VARIABILI ALEATORIE

Funzione densità discreta (pmf)

$p_X(x_i) := P(X = x_i) = p^X(\{x_i\})$

$p_X(x_i) \geq 0 \quad \forall x_i \in \mathcal{X}$

$\sum_{x_i} p_X(x_i) = 1$

Variabile aleatoria Binomiale

$X \sim Bin(n, p)$

$\mathcal{X} = \{0, 1, 2, \dots, n\}$

pmf $\rightarrow$ $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Variabile alea. geometrica

$X \sim g(p)$

$\mathcal{X} = \mathbb{N}_+$

pmf $\rightarrow$ $p_X(k) = (1-p)^{k-1} p$, $k \in \mathbb{N}_+$

$X_1 \sim G(\lambda_1)$, $X_2 \sim G(\lambda_2)$ $\quad Z = \min(X_1, X_2)$ indipendenti

$Z \sim G(1 - (1-\lambda_1)(1-\lambda_2))$

Variabile aleatoria di Bernoulli

$b(p)$, $Bern(p)$, $X \sim b(p)$, $X \in b(p)$

$\mathcal{X} = \{0, 1\}$

pmf $\rightarrow$ $p_X(1) = p$

$p_X(0) = 1 - p$

Variabili aleatorie discrete uniformi

$X \sim U(x)$

$\mathcal{X} = \{x_1, x_2, x_{\dots}, x_N\}$ finito

pmf $\rightarrow$ $p_X(x_i) = \frac{1}{N}$

Probabilità di attesa lunga

(primo successo dopo la prova k-esima)

$P(X > k) = (1-p)^k$

Variabile aleatoria di Poisson

$Y \sim P(\lambda)$ $\qquad \lambda \in \mathbb{R}$, $\lambda > 0$

$\mathcal{Y} = \{0, 1, 2, \dots\} = \mathbb{N}$

pmf $\rightarrow$ $p_Y(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ $\qquad k = 0, 1, 2 \dots$

$p_Y(k) = \lim_{n \to \infty} p_{X_n}(k)$ con $X_n \sim Bin(n, \frac{\lambda}{n})$

se X e Y v.a. Poisson indipendenti di parametro λ_1 e λ_2

$X + Y$ = v.c. di Poisson di parametro $\lambda_1 + \lambda_2$

pmf $\rightarrow$ $p_{X+Y}(k) = e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^k}{k!}$

Variabili aleatorie assolutamente continue

Variabili aleatorie uniformi

$X \sim U(a, b)$

$$f(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

Variabili aleatorie di Cauchy

parametro λ $\qquad f_X(x) = \frac{1}{\pi} \frac{\lambda}{\lambda^2 + x^2}$

Variabile aleatoria esponenziale

$X \sim Exp(\lambda)$ $\qquad f_X(x) = \lambda e^{-\lambda x} \mathbb{1}(x)$

$F_X(x) = (1 - e^{-\lambda x}) \mathbb{1}(x)$

$X_1 \sim Exp(\lambda_1)$, $X_2 \sim Exp(\lambda_2)$ indipendenti

$Z = \min(X_1, X_2) \Rightarrow Z \sim Exp(\lambda_1 + \lambda_2)$

Variabile aleatoria gaussiana

$X \sim N(\mu, \sigma^2)$ $\qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$

$\sigma^2 \to 0$ $\qquad f_X = \delta(x - \mu)$

$F(x) = \mathbb{1}(x - \mu)$

$Z = \alpha X + \beta Y$ con $X \sim N(\mu_X, \sigma_X^2)$ e $Y \sim N(\mu_Y, \sigma_Y^2)$ indipendenti

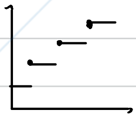
$Z \sim N(\alpha\mu_X + \beta\mu_Y, \alpha^2\sigma_X^2 + \beta^2\sigma_Y^2)$

	$E(X)$	$Var(X)$	$\varphi(\omega)$
BERNOULLI $X \sim b(p)$	p	$p(1-p)$	$1 - p + pe^{j\omega}$
GEOMETRICA $X \sim g(p)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	
BINOMIALE $X \sim Bin(n, p)$	np	$np(1-p)$	$(1 - p + pe^{j\omega})^n$
POISSON $Y \sim P(\lambda)$	λ	λ	$e^{\lambda(e^{j\omega} - 1)}$
UNIFORMI $X \sim U(a,b)$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{1}{j\omega} \frac{e^{j\omega b} - e^{j\omega a}}{b-a}$
ESPONENZIALE $X \sim Exp(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - j\omega}$
GAUSSIANA $X \sim N(\mu, \sigma)$	μ	σ^2	$e^{j\mu\omega - \frac{1}{2}\sigma^2\omega^2}$

