

## FORZA

$$F_x = F \cos \alpha$$

$$F_y = F \sin \alpha$$

## MOMENTO

$$M = d \times F = \overline{OP} \sin \alpha \cdot F \rightarrow |M| = -d \cdot F \rightarrow M = F \cdot d$$

## CARICHI DISTRIBUITI

Carico specifico o densità di carico  $q(x) = \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta F(x)}{\Delta x} \right)$

Se  $q(x)$  è  $\perp x$ :

$$R = \int_0^L q(x) dx$$

$$d = \frac{\int_0^L q(x) \cdot x dx}{\int_0^L q(x) dx}$$

## GRADI DI VINCOLO

$$V = 3i + 2(c + s) + p$$

$i$  = num. di incastri  
 $c$  = // cerniere  
 $s$  = // incastri scorrevoli  
 $p$  = // pendoli e corzelli

## REAZIONI VINCOLARI

labile se  $p - v > 0$ ,  $i = p - v$  è il grado di labilità

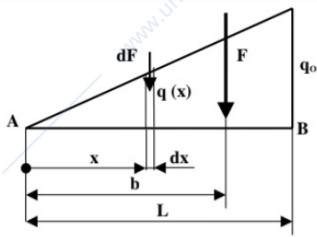
Isostatica se  $p - v = 0$

Iperstatica se  $p - v < 0$ ,  $i = v - p$  è il grado di iperstaticità

EQUAZIONI DI EQUILIBRIO:

$$\begin{cases} \text{traslazione orizz.} = 0 \\ \text{" vert.} = 0 \\ \text{rotazione intorno un punto} = 0 \end{cases}$$

## CARICO DISTRIBUITO TRIANGOLARMENTE



$$q(x) = q_0 \frac{x}{L}$$

Risultante:  $dF = q(x) dx$

Forza risultante:  $\int_0^L dF = \int_0^L q_0 \frac{x}{L} dx = \frac{q_0}{L} \int_0^L x dx = \frac{q_0}{L} \left[ \frac{x^2}{2} \right]_0^L = \frac{q_0 L^2}{2} = \frac{q_0 L^2}{2} \Rightarrow$  area del triangolo

Posizione risultante

Momento della risultante rispetto ad A:  $M = F \times b = \frac{q_0 L}{2} \times b$

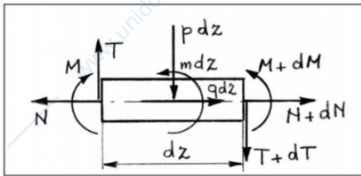
Momento elementare:  $dM = dF \times x = \frac{q_0 x}{L} \cdot x dx = \frac{q_0 x^2}{L} dx$

Momento totale:  $M = \int_0^L dM = \int_0^L \frac{q_0 x^2}{L} dx = \frac{q_0}{L} \left[ \frac{x^3}{3} \right]_0^L = \frac{q_0 L^3}{3} = \frac{q_0 L^2}{3}$

uguagliando:  $\frac{q_0 L}{2} b = \frac{q_0 L^2}{3} \Rightarrow b = \frac{2}{3} L \Rightarrow$  ascissa del BARICENTRO

### METODO ANALITICO GLOBALE

Equazioni indefinite di equilibrio per le travi piane ad asse rettilineo

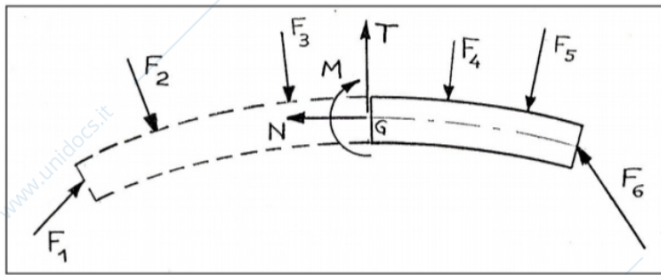


$$ds = dz \rightarrow$$

$$\begin{cases} dN + qdz = 0 \\ dT + pdz = 0 \\ dM - Tdz + mdz = 0 \end{cases}$$

$$\frac{dT}{dz} = -q \quad \frac{dM}{dz} = T \quad \frac{dN}{dz} = -p$$

### METODO DIRETTO PER IL CALCOLO DELLE AZIONI INTERNE



$$\begin{cases} N = \sum_{i=1}^3 F_{iN} \\ T = \sum_{i=1}^3 F_{iT} \\ M = \sum_{i=1}^3 M_{iG} \end{cases}$$

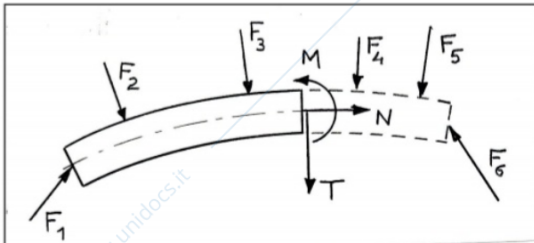
essendo:

$F_{iN}$  = Componente della forza i-esima nella direzione e nel verso di N,

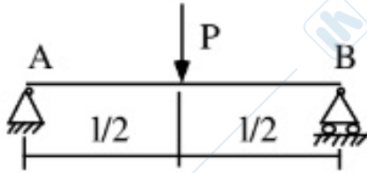
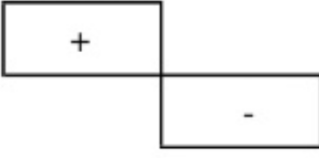

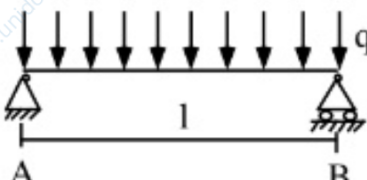
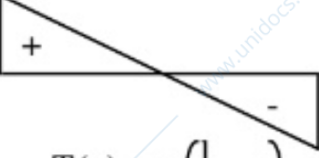
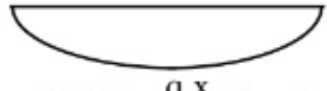
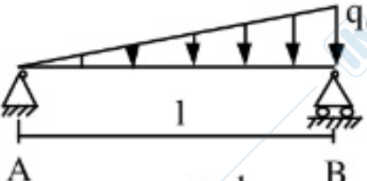
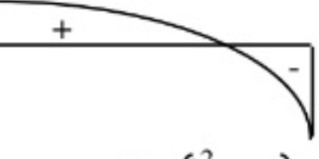
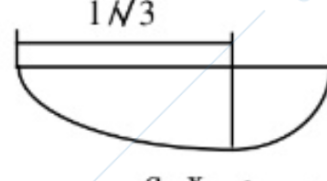
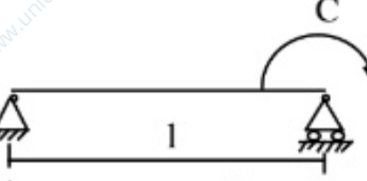


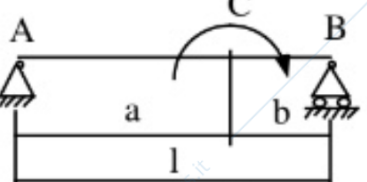
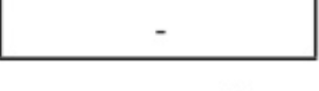

$F_{iT}$  = Componente della forza i-esima nella direzione e nel verso di T,

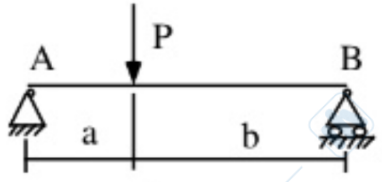
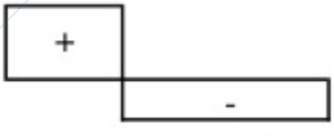

