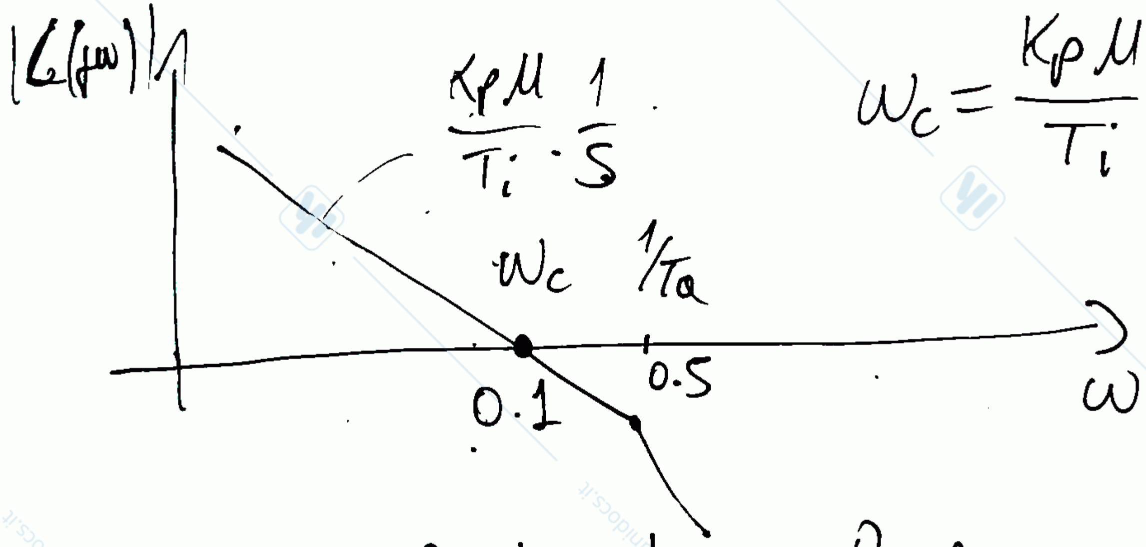
$$L(s) = C(s)\,A(s)\,G(s)$$

$$= K_p \frac{\cancel{1+sT_i}}{sT_i} \cdot \frac{1}{1+sT_a} \frac{\mu}{\cancel{1+sT_1}}$$

$T_a = 2\,s$

$T_1 = 275\,s$

$\omega_c = 0.1$ rad/s



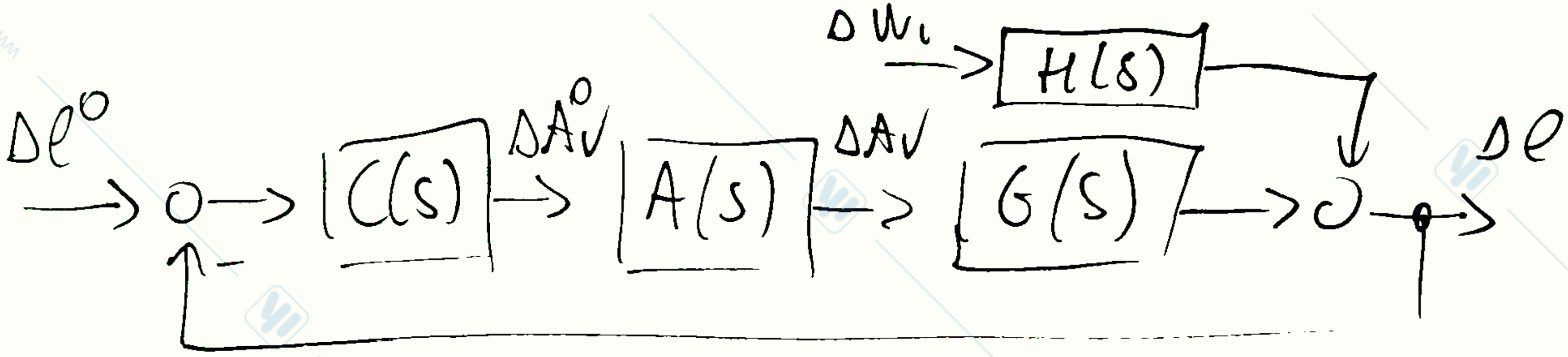
$$\omega_c = \frac{K_p\mu}{T_i} \Rightarrow K_p = \frac{\omega_c T_i}{\mu}$$