Laplace 0 $\frac{2}{\lambda^2}$

Funzione di distribuzione di probabilità (FdD)

$$F_X(x) := P(X \leq x) = \sum_{x_i < x} P_X(x_i)$$



Valore atteso (medio)

$$E(X) := \sum_{x_i \in \mathcal{X}} x_i \, P_X(x_i)$$

- $E(aX) = a\,E(X)$
- $E(X+Y) = E(X) + E(Y)$
- $E(X) \geq 0$ se X positiva
- $E(X) \leq E(Y)$ se $X \leq Y$
- $\min_{x_i \in \mathcal{X}} \{x_i\} \leq E(X) \leq \max_{x_i \in \mathcal{X}} \{x_i\}$

per le funzioni

$Y = g(X)$

$$E(Y) = \sum_{y_j \in \mathcal{Y}} y_j P_Y(y_j) = \sum_{x_i \in \mathcal{X}} g(x_i) P_X(x_i)$$

- In generale $E(g(X)) \neq g(E(X))$

Varianza di variabile aleatoria

$$\sigma_X^2 = var(X) := E\left((X - E(X))^2\right) = \sum_{x_i \in \mathcal{X}} (x_i - E(X))^2 P_X(x_i)$$

- $var(X) \geq 0$
- $var(aX) = a^2 var(X)$
- $var(X + c) = var(X)$
- $var(X) = E(X^2) - (E(X))^2$

$$var(X+Y) = var(X) + var(Y) + 2\underbrace{E((X-E(X))(Y-E(Y)))}_{cov(X,Y)}$$

Covarianza

$$cov(X,Y) := E((X-E(X))(Y-E(Y)))$$

- $cov(X,X) = var(X)$
- $cov(X,Y) = E(XY) - E(X)E(Y)$
- $cov(X,Y) \gtrless 0$

Disuguaglianza di Jensen

se g funzione convessa

$$g(E(X)) \leq E(g(X))$$

Disuguaglianza di Markov

$$P(X \geq a) \leq \frac{E(X)}{a} \quad \forall a > 0$$

Disuguaglianza di Chebyshev

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{var(X)}{\varepsilon^2}$$

$$P(|X - E(X)| < \varepsilon) \geq 1 - \frac{var(X)}{\varepsilon^2}$$

Variabili aleatorie normali / Gaussiane

$X \sim N(\mu, \sigma^2)$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$

$\sigma^2 \to 0$ $\quad f_X = \delta(x-\mu)$

$F(x) = 1\!\!1(x-\mu)$

trasformazioni lineari

$X = N(\mu, \sigma^2)$, $Y = aX + b$ $\quad a \neq 0$ e b costanti

$Y \sim N(a\mu + b, a^2\sigma^2)$

$E(Y) = aE(X) + b = a\mu + b$

$var(Y) = a^2 \, var(X) = a^2\sigma^2$

Standardizzazione

$Z \sim N(0, 1)$ → NORMALE STANDARD

$Z = \frac{X - \mu}{\sigma}$ $\quad E(Z) = 0$

$var(Z) = 1$

$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} d \quad = P(Z \leq z)$

Variabili aleatorie ass. continue

$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$

Vettori di v. a.

$$P_{XY}(x_i, y_j) = P_X(x_i)\, P_Y(y_i)$$

$$E(XY) = \sum_{i,j} x_i y_j \, P_{XY}(i,j)$$

indipendenza

$$p_{XY}(i,j) = p_X(i)\, p_Y(j)$$

scorrelazione

$$E(XY) = E(X)\,E(Y) \quad (= cov(x) = 0)$$

$$var(X+Y) = var(X) + var(Y)$$

ind $\Rightarrow$ scorr

ind $\not\Leftarrow$ scorr

FdD condizionata

$$F_{X|Y}(x|y) := \lim_{\delta \to 0} P(X \le x \mid y \le Y \le y+\delta)$$

$$f_{X|Y}(x|y) := \frac{d}{dx} F_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Vettori aleatori

PARAMETRI RIASSUNTIVI

Vettore medio

$$m_x = E(X) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_m) \end{bmatrix}$$

