① Sia $\{E_i\}_{i=0}^{n} \subseteq P_n(\mathbb{R})$

$E_0 := 1$

$E_1 := 1+x$

$E_2 := 1+x+x^2$

$E_i := 1+x+x^2+\dots+x^i = \sum_{k=0}^{i} x^k$

$E_n := 1+x+x^2+\dots+x^n$

Verificare che $\{E_i\}_{i=0}^{n}$ è una base di $P_n(\mathbb{R})$

- $L(\{E_i\}_{i=0}^{n}) = P_n(\mathbb{R})$
- $\{E_i\}_{i=0}^{n}$ è linearmente indipendente

- $B = \{\underset{U_0}{1}, \underset{U_1}{x}, \underset{U_2}{x^2}, \dots, \underset{U_n}{x^n}\}$ è una base di $P_n(\mathbb{R})$ (già dimostrato a lezione)

$$\begin{cases} U_0 = 1 \\ U_1 = x \\ U_i = x^i \\ U_n = x^n \end{cases}$$

$L(\{U_i\}_{i=0}^{n}) = P_n(\mathbb{R})$ perché è una base.

Faccio vedere che $L(\{E_i\}_{i=0}^{n}) = L(\{U_i\}_{i=0}^{n})$:

$E_0 = U_0$

$E_1 = U_0 + U_1$

$E_2 = U_0 + U_1 + U_2$

$E_i = U_0 + U_1 + \dots + U_i$

$E_n = U_0 + U_1 + \dots + U_n$

$\rightarrow \{E_i\}_{i=0}^{n} \subseteq \underbrace{L(\{U_i\}_{i=0}^{n})}_{= P_n(\mathbb{R})}$

$\{E_i\}_{i=0}^{n} \subseteq L(\{U_i\}_{i=0}^{n}) = P_n(\mathbb{R})$

$\downarrow$

$$\boxed{L(\{E_i\}_{i=0}^{n}) \subseteq L(\{U_i\}_{i=0}^{n}) = P_n(\mathbb{R})}$$

$U_0 = E_0$

$U_1 = E_1 - U_0 = E_1 - E_0$

$U_2 = E_2 - E_0 - (E_1 - E_0) = E_2 - \cancel{E_0} - E_1 + \cancel{E_0} = E_2 - E_1 \rightarrow \boxed{L(\{U_i\}_{i=0}^{n}) \subseteq L(\{E_i\}_{i=0}^{n})}$

$$L(\{E_i\}_{i=0}^{n}) = L(\{U_i\}_{i=0}^{n}) = P_n(\mathbb{R})$$

- $\lambda_0 E_0 + \lambda_1 E_1 + \dots + \lambda_n E_n = 0 \overset{?}{\Rightarrow} \lambda_0 = \lambda_1 = \dots = \lambda_n = 0$

$$\lambda_0 \cdot \underbrace{1}_{E_0} + \lambda_1 \underbrace{(1+x)}_{E_1} + \lambda_2 \underbrace{(1+x+x^2)}_{E_2} + \dots + \lambda_n \underbrace{(1+x+x^2+\dots+x^n)}_{E_n} = 0$$

$$(\lambda_0 + \lambda_1 + \lambda_2 + \dots + \lambda_n) + (\lambda_1 + \lambda_2 + \dots + \lambda_n)x + \dots + (\lambda_{n-1} + \lambda_n)x^{n-1} + \lambda_n x^n = 0$$

$$\begin{cases} \lambda_n = 0 \\ \lambda_{n-1} + \lambda_n = 0 \rightarrow \lambda_{n-1} = 0 \\ \lambda_{n-2} + \lambda_{n-1} + \lambda_n = 0 \rightarrow \lambda_{n-2} = 0 \\ \vdots \\ \lambda_0 = 0 \end{cases} \rightarrow \{E_i\}_{i=0}^n \text{ è linearmente indipendente}$$

② $V_4(\mathbb{R})$

$X = \{\underbrace{(1,1,0,0)}_{= A}, \underbrace{(-1,0,1,0)}_{= B}\}$ è linearmente indipendente.

Completare X a una base di $V_4(\mathbb{R})$.