Ⓐ By concellation

$T_i = T_1 = 275\,s$

$$\varphi_m = 180^\circ - |\varphi_c| = 180^\circ + \varphi_c = 180^\circ - 90^\circ - \arctan(\omega_c T_a)$$
$$= 69^\circ$$

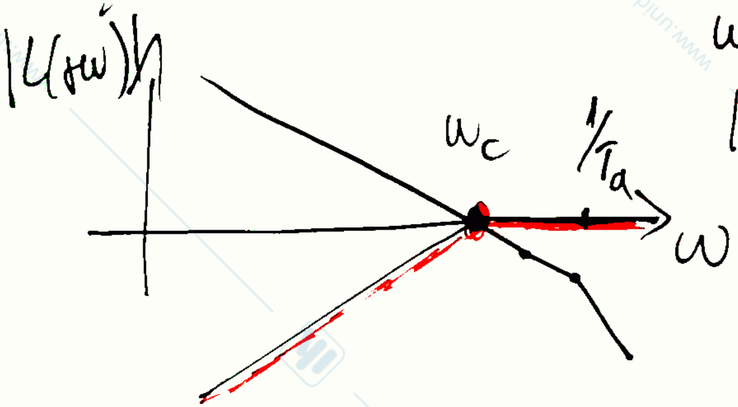
⑪



$$\frac{\Delta e}{\Delta e^0} \sim \frac{1}{1+\frac{s}{\omega_c}}$$

$$\cancel{\frac{\Delta W_i}{\Delta}} \qquad \frac{\Delta e}{\Delta W_i} = H(s) \cdot S(s)$$

$$S(s) = \frac{1}{1+L(s)}$$



$\omega \to \infty$

$|S(j\omega)| \to 1$

$$S(j\omega) \simeq \begin{cases} \frac{1}{L(j\omega)} & \omega << \omega_c \\ 1 & \omega >> \omega_c \end{cases}$$

$$S(s) \simeq \frac{s/\omega_c}{1+\frac{s}{\omega_c}} \qquad H(s) = \frac{\mu'}{1+sT_1}$$

$$|S(j\omega_c)| = \frac{1}{2\sin\frac{\varphi_m}{2}}$$

$$\frac{\Delta\theta}{\Delta\omega_i} = \frac{u'}{\omega_c} \cdot \frac{\alpha s}{1+sT_1} \frac{1}{1+\frac{s}{\omega_c}} = \alpha \frac{1}{1+sT_1} - \alpha \frac{1}{1+\frac{s}{\omega_c}}$$