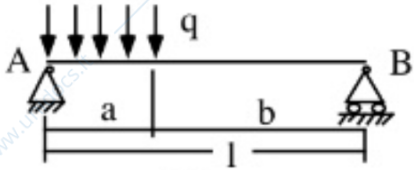


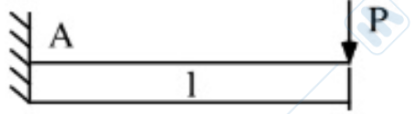
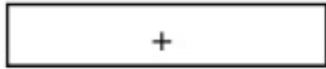

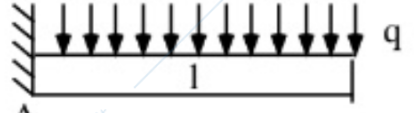


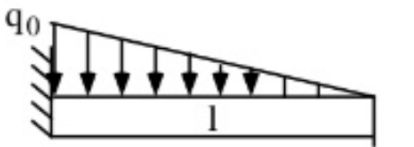
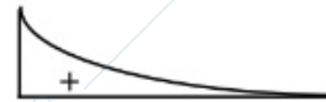

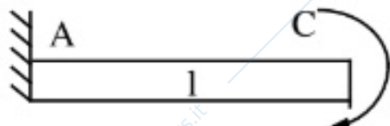


$M_{iG}$  = Momento della forza i-esima rispetto al baricentro G della sezione considerata.

Lo stesso risultato può essere ottenuto considerando LE FORZE CHE SEGUONO LA SEZIONE:

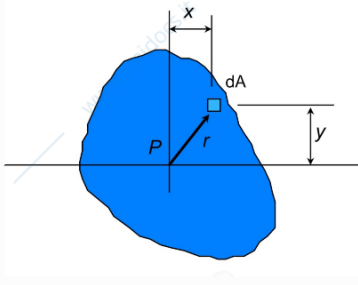


$$\begin{cases} N = \sum_{i=4}^6 F_{iN} \\ T = \sum_{i=4}^6 F_{iT} \\ M = \sum_{i=4}^6 M_{iG} \end{cases}$$

TRAVI A DUE APPOGGI		
Schema di Carico	Sforzo di Taglio	Momento Flettente
 <p> <math>V_A = V_B = \frac{P}{2}</math> </p>	 <p> <math>T(x) = +\frac{P}{2} \quad (x &lt; \frac{l}{2})</math>  <math>T(x) = -\frac{P}{2} \quad (x &gt; \frac{l}{2})</math> </p>	 <p> <math>M(x) = \frac{P}{2} x \quad (x &lt; \frac{l}{2})</math>  <math>M_{\max} = \frac{P l}{4}</math> </p>
 <p> <math>V_A = V_B = \frac{q l}{2}</math> </p>	 <p> <math>T(x) = q \left( \frac{l}{2} - x \right)</math>  <math>T_{\max} = \frac{q l}{2}</math> </p>	 <p> <math>M(x) = \frac{q x}{2} (l - x)</math>  <math>M_{\max} = q \frac{l^2}{8}</math> </p>
 <p> <math>V_A = \frac{q_0 l}{6}</math>  <math>V_B = \frac{q_0 l}{3}</math> </p>	 <p> <math>T(x) = \frac{q_0}{2 l} \left( \frac{l^2}{3} - x^2 \right)</math> </p>	 <p> <math>M_{\max} = q_0 \frac{l^2}{15,6}</math> </p>
 <p> <math>V_A = -\frac{C}{l}</math>  <math>V_B = +\frac{C}{l}</math> </p>	 <p> <math>T(x) = -\frac{C}{l}</math> </p>	 <p> <math>M(x) = \frac{C}{l} x</math> </p>
 <p> <math>V_A = -\frac{C}{l}</math>  <math>V_B = +\frac{C}{l}</math> </p>	 <p> <math>T(x) = -\frac{C}{l}</math> </p>	 <p> <math>M(x) = \frac{C}{l} x</math>  <math>M(\bar{x}) = \frac{C}{l} \bar{x}</math> </p>

TRAVI A DUE APPOGGI		
Schema di Carico	Sforzo di Taglio	Momento Flettente
 <p><math>V_A = P \frac{b}{l}</math>; <math>V_B = P \frac{a}{l}</math></p>	 <p><math>T(x) = + P \frac{b}{l}</math> <math>T(\bar{x}) = - P \frac{a}{l}</math></p>	 <p><math>M(x) = P \frac{b}{l} x</math> <math>M(\bar{x}) = P \frac{a}{l} \bar{x}</math> <math>M_{\max} = P \frac{a b}{l}</math></p>
 <p><math>V_A = \frac{q a}{2 l} (2 b + a)</math> <math>V_B = q \frac{a^2}{2 l}</math></p>	 <p><math>T(x) = q \left( a \frac{2 b + a}{2 l} - x \right)</math> <math>T(\bar{x}) = - q \frac{a^2}{2 l}</math></p>	 <p><math>M(x) = q \frac{x}{2} \left( a \frac{2 b + a}{2 l} - x \right)</math> <math>M(\bar{x}) = q \frac{a^2}{2 l} \bar{x}</math></p>
TRAVI A MENSOLA		
 <p><math>V_A = P</math>; <math>M_A = P l</math></p>	 <p><math>T(x) = + P</math></p>	 <p><math>M(x) = P x</math>; <math>M_{\max} = P l</math></p>
 <p><math>V_A = q l</math>; <math>M_A = q \frac{l^2}{2}</math></p>	 <p><math>T(x) = + q x</math>; <math>T_{\max} = q l</math></p>	 <p><math>M(x) = q \frac{x^2}{2}</math> <math>M_{\max} = q \frac{l^2}{2}</math></p>
 <p><math>V_A = \frac{q_0 l}{2}</math>; <math>M_A = \frac{q_0 l^2}{6}</math></p>	 <p><math>T(x) = q_0 \frac{x^2}{2 l}</math> <math>T_{\max} = q_0 \frac{l}{2}</math></p>	 <p><math>M(x) = q_0 \frac{x^3}{6 l}</math> <math>M_{\max} = q_0 \frac{l^2}{6}</math></p>
 <p><math>V_A = 0</math>; <math>M_A = C</math></p>	 <p><math>T = 0</math></p>	 <p><math>M(x) = C</math></p>

# GEOMETRIA DELLE AREE



• Momenti statici rispetto agli assi x e y

$$S_x = \int_A y \, dA \quad S_y = \int_A x \, dA$$

• Momenti d'inerzia

$$I_x = \int_A y^2 \, dA \quad I_y = \int_A x^2 \, dA$$

• Momento centrifugo

$$I_{xy} = \int_A x y \, dA$$

• Momento polare

$$I_P = \int_A r^2 \, dA = \int_A (x^2 + y^2) \, dA \quad \text{- Sia } I_P = I_x + I_y$$

• Raggi di inerzia

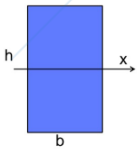
$$I_x = \rho_x^2 A \Rightarrow \rho_x = \sqrt{I_x/A} \quad \rho_y = \sqrt{I_y/A}$$