Matrice di correlazione

$$R_x = corr(X) := E(X X^T) = \begin{bmatrix} E(x_1^2) & \cdots & E(x_1 x_m) \\ \vdots & \ddots & \\ E(x_m x_1) & \cdots & E(x_m^2) \end{bmatrix}$$

$$= \Sigma_x + m_x m_x^T$$

Matrice di covarianza

$$\Sigma_x = \begin{bmatrix} E(x_1 - m_1)^2 & \cdots & E(x_1 - m_x)(x_m - m_m) \\ \vdots & \ddots & \vdots \\ E(x_m - m_m)(x_1 - m_1) & \cdots & E(x_m - m_m)^2 \end{bmatrix} = \begin{bmatrix} var(x_1) & cov(x_2 x_2) & \cdots & cov(x_n x_1) \\ cov(x_1, x_2) & \ddots & & \vdots \\ \vdots & & & \\ cov(x_1\ x_m) & \cdots & & var(x_m) \end{bmatrix}$$

$$= R_x - m_x m_x^T$$

TRASFORMAZIONI AFFINI

$$Y = AX + b$$

$$m_y = A m_x + b$$

$$\Sigma_y = A \Sigma_x A^T$$

$$R_y = A \Sigma_x A^T + [A m_x + b][A m_x + b]^T$$

$$\Sigma_y + m_y m_y^T$$

Vettori a componenti indipendenti

X e Y indipendenti

$F_{xy}(x,y) = F_y(y)\,F_x(x)$

$f_{xy}(x,y) = f_x(x)\,f_y(y)$

$f_{x|y}(x|y) = f_x(x)$

$f_{y|x}(y|x) = f_y(y)$

DENSITÀ FUNZ. SCALARI

$Z = X + Y \Rightarrow f_z(z) = [f_x * f_y](z) = \int_{-\infty}^{+\infty} f_x(z-v)\,f_y(v)\,dv$

Vettori aleatori normali (Gaussiani) standard

$\underline{Z} \sim N(\underline{0}, I_n)$

FdD: $f_{\underline{z}}(z_1, z_2 \dots z_n) = \dfrac{1}{\sqrt{(2\pi)^n}}\, e^{-\frac{1}{2}\underline{z}^T\underline{z}}$

FC: $\varphi_{\underline{z}}(\underline{\omega}) = e^{-\frac{1}{2}\|\omega\|^2}$

Vettore medio: $\underline{m}_{\underline{z}} = E(\underline{z}) = \underline{0}$

Matrice correlazione: $R_{\underline{z}} = \underline{I}_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Matrice covarianza: $\Sigma_{\underline{z}} = Cov(\underline{z}) = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Vettori aleatori normali (Gaussiani) generici

$\underline{Y} \sim N(\underline{\mu}, \Sigma_{\underline{Y}})$

$\underline{Y} = \underline{A}\,\underline{Z} + \underline{\mu}$

$\underline{Z} \sim N(\underline{0}, \underline{I}_k)$

$\underline{A} \in \mathbb{R}^{n \times k}$

$\mu \in \mathbb{R}^n$

Vettore medio: $\underline{m}_{\underline{Y}} = E(\underline{Y}) = \underline{\mu}$

Matrice covarianza: $\Sigma_{\underline{Y}} = Cov(\underline{Y}) = \underline{A}\,cov(\underline{z})\underline{A}^T = \underline{A}\,\underline{A}^T$

FC: $\varphi_{\underline{Y}}(\underline{\omega}) = e^{j\underline{\omega}^T\underline{\mu} - \frac{1}{2}\underline{\omega}^T\Sigma\,\underline{\omega}}$

FdD: ($\exists \Leftrightarrow det(\Sigma) > 0$) $f_{\underline{Y}}(\underline{y}) = \dfrac{1}{\sqrt{(2\pi)^n \cdot det(\Sigma)}}\, e^{-\frac{1}{2}(\underline{y}-\underline{\mu})^T\Sigma^{-1}(\underline{y}-\underline{\mu})}$

Indipendenza $\Leftrightarrow$ Scorrelazione $\Leftrightarrow$ $\underline{\Sigma}$ è diagonale

Trasf. lineari di v.a. normali

$\underline{W} \sim N(\underline{B}\underline{\mu} + \underline{v}, \underline{B}\,\Sigma\,\underline{B}^T)$