$T_1 = 275\,s$ ⓘ2

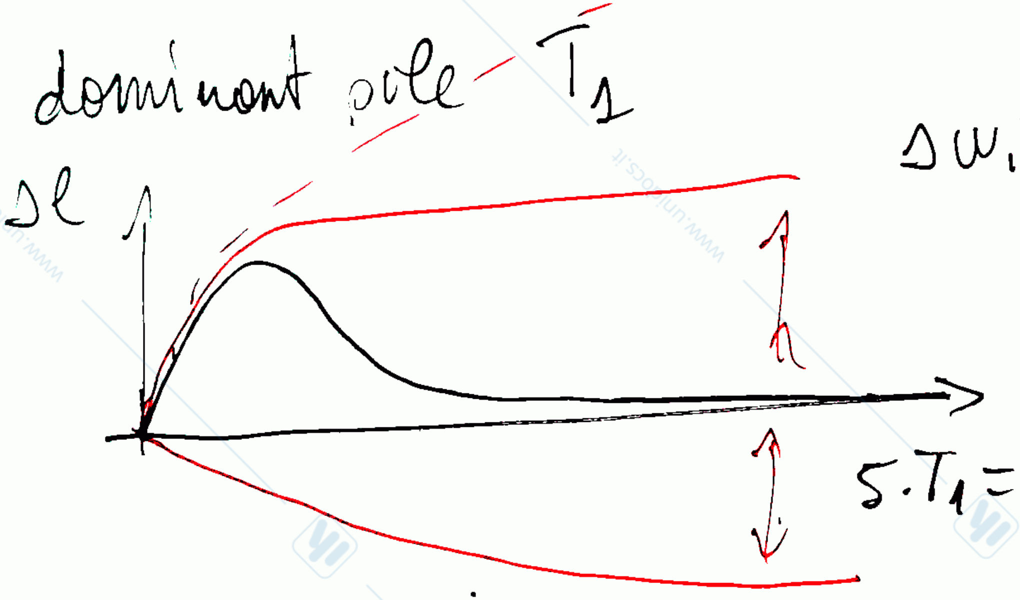
$\omega_c = 0.1\,s$

$\Delta\omega_i = \Delta\omega \cdot step(t) \rightarrow$ settling time?

dominant pole T_1

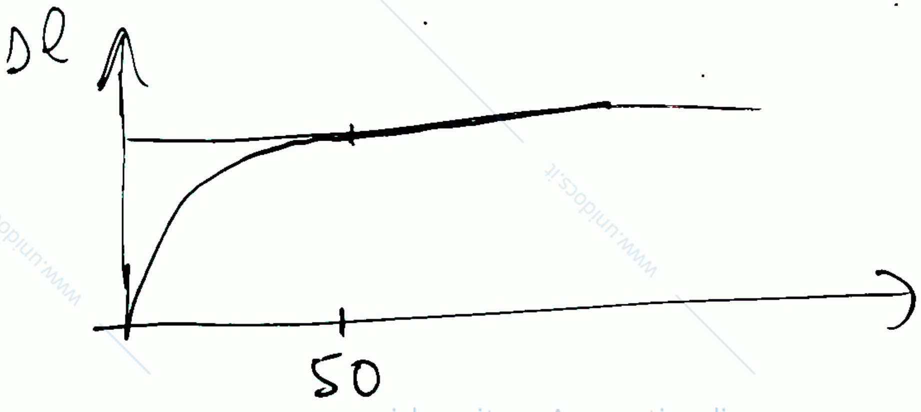
$\omega_c = 0.1$ $\frac{1}{\omega_c} = 10\,s$