Baricentro G

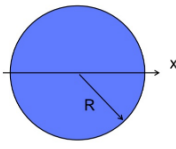
$$X_G = \frac{S_y}{A} \rightarrow S_y = X_G \cdot A$$

$$Y_G = \frac{S_x}{A} \rightarrow S_x = Y_G \cdot A$$

## MOMENTI D'INERZIA



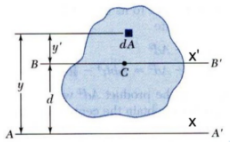
$$I_x = \frac{bh^3}{12}$$



$$I_x = \frac{\pi R^4}{4} = \frac{\pi D^4}{64}$$

## TEOREMI DI TRASPORTO

TRASLAZIONE degli assi



$$I_x = \int y^2 \, dA = \int (y' + d)^2 \, dA = \int y'^2 \, dA + 2d \int y' \, dA + d^2 \int dA$$

$$I_x = I_{x'} + Ad^2$$

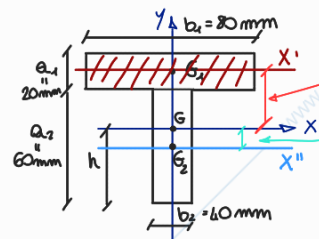
$$I_x = I_{x'} + Ad_y^2$$

$$I_y = I_{y'} + Ad_x^2$$

$$I_P = I_G + A \cdot PG^2$$

$$I_{x'y'} = I_{x'y} + Ad_x d_y$$

## Esempio



$$I_{x_1} = \frac{1}{12} a_1^3 b_1 + b_1 a_1 (a_2 + \frac{a_1}{2} - h)^2$$

$$I_{x_2} = \frac{1}{12} a_2^3 b_2 + a_2 b_2 (h - \frac{a_2}{2})^2$$

$$I_x = I_{x_1} + I_{x_2}$$

## ROTATIONE DEGLI ASSI

• Noti  $I_x = \int y^2 \, dA$ ;  $I_y = \int x^2 \, dA$ ;  $I_{xy} = \int xy \, dA$ ;  $\theta$

• Determinare  $I_{x'}$ ,  $I_{y'}$ ,  $I_{x'y'}$

$$x' = x \cos \theta + y \sin \theta$$

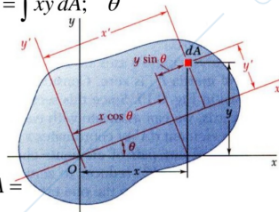
$$y' = y \cos \theta - x \sin \theta$$

$$I_{x'} = \int y'^2 \, dA = \int (y \cos \theta - x \sin \theta)^2 \, dA = \cos^2 \theta \int y^2 \, dA - 2 \cos \theta \sin \theta \int xy \, dA + \sin^2 \theta \int x^2 \, dA$$

$$I_{x'} = I_x \cos^2 \theta - 2I_{xy} \sin \theta \cos \theta + I_y \sin^2 \theta$$

$$I_{y'} = I_x \sin^2 \theta + 2I_{xy} \sin \theta \cos \theta + I_y \cos^2 \theta$$

$$I_{x'y'} = (I_x - I_y) \sin \theta \cos \theta + I_{xy} (\cos^2 \theta - \sin^2 \theta)$$



• Ricordando che

$$\sin 2\theta = 2 \sin \theta \cos \theta \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$2 \cos^2 \theta = 1 + \cos 2\theta \quad 2 \sin^2 \theta = 1 - \cos 2\theta$$

• Si ottiene

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - I_{xy} \sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\theta + I_{xy} \sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta$$

- Per quale angolo i momenti d'inerzia sono massimi o minimi?

$$0 = \frac{dI_x}{d\theta} = -(I_x - I_y) \sin 2\theta - 2I_{xy} \cos 2\theta$$

$$\tan 2\theta = \frac{-2I_{xy}}{I_x - I_y}$$

- Per quale angolo il momento centrifugo è nullo?

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\theta + I_{xy} \cos 2\theta = 0$$

$$\tan 2\theta = \frac{-2I_{xy}}{I_x - I_y}$$

- Questa relazione individua due valori di  $\theta$  distanti  $90^\circ$

- Se si assume come riferimento quello principale e si considera una rotazione degli assi di un angolo  $\theta$  che individua gli assi  $x, y$  si ha:

$$I_x = \frac{I_\xi + I_\eta}{2} + \frac{I_\xi - I_\eta}{2} \cos 2\theta$$

$$I_y = \frac{I_\xi + I_\eta}{2} - \frac{I_\xi - I_\eta}{2} \cos 2\theta$$

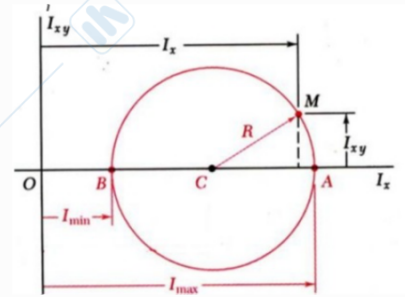
$$I_{xy} = \frac{I_\xi - I_\eta}{2} \sin 2\theta$$

Centro e raggio



$$C = \left( \frac{I_x + I_y}{2}, 0 \right)$$

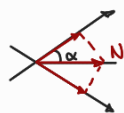
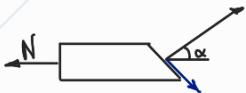
$$R = \sqrt{\left( \frac{I_x - I_y}{2} \right)^2 + I_{xy}^2}$$



### STORZO NORMALE



$$\sigma \approx \sigma_m = \frac{N}{A}$$



$$\sigma = \frac{N \cos \alpha}{A_0 / \cos \alpha}$$

$$\sigma = \frac{N \cos^2 \alpha}{A_0}$$

$$\tau = \frac{N \sin \alpha}{A_0 / \cos \alpha}$$

$$\tau = \frac{N \sin \alpha \cos \alpha}{A_0}$$

### DEFORMAZIONE UNITARIA

$$\epsilon = \frac{\delta}{L} = \frac{\Delta L}{L_0}$$

$$\delta = \epsilon \cdot L_0 = \frac{\sigma}{E} \cdot L_0 = \frac{FL}{EA}$$

### Tensione ammissibile

$$\sigma_{am} = \frac{\sigma_{erm}}{C_s} \rightarrow 1,5 \div 2,5$$

$$\sigma_{max} \leq \sigma_{am}$$

### Corpo elastico e corpo lineare

$$\sigma = E \cdot \epsilon$$

↓  
modulo di elasticità

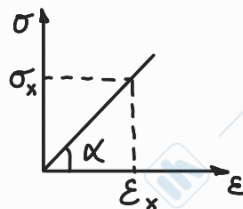
$$\tau = G \cdot \gamma$$

↓  
modulo di elasticità  
tangenziale

### LEGGE DI HOOKE

$$\sigma_x = E \epsilon_x$$

$$\tan \alpha = \frac{\sigma_x}{\epsilon_x} \Rightarrow \tan \alpha = E$$



### Coefficiente di Poisson

→ adimensionale = 0,3 per tutti i metalli

$$\epsilon_z = \epsilon_y = -\nu \epsilon_x$$

### Deformazione direzionale

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_z}{E}$$

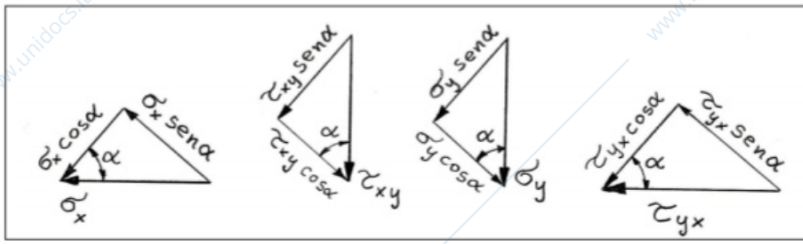
$$\epsilon_z = \frac{\sigma_z}{E} - \nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

equazioni di elasticità  
" di Gurne

$$G = \frac{E}{2(1 + 2\nu)}$$

Scriviamo le due equazioni di EQUILIBRIO alla TRASLAZIONE NELLE DIREZIONI DI  $\sigma$  E DI

$\tau$ .



