① In $V_5(\mathbb{R})$ sia $S = L(B_1, B_2, B_3)$ con $B_1 = (1,-1,0,1,-1)$
$B_2 = (0,1,2,1,0)$
$B_3 = (1,-1,2,-1,3)$

Scrivere una rappresentazione cartesiana di S.

$$\begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & -1 & 2 & -1 & 3 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & -2 & 4 \end{pmatrix}$$

$\text{rank}(B_1, B_2, B_3) = 3 = \dim(S) \Rightarrow 5 - 3 = 2$

Ogni rappresentazione cartesiana di S è formata da 2 equazioni lineari omogenee indipendenti

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ -1 & 1 & -1 & x_2 \\ 0 & 2 & 2 & x_3 \\ 1 & 1 & -1 & x_4 \\ -1 & 0 & 3 & x_5 \end{array}\right) \begin{array}{l} \\ R_2 \to R_2 + R_1 \\ \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 + R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & 0 & x_2 + x_1 \\ 0 & 2 & 2 & x_3 \\ 0 & 1 & -2 & x_4 - x_1 \\ 0 & 0 & 4 & x_5 + x_1 \end{array}\right) \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & 0 & x_2 + x_1 \\ 0 & 0 & 2 & x_3 - 2x_2 - 2x_1 \\ 0 & 0 & -2 & x_4 - x_1 - x_2 - x_1 \\ 0 & 0 & 4 & x_5 + x_1 \end{array}\right)$$

$R_4 \to R_4 + R_3$ $\quad R_5 \to R_5 - 2R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & 0 & x_2 + x_1 \\ 0 & 0 & 2 & -2x_1 - 2x_2 + x_3 \\ 0 & 0 & 0 & -2x_1 - x_2 + x_4 - 2x_1 - 2x_2 + x_3 \\ 0 & 0 & 0 & x_1 + x_5 - 2(-2x_1 - 2x_2 + x_3) \end{array}\right)$$

$$\begin{cases} -4x_1 - 3x_2 + x_3 + x_4 = 0 \\ 5x_1 + 4x_2 - 2x_3 + x_5 = 0 \end{cases}$$

RAPPRESENTAZIONE CARTESIANA DI S

$5 - 3 = 2$ equazioni

② Sia $M_{2,2}(\mathbb{R})$, sia $A = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\}$

Trovare un complemento lineare di A in $M_{2,2}(\mathbb{R})$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ è una base per A

$$\boxed{\begin{aligned} &\dim(A) = 3 \\ &\operatorname{codim}(A) = \dim(V) - \dim(A) = \\ &= 4 - 3 = 1 \end{aligned}}$$

B è un complemento lineare di A in V,

$\uparrow$

sottospazio lineare

$A, B \subseteq V$, se $V = A \oplus B$ cioè

$A \cap B = \{0\}$

e

$A + B = V$

$$B = \left\{ \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$$

$\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ è una base per B, $\boxed{\dim(B) = 1}$

- $A \cap B = \{0\}$?

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

$a_{11} = 0$

$a_{12} = 0$

$a_{12} = a \longrightarrow a = 0 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow A \cap B = \{0\}$

$a_{22} = 0$

- $A + B = M_{2,2}(\mathbb{R})$?

$A + B = L(A \cup B)$

$$L\Bigg(\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{base di } A}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{base di } B} \Bigg)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \operatorname{rank}(A+B) = 4 = \dim(M_{2,2}(\mathbb{R}))$$

Grassmann $\dim(A+B) + \dim(A \cap B) = \dim(A) + \dim(B)$

$$4 + 0 = 3 + 1$$

③ In $V_4(\mathbb{R})$ siano A e B spazi delle soluzioni di

$$A = \begin{cases} x_1 - x_3 = 0 \\ x_2 + x_4 = 0 \end{cases} \qquad B = \begin{cases} x_1 - x_2 - x_3 + x_4 = 0 \\ 2x_1 + x_2 - 2x_3 - x_4 = 0 \end{cases}$$

$\dim(A+B) = ?$ $\quad \dim(A \cap B) = ?$

Descrivere $A+B$ e $A \cap B$.