$\Delta\omega_i = step(t)$



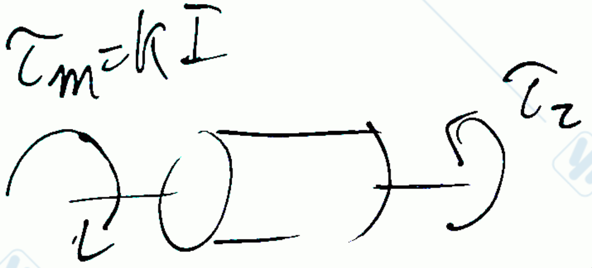
$5 \cdot T_1 = 1400\,s$

$K_I = \frac{K_p}{T_i}$

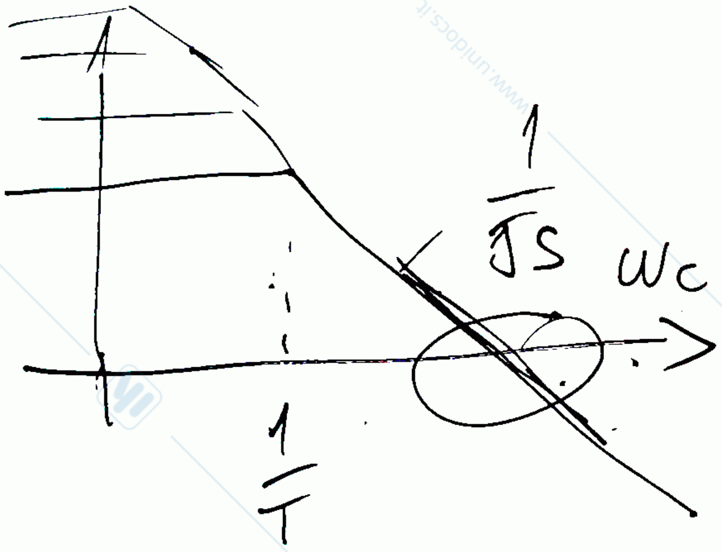


$\Delta\theta^0 = step(t)$

$T_i \downarrow\downarrow$



$$G(s) = \frac{1}{J} \frac{1}{s+\frac{D}{J}} = \frac{1}{J} \frac{1}{s+\varepsilon}$$



$$G(s) = \frac{1}{Js+D} = \frac{1}{D} \frac{1}{1+sT}$$

$$H(s) = -\frac{1}{Js+D} = -\frac{1}{D} \frac{1}{1+sT}$$

D is small

large M large T

worstcase D = 0

$$G(s) = \frac{1}{Js}$$

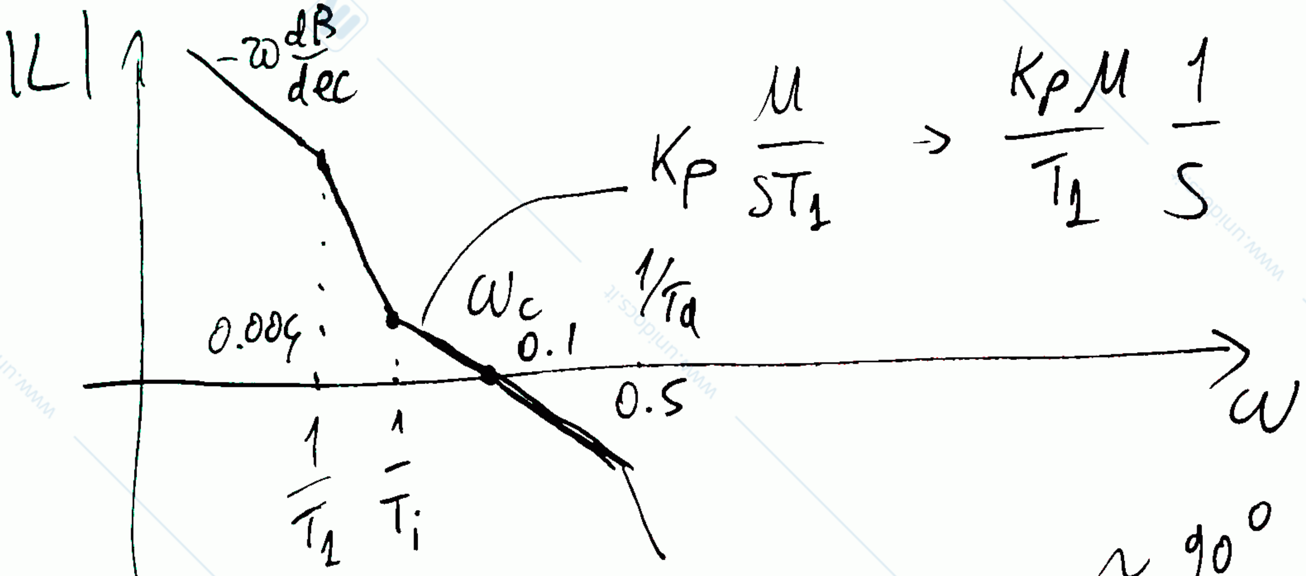
⑭

$T_1 = 275 s$

$T_a = 2 s$

$\omega_c = 0.1$ rad/s

$$L(s) = K_p \frac{1+sT_i}{sT_i} \frac{\mu}{1+sT_1} \frac{1}{1+sT_a}$$



$$K_p \frac{\mu}{sT_1} \to \frac{K_p\mu}{T_1} \frac{1}{s} \qquad \omega_c = \frac{K_p \mu}{T_1} \qquad \left(G(s) \sim \frac{\mu}{sT_1} \right)$$

$$\varphi_m = 180^\circ + \varphi_c = 180^\circ - 90^\circ - \overset{\sim 90^\circ}{\arctan(\omega_c T_1)} + \arctan(\omega_c T_i) + - \arctan(\omega_c T_a)$$

$$= \arctan(\omega_c T_i) - \underset{-11^\circ}{\arctan(\omega_c T_a)} > 60^\circ \quad > 50^\circ$$

