• In $V_3(\mathbb{R})$ siano $\mathbb{L} = \{(2, 3+t, 1-2t) : t \in \mathbb{R}\}$
$\mathbb{M} = \{(1+s, 1-s, 3) : s \in \mathbb{R}\}$. Trovare $d(\mathbb{L}, \mathbb{M})$.

$\mathbb{L} = \underbrace{(2,3,1)}_{P} + L(\underbrace{(0,1,-2)}_{N_1})$

$\mathbb{M} = \underbrace{(1,1,3)}_{Q} + L(\underbrace{(1,-1,0)}_{N_2})$

Calcolo il rango di $(N_1, N_2, P-Q)$:
se rank $= 2$ incidenti $\to d(\mathbb{L},\mathbb{M}) = 0$
se rank $= 3$ sghembe

$$\begin{matrix} 0 & 1 & -2 \\ 1 & -1 & 0 \\ 1 & 2 & -2 \end{matrix} \sim \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 1 & 2 & -2 \end{matrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -2 \end{matrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{matrix} \quad \text{rank} = 3$$

$\downarrow$ $\mathbb{L}$ e $\mathbb{M}$ sono sghembe $\downarrow$ $\exists!$ retta che è perpendicolare a $\mathbb{L}$ e $\mathbb{M}$

Calcolo d come la distanza tra due punti $P_{\mathbb{L}}$ e $P_{\mathbb{M}}$ apparten. a $\mathbb{L}$ e $\mathbb{M}$ tali che
$P_{\mathbb{L}} - P_{\mathbb{M}} \perp$ a $\mathbb{L}$ e $\mathbb{M}$

$P_{\mathbb{L}} = (2, 3+t, 1-2t)$: generico punto di $\mathbb{L}$

$P_{\mathbb{M}} = (1+s, 1-s, 3)$: generico punto di $\mathbb{M}$

$P_{\mathbb{L}} - P_{\mathbb{M}} = (1-s, 2+t+s, -2-2t)$

$\boxed{(P_{\mathbb{L}} - P_{\mathbb{M}}) \perp N_1} \wedge \boxed{(P_{\mathbb{L}} - P_{\mathbb{M}}) \perp N_2}$

$\langle P_{\mathbb{L}} - P_{\mathbb{M}}, N_1 \rangle = 0$ $\qquad$ $\langle P_{\mathbb{L}} - P_{\mathbb{M}}, N_2 \rangle = 0$

$\langle (1-s, 2+t+s, -2-2t), (0,1,-2) \rangle = 0$ $\qquad$ $\langle (1-s, 2+t+s, -2-2t), (1,-1,0) \rangle = 0$

$$\begin{cases} 2+t+s-2(-2-2t) = 0 \\ 1-s-(2+t+s) = 0 \end{cases} \to \begin{cases} 2+t+s+4+4t = 0 \\ 1-s-2-t-s = 0 \end{cases} \begin{cases} 6+5t+s = 0 \\ 1+t+2s = 0 \end{cases}$$

$$\begin{array}{cc} s & t \end{array}$$
$$\left(\begin{array}{cc|c} 1 & 5 & -6 \\ 2 & 1 & -1 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 5 & -6 \\ 0 & -9 & 11 \end{array}\right) \xrightarrow{R_2 \to (-\frac{1}{9})R_2} \left(\begin{array}{cc|c} 1 & 5 & -6 \\ 0 & 1 & -\frac{11}{9} \end{array}\right) \xrightarrow{R_1 \to R_1 - 5R_2} \left(\begin{array}{cc|c} 1 & 0 & 1/9 \\ 0 & 1 & -11/9 \end{array}\right)$$

$s = \frac{1}{9}$; $t = -\frac{11}{9}$ $\to$ $P_{\mathbb{L}} = (2, \frac{16}{9}, \frac{31}{9})$, $P_{\mathbb{M}} = (\frac{10}{9}, \frac{8}{9}, 3)$; $P_{\mathbb{L}} - P_{\mathbb{M}} = (\frac{8}{9}, \frac{8}{9}, \frac{4}{9})$

$$d(\mathbb{L},\mathbb{M}) = d(P_{\mathbb{L}}, P_{\mathbb{M}}) = \|P_{\mathbb{L}} - P_{\mathbb{M}}\| = \sqrt{\left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^2 + \left(\frac{4}{9}\right)^2} = \frac{12}{9}$$

• Sia $\varphi : V_3(\mathbb{R}) \to V_4(\mathbb{R})$. Calcolare null$(\varphi)$, rank$(\varphi)$. Descrivere Im$(\varphi)$ e Ker(φ).