$\dim(A+B) = 3$; $\dim(A \cap B) = 1$; $\dim(A) = \dim(B) = 2$

(svolto a lezione)

$A \cap B$: RAPPRESENT. CARTESIANA $4 - 1 = 3$ eq.

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_4 = 0 \\ x_4 = 0 \end{cases}$$

$A \cap B = \{(x_3, 0, x_3, 0) : x_3 \in \mathbb{R}\} = L((1,0,1,0))$

RAPPRESENTAZIONE PARAMETRICA

$A + B = L(A \cup B)$

$A = \{(x_3, -x_4, x_3, x_4) : x_3, x_4 \in \mathbb{R}\}$; base di A: $\{(1,0,1,0), (0,-1,0,1)\}$

$$B: \begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & 1 & -2 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 3 & 0 & -3 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{pmatrix} 1 & \boxed{-1} & -1 & 1 \\ \boxed{0} & 1 & 0 & -1 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \qquad \begin{cases} x_1 - x_3 = 0 \\ x_2 - x_4 = 0 \end{cases}$$

$B = \{(x_1, x_2, x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$

base di B: $\{(1,0,1,0), (0,1,0,1)\}$

$$A + B = L(A \cup B) = L(\underbrace{(1,0,1,0), (0,-1,0,1)}_{\text{base di A}}, \underbrace{\cancel{(1,0,1,0)}, (0,1,0,1)}_{\text{base di B}})$$

$$= L(\underbrace{(1,0,1,0)}_{A_1}, \underbrace{(0,-1,0,1)}_{A_2}, \underbrace{(0,1,0,1)}_{A_3}) = \{(a_1, -a_2 + a_3, a_1, a_2 + a_3) : a_1, a_2, a_3 \in \mathbb{R}\} =$$

$$= \{a_1 \cdot A_1 + a_2 \cdot A_2 + a_3 \cdot A_3 : a_1, a_2, a_3 \in \mathbb{R}\}$$

RAPPRESENTAZIONE PARAMETRICA

$$A + B: \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & -1 & 1 & x_2 \\ 1 & 0 & 0 & x_3 \\ 0 & 1 & 1 & x_4 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & -1 & 1 & x_2 \\ 0 & 0 & 0 & x_3 - x_1 \\ 0 & 1 & 1 & x_4 \end{array}\right) \to x_3 - x_1 = 0$$

RAPPRES. CART.

$4 - 3 = 1$ eq.

④ In $V_5(\mathbb{R})$

$$S: \begin{cases} x_1 - x_2 + x_3 + 2x_4 - x_5 = 1 \\ x_2 - 2x_3 + x_4 - 3x_5 = 0 \end{cases}$$

$$S': \begin{cases} x_1 + x_3 - x_5 = 1 \\ x_2 + x_4 + 3x_5 = 0 \\ x_3 - x_4 + 2x_5 = 1 \end{cases}$$

$S \cap S'$?

$$S \cap S': \begin{cases} x_1 - x_2 + x_3 + 2x_4 - x_5 = 1 \\ x_2 - 2x_3 + x_4 - 3x_5 = 0 \\ x_1 + x_3 - x_5 = 1 \\ x_2 + x_4 + 3x_5 = 0 \\ x_3 - x_4 + 2x_5 = 1 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 1 & -3 & 0 \\ 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccccc|c} 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -2 & 1 & -3 & 0 \\ 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

$R_2 \to R_2 + R_1 \quad R_4 \to R_4 + R_1$

$$\left(\begin{array}{ccccc|c} 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 & -3 & 0 \\ 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right) \xrightarrow{R_5 \to 2R_5 + R_2} \begin{array}{c} \left(\begin{array}{ccccc|c} 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 & -3 & 0 \\ 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array}\right) \end{array} \begin{array}{l} 2° \\ 3° \\ 1° \\ 4° \\ 5° \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 & -3 & 0 \\ 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array}\right) \xrightarrow{R_5 \to -3R_5 + R_4} \left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 & -3 & 0 \\ 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 \end{array}\right)$$

rank 4

rank 5

$\text{rank}(M_c) = \text{rank}(M_i) + 1$ il sistema è impossibile (da Rouché Capelli)

$$Sol(S \cap S') = \emptyset$$

⑤ In $V_5(\mathbb{R})$

$$S: \begin{cases} x_1 - x_2 + x_3 + 2x_4 - x_5 = 1 \\ x_2 - 2x_3 + x_4 - 3x_5 = 0 \end{cases} \leftarrow \text{RAPP. CART.}$$

$S': P + L(A,B)$ dove $P = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 1 \end{pmatrix}$ $A = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$

RAPP. PAR. ↑

$B = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

$S \cap S'$?