**DIREZIONE  $\sigma$ :**

$$\sigma = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + \tau_{yx} \sin \alpha \cos \alpha + \tau_{xy} \sin \alpha \cos \alpha =$$

$$= \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + 2 \tau_{xy} \sin \alpha \cos \alpha$$

essendo:  $\tau_{xy} = \tau_{yx}$

Ricordando che:  $2 \sin \alpha \cos \alpha = \sin 2 \alpha$   
 si ottiene:

$$\sigma = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + \tau_{xy} \sin 2 \alpha \quad (1)$$

**DIREZIONE  $\tau$ :**

$$\tau = \sigma_x \sin \alpha \cos \alpha - \sigma_y \cos \alpha \sin \alpha - \tau_{xy} \cos^2 \alpha + \tau_{yx} \sin^2 \alpha =$$

$$= (\sigma_x - \sigma_y) \sin \alpha \cos \alpha - \tau_{xy} (\cos^2 \alpha - \sin^2 \alpha)$$

Ricordando che:  $\sin \alpha \cos \alpha = \frac{1}{2} \sin 2 \alpha$

e che:  $\cos^2 \alpha - \sin^2 \alpha = \cos 2 \alpha$   
 si ottiene:

$$\tau = \frac{1}{2} (\sigma_x - \sigma_y) \sin 2 \alpha - \tau_{xy} \cos 2 \alpha \quad (2)$$

$\tan 2\alpha = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}$

$\tan 2\alpha$  assume questo valore im corris. di due valori dell'angolo:

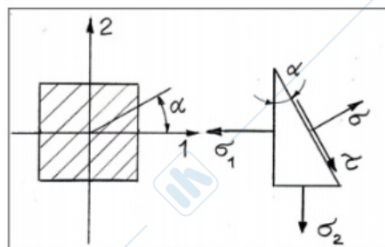
$2\alpha$  e  $2\alpha + \pi$

**RISPETTO AI DUE ASSI PRINCIPALI 1 e 2**

$$\sigma = \sigma_1 \cos^2 \alpha + \sigma_2 \sin^2 \alpha$$

$$\tau = \frac{1}{2} (\sigma_1 - \sigma_2) \sin 2 \alpha \quad (4)$$

essendo  $\alpha$  l'angolo formato dalla normale al piano dell'elemento su cui si calcolano le  $\sigma$  e  $\tau$  con la direzione principale 1.



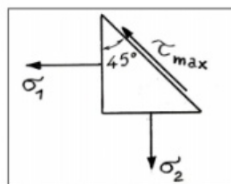
Osservando le espressioni (4) si vede che, ovviamente:

$\sigma = \sigma_1$  per  $\alpha = 0$   
 $\sigma = \sigma_2$  per  $\alpha = \frac{\pi}{2}$

Inoltre la  $\tau$  sar  MASSIMA quando   massimo:  $\sin 2 \alpha$ , cio :

$\tau = \tau_{max}$  per  $\sin 2 \alpha = \pm 1$ , cio  per:  $2 \alpha = \pm \frac{\pi}{2}$ , cio  per:  $\alpha = \pm \frac{\pi}{4}$

Quindi:  $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$  per  $\alpha = \pm 45^\circ$



## CIRCOLO DI MOHR

Le precedenti relazioni (4) possono essere trasformate, tenendo conto delle note relazioni trigonometriche:

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \quad \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

La prima delle (4) diventa:

$$\sigma = \sigma_1 \frac{1 + \cos 2\alpha}{2} + \sigma_2 \frac{1 - \cos 2\alpha}{2} = \frac{\sigma_1}{2} + \frac{\sigma_1}{2} \cos 2\alpha + \frac{\sigma_2}{2} - \frac{\sigma_2}{2} \cos 2\alpha$$

da cui:

$$\sigma = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\alpha$$

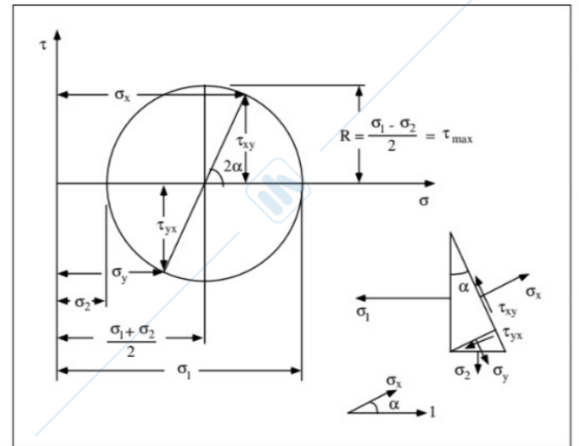
e inoltre:

$$\tau = \frac{\sigma_1 - \sigma_2}{2} \sin 2\alpha$$

(5)

Le (5) rappresentano, in coordinate parametriche  $\sigma, \tau$ , un CERCHIO (CIRCOLO DI MOHR):

- CENTRO sull'ASSE  $\sigma$ , all'ASCISSA:  $\sigma_c = \frac{\sigma_1 + \sigma_2}{2}$
- RAGGIO:  $R = \frac{\sigma_1 - \sigma_2}{2} = \tau_{\max}$



Uso del circolo di Mohr

- se ho  $\sigma_1$  e  $\sigma_2$  la costruzione è immediata
- se ho  $\sigma_x, \sigma_y$  e  $\tau_{xy}$ , si può ricavare:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

ascisse del  $\pm$  Raggio  
centro

Raggio

Osservando la figura, si può scrivere:

$$\tau_{xy} = \frac{\sigma_x - \sigma_y}{2} \operatorname{tg} 2\alpha$$

da cui:

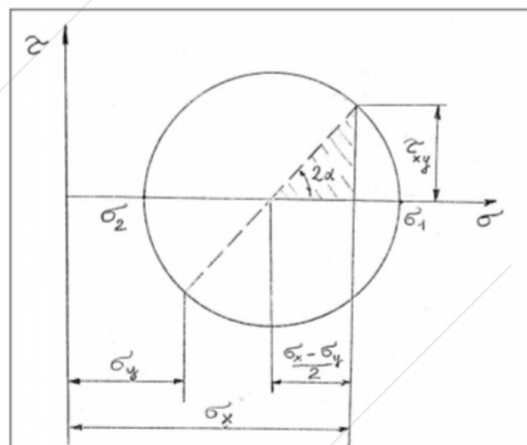
$$\operatorname{tg} 2\alpha = 2 \frac{\tau_{xy}}{\sigma_x - \sigma_y}$$

da cui:

$$2\alpha = \operatorname{arctg} \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}$$

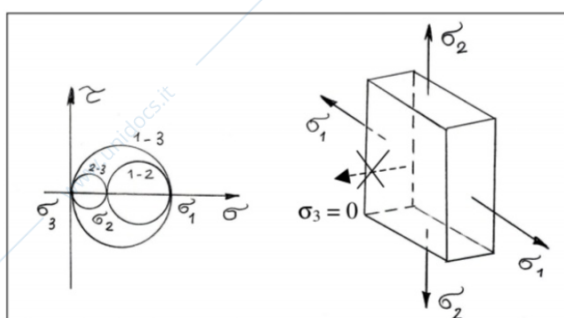
e infine:

$$\alpha = \frac{1}{2} \operatorname{arctg} \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}$$



OSSERVAZIONE  $\rightarrow$  la coordinata del centro:  $\frac{\sigma_1 + \sigma_2}{2}$   $\sigma$   $\frac{\sigma_x + \sigma_y}{2}$

I tre circoli di Mohr relativi all'elemento possono essere rappresentati CONTEMPORANEAMENTE NELLO STESSO GRAFICO:



PIANO 1-2

$$\sigma_1, \sigma_2 \neq 0 \quad \sigma_3 = 0$$

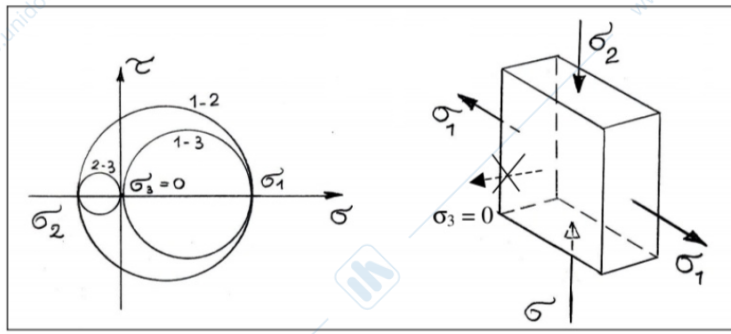
PIANO 1-3

$$\sigma_1 \neq 0 \quad \sigma_3 = 0$$

PIANO 2-3

$$\sigma_2 \neq 0 \quad \sigma_3 = 0$$

Quando invece la tensione principale  $\sigma_1$  è positiva e la tensione principale  $\sigma_2$  è negativa, la  $\tau_{max}$  più elevata è quella relativa al piano 1-2, come illustrato nella figura seguente.