$\{U_1, U_2, U_3, U_4\}$ base di $V_4(\mathbb{R})$

- $U_1 \overset{?}{\in} L(A,B)$

$$\begin{array}{c} U_1 = \\ A = \\ B = \end{array}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \begin{array}{c} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\text{rank}(A,B,U_1) = 3 \Rightarrow U_1 \notin L(A,B)$

$\{A, B, U_1\}$

- $U_2 \overset{?}{\in} L(A,B,U_1)$

$$\begin{array}{c} U_2 \\ A \\ B \\ U_1 \end{array}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \\ \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - R_1 \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{array}{c} \\ \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 + R_2 \end{array}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

$\text{rank}(U_2, A, B, U_1) = 3 \Rightarrow U_2 \in L\{A, B, U_1\}$

- $U_3 \overset{?}{\in} L(A,B,U_1)$ <u>sì</u>, verificare
- $U_4 \overset{?}{\in} L(A,B,U_1)$

$$\begin{array}{c} U_4 = \\ A = \\ B = \\ U_1 = \end{array}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

verificare che $\text{rank}(U_4, A, B, U_1) = 4 \Rightarrow U_4 \notin L(A,B,U_1)$

$\{A, B, U_1, U_4\}$ è linearmente indipendente

$$L(A,B,U_1,U_4) = L(U_1,U_2,U_3,U_4) = V_4(\mathbb{R})$$

$\Longrightarrow \{A, B, U_1, U_4\}$ base di $V_4(\mathbb{R})$

③ Sia $V = V_4(\mathbb{R})$ e sia $\mathbb{U} = (U_1, U_2, U_3, U_4)$ la base ordinata dei versori di V.

Sia $\mathbb{E} = (E_1, E_2, E_3, E_4)$ dove:

$$\left.\begin{array}{l} E_1 = (1,0,1,2) \\ E_2 = (-1,2,2,1) \\ E_3 = (3,1,3,0) \\ E_4 = (1,-1,1,1) \end{array}\right) \text{ coordinate rispetto alla base dei versori}$$

1. Verificare che $\{E_1, E_2, E_3, E_4\}$ è una base di V
2. Scrivere la matrice M che porta $\mathbb{U}$ in $\mathbb{E}$
3. Determinare M^{-1}
4. Sia $X \in V$: $(X)_{\mathbb{U}} = (1,-1,0,1)$.
   Scrivere le coordinate $(X)_{\mathbb{E}}$ di X rispetto alla base $\mathbb{E}$.

1.
$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ -1 & 2 & 2 & 1 \\ 3 & 1 & 3 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 3 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & -1 & 0 & -1 \end{pmatrix} R_4 \to (-1)R_4 \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 2 & 3 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 3 & 15 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

$\text{rank}(E_1, E_2, E_3, E_4) = 4 \Rightarrow \{E_1, E_2, E_3, E_4\}$ linearmente indipendenti
$\Downarrow$
$\{E_1, E_2, E_3, E_4\}$ è una base di V

2. $\mathbb{U} \boxed{M} = \mathbb{E}$

$$\underset{\mathbb{U} \to \mathbb{E}}{M} = \left(\underset{(E_1)_{\mathbb{U}}}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}} \underset{(E_2)_{\mathbb{U}}}{\begin{pmatrix} -1 \\ 2 \\ 2 \\ 1 \end{pmatrix}} \underset{(E_3)_{\mathbb{U}}}{\begin{pmatrix} 3 \\ 1 \\ 3 \\ 0 \end{pmatrix}} \underset{(E_4)_{\mathbb{U}}}{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}} \right)$$

$(X)_{\mathbb{U}} = (x_1, x_2, x_3, x_4)$

$(X)_{\mathbb{E}} = (y_1, y_2, y_3, y_4)$

$$\begin{cases} x_1 = 1 \cdot y_1 - 1 \cdot y_2 + 3y_3 + 1 \cdot y_4 \\ x_2 = 2y_2 + y_3 - y_4 \\ x_3 = y_1 + 2y_2 + 3y_3 + y_4 \\ x_4 = 2y_1 + y_2 + y_4 \end{cases}$$

$(X)_{\mathbb{U}} = M (X)_{\mathbb{E}}$

$(X)_{\mathbb{E}} = M^{-1} (X)_{\mathbb{U}}$

• 3

M^{-1}

$E \to U$

$$(M \mid I) \sim (I \mid M^{-1})$$

$$\begin{array}{cccc|cccc} & & & & x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to \frac{R_2 - R_1}{3} \\ \\ R_4 \to R_4 - 2R_1 \end{array} \left(\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3/3 & 0 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -6 & -1 & -2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1 & 2/3 & 1 & -2/3 & 0 \\ 0 & 0 & -6 & -1 & -1 & 0 & -1 & 1 \end{array}\right) R_4 \to R_4 + 6R_3 \left(\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1 & 2/3 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & -7 & 3 & 6 & -5 & 1 \end{array}\right) R_4 \to \left(-\frac{1}{7}\right) R_4$$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & -1 & 2/3 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 & -3/7 & -6/7 & 5/7 & -1/7 \end{array}\right) \begin{array}{l} R_1 \to R_1 - R_4 \\ \\ R_3 \to R_3 + R_4 \end{array} \left(\begin{array}{cccc|cccc} 1 & -1 & 3 & 0 & 10/7 & 6/7 & -5/7 & 1/7 \\ 0 & 1 & 0 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 5/21 & 1/7 & 1/21 & -1/7 \\ 0 & 0 & 0 & 1 & -3/7 & -6/7 & 5/7 & -1/7 \end{array}\right)$$