$x = P + t \cdot A + s \cdot B \in S'$

$x_1 = 1 + 3t + s$

$x_2 = t + 2s$

$x_3 = -1 + s$

$x_4 = 3 - t$

$x_5 = 1 + 2t - s$

$$\begin{cases} (1+3t+s) - (t+2s) + (-1+s) + 2(3-t) - (1+2t-s) = 1 \\ (t+2s) - 2(-1+s) + (3-t) - 3(1+2t-s) = 0 \end{cases}$$

$$\begin{cases} -2t + s = -4 \\ -6t + 3s = -2 \end{cases} \quad \begin{cases} 2t - s = 4 \\ 2t - s = \frac{2}{3} \end{cases} \quad \text{NO!} \quad S \cap S' = \emptyset$$

⑥ In $V_5(\mathbb{R})$

$S = P + L(A_1, A_2, A_3)$

$S' = Q + L(B_1, B_2)$

$$P = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad A_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 \\ -1 \\ 3 \\ -1 \\ 2 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 1 \end{pmatrix} \quad B_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \\ 2 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$S \cap S'$?

$$\begin{matrix} 3 & 1 & 0 & 1 & 1 \\ 0 & -1 & 3 & -1 & 2 \\ 1 & -1 & 2 & 0 & 1 \\ 3 & 1 & 0 & -1 & 2 \\ 1 & 2 & 1 & 0 & 0 \end{matrix}$$

$\operatorname{rank} K = 5$ (verificare!) $\rightarrow$ qualsiasi altra equazione sarà linearm. dipend. da queste

Quindi $Q - P \in L(A_1, A_2, A_3, B_1, B_2) = V_5(\mathbb{R})$

$S \cap S' \neq \emptyset$ anzi è un punto (Grassmann)

poiché $\dim(S \cap S') = \dim(S) + \dim(S') - \underbrace{\dim(S + S')}_{\dim(L(S \cup S'))}$

$= 3 + 2 - 5$

$P + t_1 \cdot A_1 + t_2 \cdot A_2 + t_3 \cdot A_3 = Q + s_1 B_1 + s_2 B_2$

$-1 + 3t_1 + 0 \cdot t_2 + 1 \cdot t_3 = 1 + 3 s_1 + 1 \cdot s_2$

$0 + 1 \cdot t_1 - 1 \cdot t_2 - 1 \cdot t_3 = 0 + 1 \cdot s_1 + 2 s_2$

$1 + 0 \cdot t_1 + 3 \cdot t_2 + 2 \cdot t_3 = -1 + 0 \cdot s_1 + 1 \cdot s_2$

$1 + 1 \cdot t_1 - 1 \cdot t_2 + 0 \cdot t_3 = 3 - 1 \cdot s_1 + 0 \cdot s_2$

$0 + 1 \cdot t_1 + 2 t_2 + 1 \cdot t_3 = 1 + 2 \cdot s_1 + 0 \cdot s_2$

$$\left(\begin{matrix} 3 & 0 & 1 & -3 & -1 \\ 1 & -1 & -1 & -1 & -2 \\ 0 & 3 & 2 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 1 & -2 & 0 \end{matrix} \left| \begin{matrix} 2 \\ 0 \\ -2 \\ 2 \\ 1 \end{matrix} \right.\right) \begin{matrix} R_1 \to R_1 - 3R_4 \\ R_2 \to R_2 - R_4 \\ \\ \\ R_5 \to R_5 - R_4 \end{matrix} \quad \left(\begin{matrix} 0 & 3 & 1 & -6 & -1 \\ 0 & 0 & -1 & -2 & -2 \\ 0 & 3 & 2 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 3 & 1 & -3 & 0 \end{matrix} \left| \begin{matrix} -4 \\ -2 \\ -2 \\ 2 \\ -1 \end{matrix} \right.\right)$$