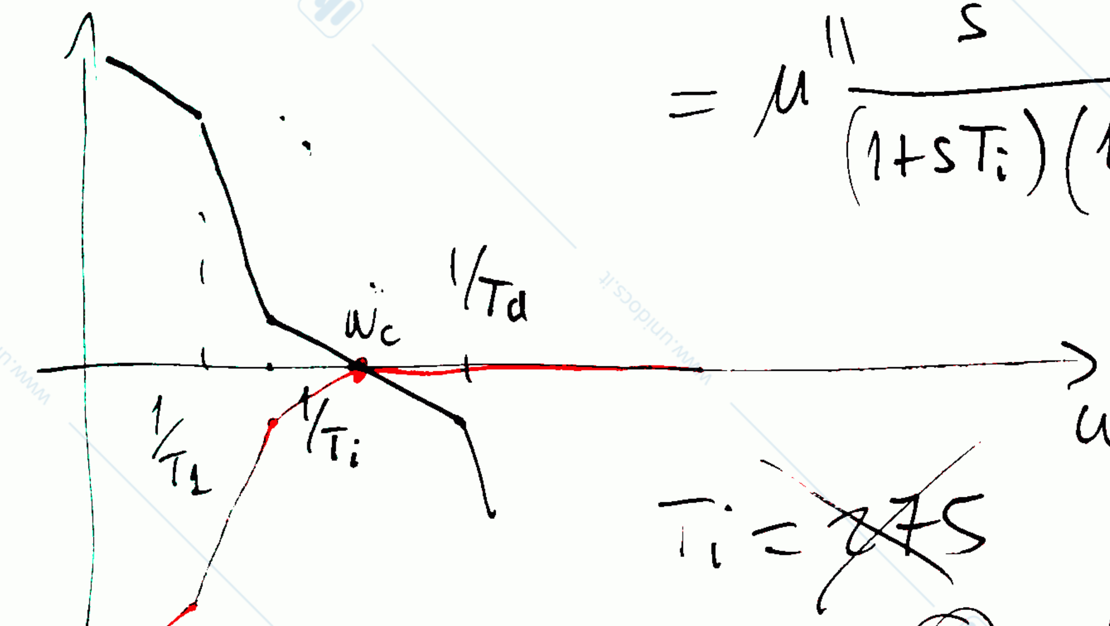
⑮

$$\frac{\Delta e}{\Delta w_i} = H(s)\cdot S(s) = \frac{\mu'}{\cancel{1+sT_1}}\cdot\frac{s\cancel{(1+sT_1)}}{1+sT_i}\;\frac{1}{1+s/\omega_c}\;\frac{T_i}{T_1\omega_c}$$

$$= \mu''\frac{s}{(1+sT_i)(1+s/\omega_c)}$$

$T_i > \frac{1}{\omega_c}$

10



$T_i =$ ~~275~~

$\textcircled{T_1}$

high φ_m $T_i = \textcircled{50}$

$\varphi_m = 68°$

low φ_m $T_i = 30$

$\varphi_m = 58°$

good φ_m

~~long~~ settling time

$T_i = \textcircled{70}$

$\varphi_m = 45°$

what happens change operating point

not much

W_c, φ_m $\Leftrightarrow$ $G'(s) = \frac{\mu}{sT_1}$

$\frac{\mu}{T_1}$ does not depend on W_u

→ fixed parameter PI

$\frac{\mu}{T_i} \propto \sqrt{\bar{e}+L_2}$

since $\bar{e}$ can change ∓ 1 m

$\sqrt{\bar{e}+L_2} \sim \sqrt{6} \div \sqrt{8}$

the gain of $G'(s)$ changes by $\mp 7\%$

→ no big deal