## FLESSIONE

$$L = \rho\theta$$

$$L' = (\rho - y)\theta$$

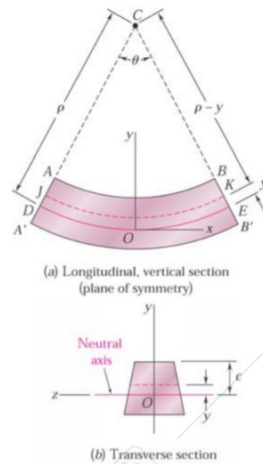
$$\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\epsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta}$$

$$\boxed{\epsilon_x = -\frac{y}{\rho}}$$

$$\epsilon_{max} = \frac{c}{\rho} \quad \rho = \frac{c}{\epsilon_{max}}$$

$$\epsilon_x = -\frac{y}{c} \epsilon_{max}$$



$$\sigma_x = E\epsilon_x = -\frac{y}{c} E\epsilon_{max} = -\frac{y}{c} \sigma_{max}$$

$$0 = \int \sigma_x dA = \int -\frac{y}{c} \sigma_{max} dA = -\frac{\sigma_{max}}{c} \int y dA$$

$$\int y dA = 0 \rightarrow \text{momento statico rispetto all'asse baricentrico}$$

$$M = \int -y\sigma_x dA = \int -y\left(-\frac{y}{c} \sigma_{max}\right) dA = \frac{\sigma_{max}}{c} \int y^2 dA = \frac{\sigma_{max} I}{c}$$

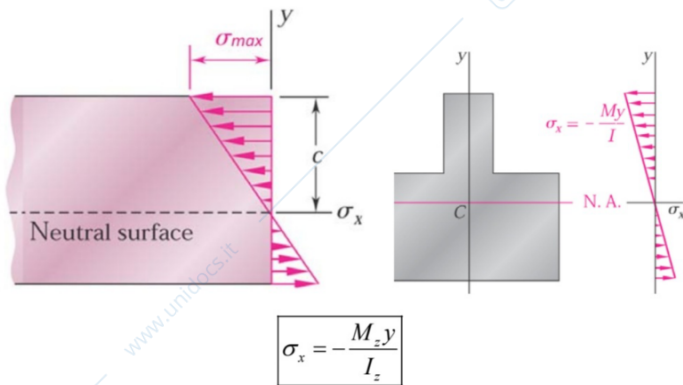
$$\sigma_{max} = \frac{Mc}{I} = \frac{M}{W} \quad W = \frac{I}{c}$$

si può anche ignorare, basta  $\frac{Mc}{I}$

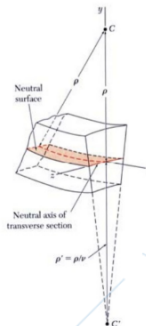
$$\sigma_x = -\frac{y}{c} \sigma_{max}$$

$$\boxed{\sigma_x = -\frac{My}{I}}$$

## DEFORMAZIONE



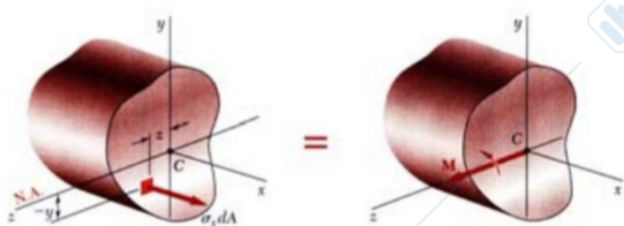
$$\boxed{\sigma_x = -\frac{M_2 y}{I_z}}$$



$$\frac{1}{\rho} = \frac{\epsilon_{max}}{c} = \frac{\sigma_{max}}{Ec} = \frac{1}{Ec} \frac{Mc}{I}$$

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

## FLESSIONE ASIMMETRICA

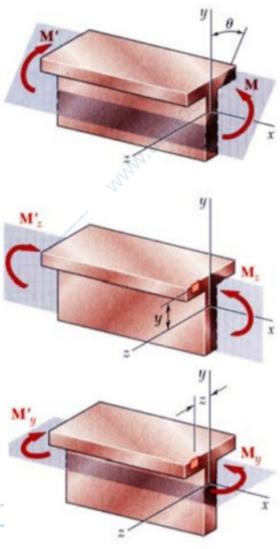


$$F_x = 0 = M_y \quad M_z = M = \text{coppia applicata}$$

$$0 = F_x = \int \sigma_x dA = \int \left(-\frac{y}{c} \sigma_{max}\right) dA \quad \& \quad 0 = \int y dA$$

$$M = M_z = -\int y \left(-\frac{y}{c} \sigma_{max}\right) dA \quad \& \quad M = \frac{\sigma_{max} I}{c} \quad I = I_{max} = \text{momento di inerzia}$$

$$0 = M_y = \int z \sigma_x dA = \int z \left(-\frac{y}{c} \sigma_{max}\right) dA \quad \& \quad 0 = \int yz dA = I_{yz} = \text{prodotto di inerzia}$$



• Scomporre il vettore coppia in componenti lungo gli assi baricentrici principali  
 $M_z = M \cos \theta$      $M_y = M \sin \theta$

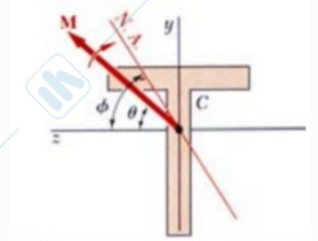
• Sovrapposizione e distribuzioni delle sollecitazioni dei componenti

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

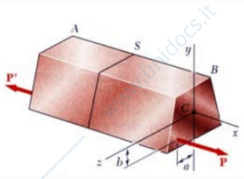
• Lungo l'asse neutro

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = -\frac{(M \cos \theta) y}{I_z} + \frac{(M \sin \theta) z}{I_y}$$

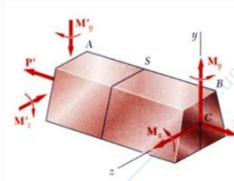
$$\tan \varphi = \frac{y}{z} = \frac{I_z}{I_y} \tan \theta$$



### SOVRAPPOSIZIONE DEGLI EFFETTI



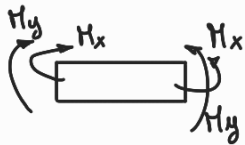
P = forza centrale  
 $M_y = P a$      $M_z = P b$



$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

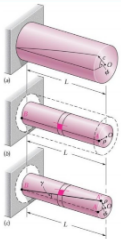
• Se l'asse neutro si trova sulla sezione:  
 $\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$