$\underline{W} = \underline{B}\,\underline{Y} + \underline{v}$

$\underline{m}_{w} = \underline{B}\,\underline{\mu} + \underline{v}$

$\Sigma_w = \underline{B}\,\Sigma\,\underline{B}^T$

LLN

se ho $\{X_n\}$ di v.a. con media μ e varianza σ^2

$\bar{X}_n \longrightarrow \mu$

$$\lim_{n\to+\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) = 0 \qquad \forall \varepsilon > 0$$

sequenza standardizzata

$\{X_n\}$ con μ valore atteso

σ^2 varianza

$\hookrightarrow W = \dfrac{\bar{X}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \dfrac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu)$ $\quad n = \#$ elementi nella sequenza

$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} x_k$

Processi aleatori notevoli

$m_X(s) = E(X_s)$ media

$M_X(s) = E(X_s^2)$ potenza statistica

$\sigma_X^2(s) = E((X_s - m_X(s))^2)$ varianza

$r_X(s_1, s_2) = E(X_{s_1}, X_{s_2})$ autocorrelazione

$k_X(s_1, s_2) = E((X_{s_1} - m_X(s_1))(X_{s_2} - m_X(s_2)))$ autocovarianza

densità spettrale di potenza $\longrightarrow$ TdF della correlazione

$K_X(k) = r_X(k) - m_X^2$

White Gaussian Noise - WGN

$W_n, \ n \in \mathbb{Z} \qquad W_n \sim N(0, \sigma^2)$

$\hookrightarrow$ intensità

- $m_w(n) = E(W_n) = 0 \qquad \forall n \in \mathbb{Z}$
- $M_w(n) = E(W_n^2) = \sigma^2$
- $\sigma^2(n) = \sigma^2$
- correlazione $r_w(n+k, n) = \begin{cases} \sigma^2 & k=0 \\ 0 & k \neq 0 \end{cases} = \sigma^2 \delta(k)$
- covarianza $k_w(k) = r_w(k) = \sigma^2 \delta(k)$

Gaussian Random Walk

$Y_{n+1} = Y_n + W_n$

$\hookrightarrow \sim WGN(\sigma^2) \qquad Y_0 = 0$

$Y_1 = W_0$

$Y_2 = W_0 + W_1$

$\vdots$

$Y_n = \sum_{i=0}^{n-1} W_i$

- $m_Y(n) = E(Y_n) = E(\Sigma W_n) = 0 \qquad \forall n \in \mathbb{Z}$
- $M_w(n) = E(Y_n^2) = n\sigma^2$
- correlazione $r_Y(n_1, n_2) = \sigma^2 \min\{n_1, n_2\}$
- covarianza $k_Y(n_1, n_2) = r_Y(k) = \sigma^2 \min\{n_1, n_2\}$

First order Moving Average - MA(1)

$Y_n = W_n + b W_{n-1} \qquad b \neq 0$

- $m_Y(n) = 0$
- $M_Y(n) = \sigma^2 + b^2\sigma^2 = \sigma^2(1+b^2)$
- $r_Y(n+k, n) = (1+b^2)\sigma^2 \delta(k) + b\sigma^2\delta(k+1) + b\sigma^2\delta(k-1)$
- $k_Y(k) = r_Y(k)$

First Order Autoregressive AR(1)

$Y_{n+1} = a Y_n + W_n \qquad |a| < 1 \qquad Y_0 \sim N(0, \sigma_0^2) \quad W_n \sim WGN(\sigma^2)$

indipendenti

- $m_Y(n) = 0$
- $M_Y(n) = \sigma_0^2 a^{2n} + \sigma^2 \dfrac{1 - a^{2n}}{1 - a^2} = \dfrac{\sigma^2}{1 - a^2}$ $\quad \left(\sigma_0^2 = \dfrac{\sigma^2}{1-a^2}\right)$
- $r_Y(n+k, n) = a^{|k|} \dfrac{\sigma^2}{1 - a^2}$

processi stazionari

- in media

$m_x(t+h) = m_x(h) \quad \forall h \in T$

$m_x(t) = E(X_t) = m_x$ costante $\leftarrow$ $m_x(t)$ non dipende da t

- in correlazione

$r_x(t_1+h, t_2+h) = r_x(t_1, t_2) \quad \forall h$

$E(X_{t_1+h}\ X_{t_2+h}) = E(X_{t_1}, X_{t_2})$ $\leftarrow$ $r(t)$ non dipende da t

- in senso lato

processo stazionario in media e correlazione