$$M_{E,F}(\varphi) = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \\ 2 & 3 & 3 \end{pmatrix} = M$$

dim(dominio) = rank(φ)

φ è iniettiva ma non suriettiva $\to$ dim(codom) $\neq$ rank(φ)

$$M^T = \begin{pmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & 1 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -7 & -1 \\ 0 & 5 & 5 & 5 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{5}R_3} \begin{pmatrix} 1 & 3 & 4 & 2 \\ 0 & -7 & -7 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -7 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 7R_2} \begin{pmatrix} 1 & 3 & 4 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

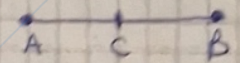
rank$(M) = 3 \Rightarrow$ rank$(\varphi) = 3$. $\underset{3}{\dim(V)} - \underset{3}{\text{rank}(\varphi)} = \underset{0}{\text{null}(\varphi)}$

Im$(\varphi) = L((1,3,4,2), (0,1,1,1), (0,0,0,6))$. Ker$(\varphi) = \{0\}$

• In $V_2(\mathbb{R})$. Esprimere le coordinate del punto medio $C=(c_1,c_2)$ del segmento di estremi $A=(a_1,a_2)$ e $B=(b_1,b_2)$ in funzione delle coordinate di A e B.

$$\|(a_1,a_2)-(c_1,c_2)\| = \|(c_1,c_2)-(b_1,b_2)\|$$ per definizione di punto medio

$$\|(a_1-c_1,a_2-c_2)\| = \|(c_1-b_1,c_2-b_2)\|$$



poiché $\|x\| = \sqrt{\langle x,x\rangle}$:

$$\sqrt{\langle(a_1-c_1,a_2-c_2),(a_1-c_1,a_2-c_2)\rangle} = \sqrt{\langle(c_1-b_1,c_2-b_2),(c_1-b_1,c_2-b_2)\rangle}$$

sse $\langle(a_1-c_1,a_2-c_2),(a_1-c_1,a_2-c_2)\rangle = \langle(c_1-b_1,c_2-b_2),(c_1-b_1,c_2-b_2)\rangle$

il prodotto scalare di un vettore per se stesso è il prodotto dei moduli, quindi: $(a_1-c_1)^2+(a_2-c_2)^2 = (c_1-b_1)^2+(c_2-b_2)^2$

$$a_1^2+\cancel{c_1^2}-2a_1c_1+a_2^2+\cancel{c_2^2}-2a_2c_2 = \cancel{c_1^2}+b_1^2-2c_1b_1+\cancel{c_2^2}+b_2^2-2c_2b_2$$

poiché $C\in[A,B]$ posso scriverlo come $C = t\cdot A+(1-t)\cdot B$ con $t\in[0,1]$

(quando $t=0 \rightarrow C=B$; quando $t=1\rightarrow C=A$)

$$(c_1,c_2) = t(a_1,a_2)+(1-t)(b_1,b_2)$$
$$c_1 = ta_1+(1-t)b_1 = t(a_1-b_1)+b_1$$
$$c_2 = ta_2+(1-t)b_2 = t(a_2-b_2)+b_2$$

$$a_1^2-2a_1(t(a_1-b_1)+b_1)+a_2^2-2a_2(t(a_2-b_2)+b_2) = b_1^2-2b_1(t(a_1-b_1)+b_1) - 2b_2(t(a_2-b_2)+b_2) \quad \dots$$

$$(a_1-b_1)^2(1-2t)+(a_2-b_2)^2(1-2t)=0$$

$$(1-2t)\underbrace{\left[(a_1-b_1)^2+(a_2-b_2)^2\right]}=0$$

$=0$? $(a_1-b_1)^2+(a_2-b_2)^2=0$ sse $a_1=b_1$ e $a_2=b_2$

No! perché devono essere punti distinti

$=0$: unica alternativa

$1-2t=0 \quad t=1/2$

$$c_1 = \frac{1}{2}(a_1-b_1)+b_1 = \frac{a_1+b_1}{2}$$
$$c_2 = \frac{1}{2}(a_2-b_2)+b_2 = \frac{a_2+b_2}{2}$$