### RIASSUNTO FLESSIONE



$$\sigma_z = \frac{M_x}{I_x} y - \frac{M_y}{I_y} x$$

### TORSIONE



$$L\gamma = \rho\phi \Rightarrow \gamma = \frac{\rho\phi}{L}$$

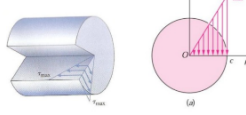
$$\gamma_{\max} = \frac{c\phi}{L} \quad \gamma = \frac{\rho}{c} \gamma_{\max}$$

### TENSIONI IN CAMPO ELASTICO

• Elasticità

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$

$$\tau = \frac{\rho}{c} \tau_{\max}$$

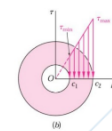


• Equilibrio

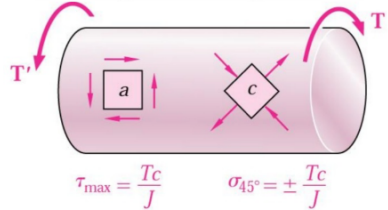
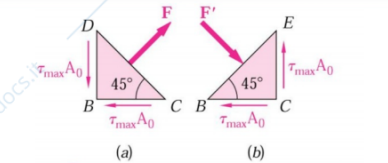
$$T = \int \rho \tau dA = \frac{\tau_{\max}}{c} \int \rho^2 dA = \frac{\tau_{\max}}{c} J$$

$$\tau_{\max} = \frac{Tc}{J} = \frac{TD}{2J}$$

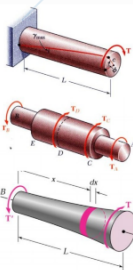
$$\tau = \frac{T\rho}{J}$$



a 45° ci sono le tensioni principali



### ANGOLO DI TORSIONE

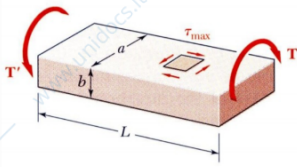


$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{GJ} \quad \phi = \gamma_{\max} \frac{L}{c} = \frac{TL}{GJ}$$

$$\phi = \sum \frac{T_i L_i}{J_i G_i}$$

$$\phi = \int_0^L \frac{T dx}{GJ}$$

### FORMULA DEL RETTANGOLO



$$\tau_{max} = \frac{T}{c_1 ab^2} \quad \varphi = \frac{TL}{c_2 ab^3 G}$$

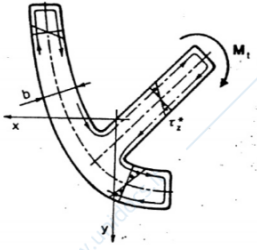
### SEZIONE RETTANGOLARE SOTTILE



$$\tau_{max} = \frac{T}{c_1 ab^2} = \frac{3T}{ab^2} = \frac{3Tb}{ab^3} = \frac{T}{J_1} b$$

$$\varphi = \frac{TL}{c_2 ab^3 G} = \frac{3TL}{ab^3 G} = \frac{TL}{GJ_1}$$

### SEZIONI SOTTILI APERTE



$$\tau_{max} = \frac{T}{J_1} b$$

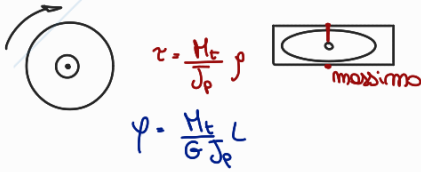
$$J_1 = \frac{1}{3} \int b^3 ds$$

Circle:  $t_{amm} = \frac{T \cdot b}{\frac{\pi b^2}{32} \cdot 2}$

Square:  $t_{max} = t_{amm} = \frac{T}{c_1 ab^2} = \frac{T}{c_1 b^3}$

Rectangle:  $t_{max} = t_{amm} = \frac{T}{c_1 ab^2} = \frac{T}{c_1 2bb^2}$

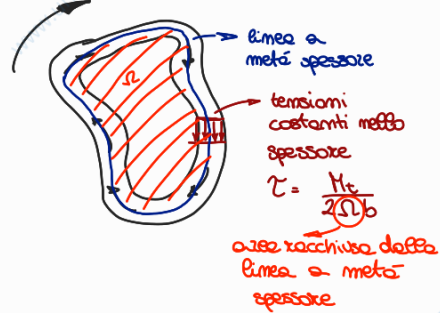
### RIASSUNTO



Thin section:  $\tau_{max} = \frac{M_t}{J_t} b$

$$J_t = \frac{1}{3} \int b^3 ds$$

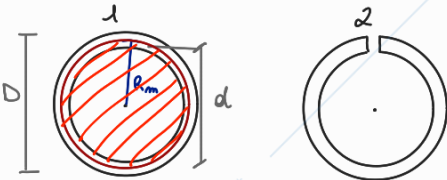
$$\varphi = \frac{M_t L}{G J_t}$$



$$\varphi = \frac{M_t L}{G J_t}$$

$$J_t = \frac{4 \Omega^2}{\oint \frac{1}{b} ds}$$

### CONFRONTO SEZIONE APERTA e CHIUSA



$$\tau_1 = \frac{M_t}{2 \Omega b} = \frac{M_t}{2 b \pi R_m^2}$$

$$\varphi_1 = \frac{M_t L}{G J_t}$$

$$J_t = \frac{4 \Omega^2}{\oint \frac{1}{b} ds} = \frac{4 \pi R_m^2}{\frac{1}{b} 2 \pi R_m}$$

$$\tau_2 = \frac{M_t}{J_t} b = \frac{M_t b}{\frac{b^3}{3} 2 \pi R_m}$$

$$\varphi_2 = \frac{M_t L}{G J_t}$$

$$J_t = \frac{b^3}{3} 2 \pi R_m$$

$R_m$  = raggio medio

### CONFRONTO

$$\frac{\tau_2}{\tau_1} = \frac{3 R_m}{b}$$

$$\frac{\varphi_2}{\varphi_1} = \frac{3 R_m^2}{b^2}$$

$$d/b = 0,99$$

$$0,9$$

$$\tau_2/\tau_1 = 299$$

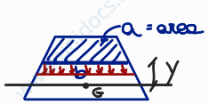
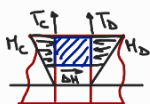
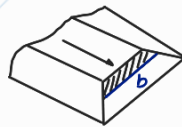
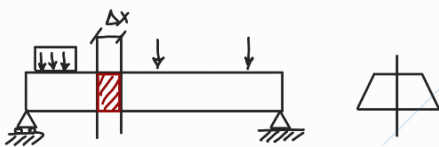
$$29$$

$$\varphi_2/\varphi_1 = 29701$$

$$271$$

$$\tau = \frac{M_t}{2 \Omega b}$$

### TAGLIO



$$\Delta H + \int_a (\tau_c - \tau_b) dA = 0$$

$$\Delta H = \frac{M_b - M_c}{I} \int y dA$$

$$\Delta H + \frac{M_c - M_b}{I} \int y dA = 0$$

$I_x$  momento d'inerzia

$$M_b - M_c = \frac{dM}{dx} \Delta x = T \Delta x$$

$T$  taglio

$$\Rightarrow \tau = \frac{\Delta H}{\Delta x b} = \frac{T \Delta x}{I \Delta x b} \Rightarrow \tau = \frac{TS}{Ib}$$

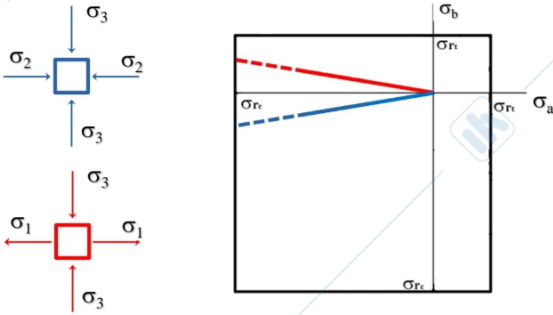
## CRITERIO DELLA MAX TENSIONE NORMALE (RANKINE - GALILEO)

$$\sigma_{id}^+ = \max(\sigma_1, \sigma_2, \sigma_3)$$

$$\sigma_{id}^- = \min(\sigma_1, \sigma_2, \sigma_3)$$

materiale fragile

- Stato di tensione piano (riferimento tensioni principali)