$R_1 \to R_1 - 3R_3 + R_2$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 8/21 & 3/7 & -11/21 & 4/7 \\ 0 & 1 & 0 & 0 & -1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 5/21 & 1/7 & 1/21 & -1/7 \\ 0 & 0 & 0 & 1 & -3/7 & -6/7 & 5/7 & -1/7 \end{array}\right)$$

M^{-1}

$$\begin{cases} y_1 = \frac{8}{21} x_1 + \frac{3}{7} x_2 - \frac{11}{21} x_3 + \frac{4}{7} x_4 \\ y_2 = -\frac{1}{3} x_1 + \frac{1}{3} x_3 \\ y_3 = \frac{5}{21} x_1 + \frac{1}{7} x_2 + \frac{1}{21} x_3 - \frac{1}{7} x_4 \\ y_4 = -\frac{3}{7} x_1 - \frac{6}{7} x_2 + \frac{5}{7} x_3 - \frac{1}{7} x_4 \end{cases}$$

• 4 $(x)_{\mathcal{U}} = \underset{x_1}{(1}, \underset{x_2}{-1}, \underset{x_3}{0}, \underset{x_4}{1})$

$$(x)_{\mathcal{B}} = \left(\frac{8}{21} - \frac{3}{7} + \frac{4}{7}, -\frac{1}{3}, \frac{5}{21} - \frac{1}{7} - \frac{1}{7}, -\frac{3}{7} + \frac{6}{7} - \frac{1}{7}\right)$$

$$\begin{pmatrix} \frac{8}{21} - \frac{3}{7} + \frac{4}{7} \\ -\frac{1}{3} \\ \frac{5}{21} - \frac{1}{7} - \frac{1}{7} \\ -\frac{3}{7} + \frac{6}{7} - \frac{1}{7} \end{pmatrix} = \begin{pmatrix} 11/7 \\ -1/3 \\ -1/21 \\ 2/7 \end{pmatrix}$$

④ Verificare che l'insieme delle matrici 2×2 con traccia 0 è un sottospazio lineare di $M_2(\mathbb{R})$.
Scriverne una base e calcolarne la dimensione.

$$\left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : Tr(A) = a_{11} + a_{22} = 0 \right\}$$

$\forall \lambda, \mu \in \mathbb{R} \quad \forall a, b \in A$

$$\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \mu \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \overset{?}{\in} A$$

$$\begin{pmatrix} \lambda a_{11} + \mu b_{11} & \lambda a_{12} + \mu b_{12} \\ \lambda a_{21} + \mu b_{21} & \lambda a_{22} + \mu b_{22} \end{pmatrix} \overset{?}{\in} A \qquad \lambda a_{11} + \mu b_{11} + \lambda a_{22} + \mu b_{22} \overset{?}{=} 0$$

$$\lambda \underbrace{(a_{11} + a_{22})}_{0} + \mu \underbrace{(b_{11} + b_{12})}_{0} \overset{?}{=} 0 \qquad \text{OK. sottospazio lineare}$$

$$a_{11} + a_{22} = 0 \rightarrow a_{11} = -a_{22}$$

$$\begin{pmatrix} -a_{22} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{22} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$ è una base per A

$\dim(A) = 3$

5. Verificare che l'insieme delle matrici triangolari superiori di ordine 3 è un sottospazio lineare di $M_3(\mathbb{R})$.
Scrivere una base e calcolarne la dimensione.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

$\forall \lambda, \mu \in \mathbb{R} \quad \forall a, b \in A:$

$$\lambda \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} + \mu \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} \overset{?}{\in} A$$

$$\begin{pmatrix} \lambda a_{11} + \mu b_{11} & \lambda a_{12} + \mu b_{12} & \lambda a_{13} + \mu b_{13} \\ 0 & \lambda a_{22} + \mu b_{22} & \lambda a_{23} + \mu b_{23} \\ 0 & 0 & \lambda a_{33} + \mu b_{33} \end{pmatrix} \in A \quad \text{OK}$$

$$a_{11} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{v_1} + a_{12} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{v_2} + a_{13} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{v_3} + a_{22} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{v_4} +$$

$$+ a_{23} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{v_5} + a_{33} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{v_6}$$

$\{v_1, v_2, v_3, v_4, v_5, v_6\}$ è una base per A
$\dim(A) = 6$.