① a) $(X)_{\mathbb{E}} = (1,1,0,0)$

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \qquad \begin{cases} x_1 = y_2 + y_3 \\ x_2 = y_1 - y_4 \\ x_3 = y_1 \\ x_4 = y_2 - y_3 \end{cases} \quad \dots \quad \begin{cases} y_1 = x_3 \\ y_2 = \frac{1}{2}x_1 + \frac{1}{2}x_4 \\ y_3 = \frac{1}{2}x_1 - \frac{1}{2}x_4 \\ y_4 = x_3 - x_2 \end{cases} \quad \begin{cases} y_1 = 1 \\ y_2 = 1 \\ y_3 = 0 \\ y_4 = 0 \end{cases}$$

$\uparrow E_1 \quad \uparrow E_2 \quad \uparrow E_3 \quad \uparrow E_4$

b) $v = 1\cdot E_1 + 1\cdot E_2 + 1\cdot E_3 + 0\cdot E_4$ (perché $v_{\mathbb{E}} = (1,1,1,0)$)

$v = E_1 + E_2 + E_3$

c) $$M_{\mathbb{E}\to\mathbb{F}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$ con $\mathbb{F} = (E_4, E_3, E_1, E_2)$

②

$A + L(B-A, C-A)$

$B-A = (-1,1,-1,1) \leftarrow s$

$C-A = (0,1,-1,0) \leftarrow t$

$\{(1,0,1,0) + (-s,s,-s,s) + (0,t,-t,0)\} = \{(1-s, s+t, 1-s-t, s) : s,t \in \mathbb{R}\}$

a)
$$\begin{matrix} -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 \end{matrix} \quad R_3 \to R_3 - R_2 + R_1 \quad \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$$

S non è lineare perché $L(B-A, C-A)$ non contiene il punto di appoggio

b) A, B, C

③ $\dim(A^\perp) = 4 - 3 = 1$

$$A^\perp = \begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_3 = 0 \\ x_1 + x_4 = 0 \end{cases}$$

④ $x + y + z = 1 \qquad x = -y - z + 1 \qquad \{(-y-z+1, y, z) : x,y,z \in \mathbb{R}\} =$

$= (1,0,0) + L((-1,1,0), (-1,0,1))$

$L((-1,1,0))$: generica retta // non contenuta nel piano.

⑤ $$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$F: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ rango = 2

dominio (3 colonne) codominio (2 righe)

• suriettiva? sì $\dim(\text{codom}) = \text{rank}$ (2 = 2)

• iniettiva? NO $\dim(\text{dom}) \neq \text{rank}$ (3 ≠ 2)

• $A = \mathcal{L}(A, B, C) = \mathcal{L}\left(\underset{A}{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}, \underset{B}{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}, \underset{C}{\begin{pmatrix}0\\1\\-1\\1\end{pmatrix}}\right)$ rappresentazione parametrica di A in $V_4(\mathbb{R})$

$\dim(A) = 3$

cartesiana di $A^\perp$:

$$\underset{A^\perp}{\underset{\shortparallel}{(\mathcal{L}(A,B,C))^\perp}} = \begin{cases} \langle X, A\rangle = 0 \\ \langle X, B\rangle = 0 \\ \langle X, C\rangle = 0 \end{cases} \begin{cases} (x_1,x_2,x_3,x_4)\cdot(1,0,1,0) = 0 \\ (x_1,x_2,x_3,x_4)\cdot(0,0,0,1) = 0 \\ (x_1,x_2,x_3,x_4)\cdot(0,1,-1,1) = 0 \end{cases} \begin{cases} x_1 + x_3 = 0 \\ x_4 = 0 \\ x_2 - x_3 + x_4 = 0 \end{cases}$$

Scrivo A in rappresentazione cartesiana (da A in parametrica):

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 1 & 0 & -1 & x_3 \\ 0 & 1 & 1 & x_4 \end{array}\right) R_3 \to R_3 + R_2 \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 1 & 0 & 0 & x_3 + x_2 \\ 0 & 1 & 1 & x_4 \end{array}\right) R_3 \to R_3 - R_1 \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & x_3 + x_2 - x_1 \\ 0 & 1 & 1 & x_4 \end{array}\right)$$

$x_3 + x_2 - x_1 = 0$: rappresentazione cartesiana di A (infatti ho 1 equazione)

$4-3$

da qui

se voglio scrivere $A^\perp$ è immediato: $\mathcal{L}\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}$ (infatti $\dim(A^\perp) = 1$)

Se voglio passare da $A^\perp$ cartesiana a $A^\perp$ parametrica:

$$\begin{cases} x_1 = -x_3 \\ x_4 = 0 \\ x_2 = x_3 - x_4 \to x_2 = x_3 \end{cases} \qquad \{x_3(-1,1,1,0) : x_3 \in \mathbb{R}\} = \mathcal{L}((-1,1,1,0))$$