- Casi particolari

- stato di tensione piano in un sistema di riferimento non principale

$$\sigma_{id}^+ = \max \left( \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \right)$$

$$\sigma_{id}^- = \min \left( \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \right)$$

- Flessione e taglio

$$\sigma_{id}^+ = \max \left( \frac{\sigma}{2} \pm \sqrt{\left( \frac{\sigma}{2} \right)^2 + \tau^2} \right)$$

$$\sigma_{id}^- = \min \left( \frac{\sigma}{2} \pm \sqrt{\left( \frac{\sigma}{2} \right)^2 + \tau^2} \right)$$

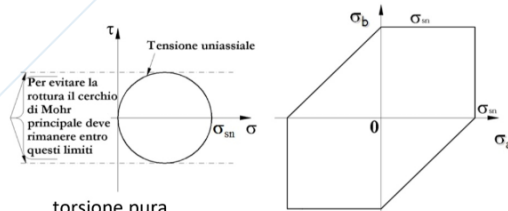
## CRITERIO DELLA max TENSIONE TANGENZIALE (TRESCA)

materiale duttile

- Prova di trazione, al limite  $\tau_{max} = \frac{\sigma_1}{2} = \frac{\sigma_{sn}}{2}$
- Caso generico, al limite  $\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{sn}}{2}$
- Per cui:  $\sigma_{id} = \sigma_1 - \sigma_3$

- Stato di tensione biassiale

- Se  $\sigma_a \geq \sigma_b \geq 0$   $\sigma_1 = \sigma_a; \sigma_3 = 0$   $\sigma_a = \sigma_{sn}$
- Se  $\sigma_a \geq 0 \geq \sigma_b$   $\sigma_1 = \sigma_a; \sigma_3 = \sigma_b$   $\sigma_a - \sigma_b = \sigma_{sn}$
- Se  $0 \geq \sigma_a \geq \sigma_b$   $\sigma_1 = 0; \sigma_3 = \sigma_b$   $\sigma_b = -\sigma_{sn}$



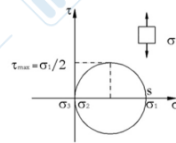
- Caso particolare: flessione e taglio

$$\tau_{max} = \sqrt{\left( \frac{\sigma_x}{2} \right)^2 + \tau_{xy}^2}$$

$$\sigma_{sn} = 2\tau_{sn}$$

$$\sigma_{max} = \sigma_{id} = \sqrt{\sigma_x^2 + 4\tau_{xy}^2}$$

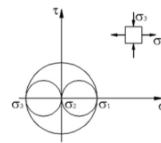
- trazione semplice



$$\tau_{max} = \frac{\sigma_1}{2}$$

$$\sigma_{id} = 2\tau_{max} = \sigma_1$$

- torsione pura

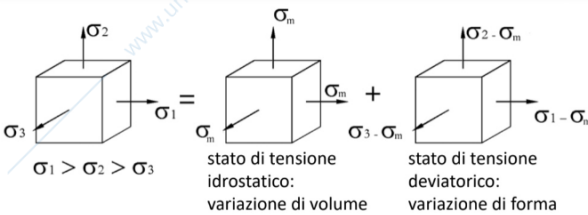


$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

$$\sigma_{id} = 2\tau_{max} = \sigma_1 - \sigma_3$$

## CRITERIO DELLA max ENERGIA DI DISTORSIONE (VON MISES)

materiale duttile



- Lavoro per unità di volume (L)

$$L = \frac{1}{2}(\sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 + \sigma_3 \epsilon_3)$$

- Ricordando che:

$$\epsilon_1 = \frac{\sigma_1}{E} - \frac{\nu \sigma_2}{E} - \frac{\nu \sigma_3}{E}$$

$$\epsilon_2 = -\frac{\nu \sigma_1}{E} + \frac{\sigma_2}{E} - \frac{\nu \sigma_3}{E}$$

$$\epsilon_3 = -\frac{\nu \sigma_1}{E} - \frac{\nu \sigma_2}{E} + \frac{\sigma_3}{E}$$

$$L = \frac{1}{2}(\sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 + \sigma_3 \epsilon_3) =$$

$$= \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1)]$$

- Tensore idrostatico:

$$\sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

- L'energia associata alla variazione di volume senza distorsioni:

$$L_v = \frac{1}{2E} [3\sigma_m^2 - 2\nu(3\sigma_m^2)] = \frac{3\sigma_m^2}{2E} (1 - 2\nu)$$

- Da cui:

$$L_v = \frac{1 - 2\nu}{6E} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1 \sigma_2 + 2\sigma_2 \sigma_3 + 2\sigma_3 \sigma_1)$$

- L'energia di distorsione si determina dalla sottrazione tra l'energia totale e quella che compete alla variazione di volume:

$$L_d = L - L_v = \frac{1 + \nu}{3E} \left( \frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right)$$

- Nel caso di trazione semplice, allo snervamento si ha:

$$\sigma_1 = \sigma_{sn}$$

$$\sigma_2 = \sigma_3 = 0$$

$$L_d = \frac{1 + \nu}{3E} \sigma_{sn}^2$$

- La condizione imposta dal criterio si ottiene uguagliando le precedenti equazioni:

$$2\sigma_{sn}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

$$\sigma_{id} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

- Condizione di inizio snervamento in uno stato di tensione triassiale
- In termini di tensioni non principali

$$\sigma_{id} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

- Casi particolari:

- stato di tensione piano

$$\sigma_{id} = \sqrt{\sigma_a^2 + \sigma_b^2 - \sigma_a \sigma_b} \quad (\text{equazione di un'ellisse})$$

$$\sigma_{id} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2}$$

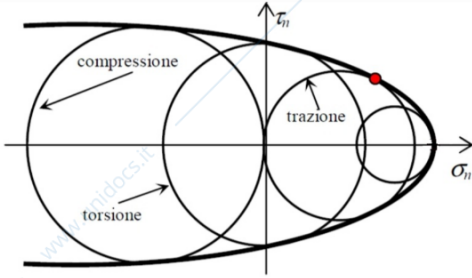
- Flessione e taglio

$$\sigma_{id} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2}$$

- torsione pura

$$\sigma_{id} = \sqrt{3} \cdot \sigma_1 = \sqrt{3} \cdot \tau$$

## CRITERIO DI MOHR-COULOMB



- Casi particolari (tensioni non principali)

- Stato di tensione piano

$$\frac{k-1}{k} \frac{\sigma_x + \sigma_y}{2} + \frac{k+1}{k} \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma_{rt}}{c_s} \quad k = \frac{\sigma_{rc}}{\sigma_{rt}}$$

- Torsione + trazione

$$\frac{k-1}{2k} \sigma_x + \frac{k+1}{2k} \sqrt{\sigma_x^2 + 4\tau_{xy}^2} = \frac{\sigma_{rt}}{c_s}$$