$R_1 \to R_1 - R_3 \quad R_5 \to R_5 - R_3$

$$\left(\begin{matrix} 0 & 0 & -1 & -6 & 0 \\ 0 & 0 & -1 & -2 & -2 \\ 0 & 3 & 2 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -3 & 1 \end{matrix} \left| \begin{matrix} -2 \\ -2 \\ -2 \\ 2 \\ 1 \end{matrix} \right.\right) \begin{matrix} \\ R_2 \to R_2 - R_1 \\ \\ \\ R_5 \to R_5 - R_1 \end{matrix} \quad \begin{array}{ccccc|c} 0 & 0 & -1 & -6 & 0 & -2 \\ 0 & 0 & 0 & \boxed{4} & \boxed{-2} & \boxed{0} \\ 0 & 3 & 2 & 0 & -1 & -2 \\ 1 & -1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & \boxed{3} & \boxed{1} & \boxed{3} \\ t_1 & t_2 & t_3 & s_1 & s_2 & \end{array}$$

$$\begin{cases} 4S_1 - 2S_2 = 0 \\ 3S_1 + S_2 = 3 \end{cases} \rightarrow \begin{cases} S_2 = 2S_1 \\ 5S_1 = 3 \end{cases} \rightarrow S_1 = \frac{3}{5} \quad S_2 = \frac{6}{5}$$

$$S \cap S' = Q + \frac{3}{5}B_1 + \frac{6}{5}B_2 = Q + \frac{3}{5}(B_1 + 2B_2) = (1,0,-1,3,1) + $$
$$+ \frac{3}{5}(5,5,2,-1,2) = \left(4, 3, \frac{1}{5}, \frac{12}{5}, \frac{11}{5}\right)$$

(7) In $V_4(\mathbb{R})$

$$A = \mathcal{L}\left(\begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}\right) = \left\{\begin{pmatrix} 2t+s \\ -t+2s \\ 2t+s \\ t-2s \end{pmatrix} : t,s \in \mathbb{R}\right\}$$

$$B = \begin{cases} x_1 - x_2 - x_3 + x_4 = 0 \\ 2x_1 + x_2 - 2x_3 - x_4 = 0 \end{cases} \qquad \dim(B) = 4 - \underset{\#\text{ eq.}}{2} = 2$$

Descrivere $A+B$ e $A \cap B$.

$\boxed{A \cap B}$

$$\begin{cases} 2t+s-(-t+2s)-(2t+s)+t-2s = 0 \\ 2(2t+s)+(-t+2s)-2(2t+s)-(t-2s) = 0 \end{cases}$$

$$\begin{cases} 2t - 4s = 0 \\ -2t + 4s = 0 \end{cases} \rightarrow 2t = 4s \rightarrow t = 2s$$

$$A \cap B: \begin{pmatrix} 2\cdot 2s + s \\ -2s + 2s \\ 2\cdot 2s + s \\ 2s - 2s \end{pmatrix} = \begin{pmatrix} 5s \\ 0 \\ 5s \\ 0 \end{pmatrix} = 5s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad A \cap B = \mathcal{L}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\dim(A \cap B) = 1$$

$$\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B) = 2 + 2 - 1 = 3$$

$\boxed{A+B}$

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 2 & 1 & -2 & -1 \end{pmatrix} R_2 \to R_2 - 2R_1 \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 3 & 0 & -3 \end{pmatrix} R_2 \to \frac{1}{3}R_2$$

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} R_1 \to R_1 + R_2 \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_4 \end{cases}$$

$$B = \{(x_1, x_2, x_1, x_2) : x_1, x_2 \in \mathbb{R}\} = \{x_1 \underbrace{(1,0,1,0)}_{\in A \cap B} + x_2(0,1,0,1)\}$$

$$A + B = \mathcal{L}\left(\begin{pmatrix} 2 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right)$$