- Solo taglio

$$\frac{k+1}{k} \tau_{xy} = \frac{\sigma_{rt}}{c_s}$$

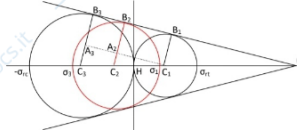
## materiali fragili

- Le proporzioni fra i raggi dei cerchi rispetto alle distanze dei loro centri fornisce la seguente relazione:

$$\frac{A_2 C_2}{A_3 C_3} = \frac{C_2 C_1}{C_3 C_1}; \quad \frac{B_2 C_2 - B_3 A_2}{B_3 C_3 - B_3 A_3} = \frac{HC_2 + HC_1}{HC_3 + HC_1}$$

$$\frac{\frac{\sigma_1 - \sigma_3}{2} - \frac{\sigma_{rt}}{2}}{\frac{\sigma_{rc} - \sigma_{rt}}{2}} = \frac{\frac{\sigma_{rt} - \sigma_1 + \sigma_3}{2}}{\frac{\sigma_{rc} + \sigma_{rt}}{2}}$$

$$\frac{\sigma_1 - \sigma_3}{\sigma_{rc}} = 1$$



- Introducendo un coefficiente di sicurezza  $\frac{\sigma_1 - \sigma_3}{\sigma_{rc}} = \frac{1}{c_s}$

- Stato di tensione biassiale

- Se  $\sigma_a > \sigma_b > 0$

$$\sigma_1 = \sigma_a; \quad \sigma_3 = 0$$

$$\sigma_a = \sigma_{rt}$$

- Se  $\sigma_a > 0 > \sigma_b$

$$\sigma_1 = \sigma_a; \quad \sigma_3 = \sigma_b$$

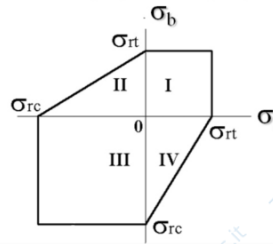
$$\sigma_a / \sigma_{rt} - \sigma_b / \sigma_{rt} = 1$$

- Se  $0 > \sigma_a > \sigma_b$

$$\sigma_1 = 0; \quad \sigma_3 = \sigma_b$$

$$\sigma_b = \sigma_{rc}$$

$$\frac{\sigma_1 - \sigma_3}{\sigma_{rc}} = 1$$



## FATICA

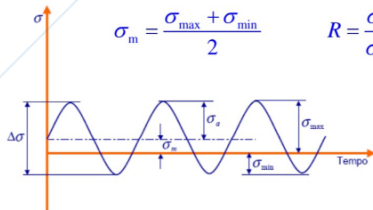
### NOMENCLATURA

Ciclo monotonnale e variabile

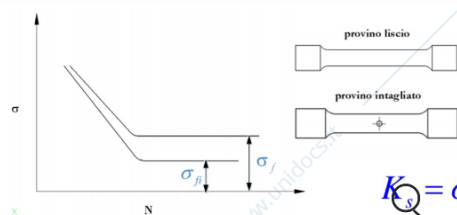
- Ciclo di ampiezza costante

$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2} \quad \Delta\sigma = \sigma_{max} - \sigma_{min}$$

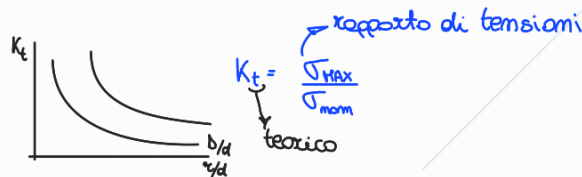
$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} \quad R = \frac{\sigma_{min}}{\sigma_{max}}$$



### FATICA AD EFFETTO DI INTAGLIO



$K_t = \sigma_f / \sigma_{fi}$  → lim. fat. provino intagliato  
 lim. fat. provino liscio  
 sperimentale



- Si definisce il fattore q di sensibilità all'intaglio

$$q = \frac{(K_s - 1)}{(K_t - 1)}$$

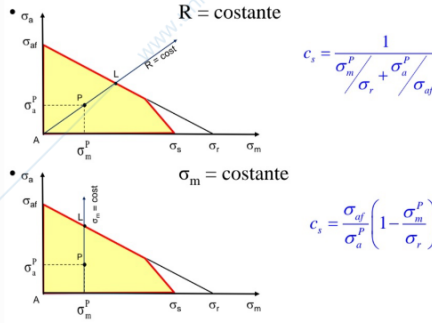
In campo plastico, nei materiali duttili  $K_t > K_s$   
 In campo elastico, nei materiali fragili  $K_t = K_s$

## TENSIONE AMMISSIBILE A FATICA

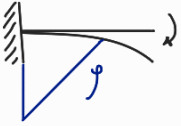
$$\sigma_{am} = \frac{\sigma_f K_a K_d K_b}{c_s K_s}$$

- $\sigma_f$  limite di fatica del provino lucidato
- $c_s$  coefficiente di sicurezza  $>1$
- $K_a$  coefficiente finitura superficiale  $<1$
- $K_d$  coefficiente dimensionale (nel caso di flessione o torsione)  $<1$
- $K_b$  coefficienti tipo di carico
- $K_s$  coefficiente d'intaglio sperimentale  $>1$

## Tensione equivalente



## LINEA ELASTICA



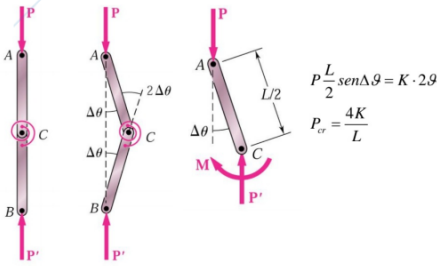
$$\frac{1}{f} = \frac{M}{EI}$$

CURVATURA =  $\frac{1}{f} = \frac{d^2 y}{dx^2} \Rightarrow \frac{1}{f} = \frac{d^2 y}{dx^2} = \frac{M}{EI}$

*trascuabile*

nel momento flettente

## STRUTTURA CON ELASTICITÀ CONCENTRATA



rigidezza, dipendente dalla molla  
 ampolo di cui ruota, ampolo relativo

## TRAVE APPOGGIATA IN COMPRESSIONE

$$M = Py$$

$$\frac{d^2 y}{dx^2} = \frac{-M}{EI} = \frac{-Py}{EI}$$

$$\frac{d^2 y}{dx^2} + \frac{Py}{EI} = 0$$

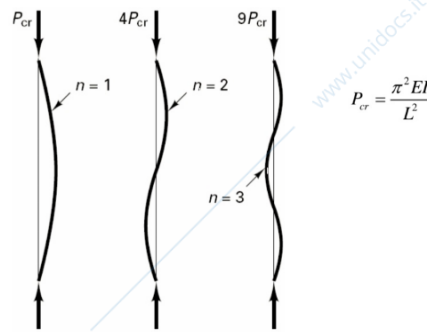
$$\alpha = \sqrt{\frac{P}{EI}}$$

$$y = A \sin(\alpha x) + B \cos(\alpha x)$$

per  $x=0$  e  $x=L \Rightarrow y=0$

$$B=0; A \sin(\alpha L) = 0 \Rightarrow A=0; \sin(\alpha L) = 0; \alpha L = n\pi$$

$$P_{cr} = \frac{n^2 \pi^2 EI}{L^2}$$



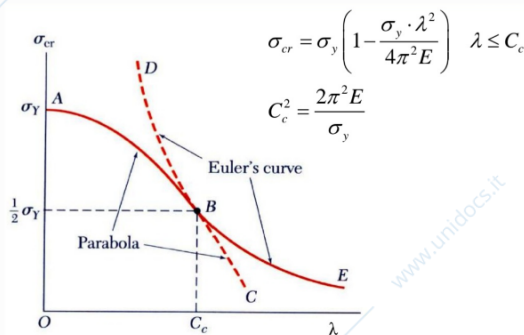
## CARICO CRITICO

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2} = \frac{\pi^2 EA \rho^2}{AL^2} = \frac{\pi^2 E}{L^2 / \rho^2} = \frac{\pi^2 E}{\lambda^2}$$

$\lambda = L / \rho$  raggio d'inerzia

$$\rho = \sqrt{I/A}$$

## FORMULA PARABOLICA DI JOHNSON



$$\frac{P_{cr}}{A} = \frac{\pi^2 EI_y}{AL^2}$$