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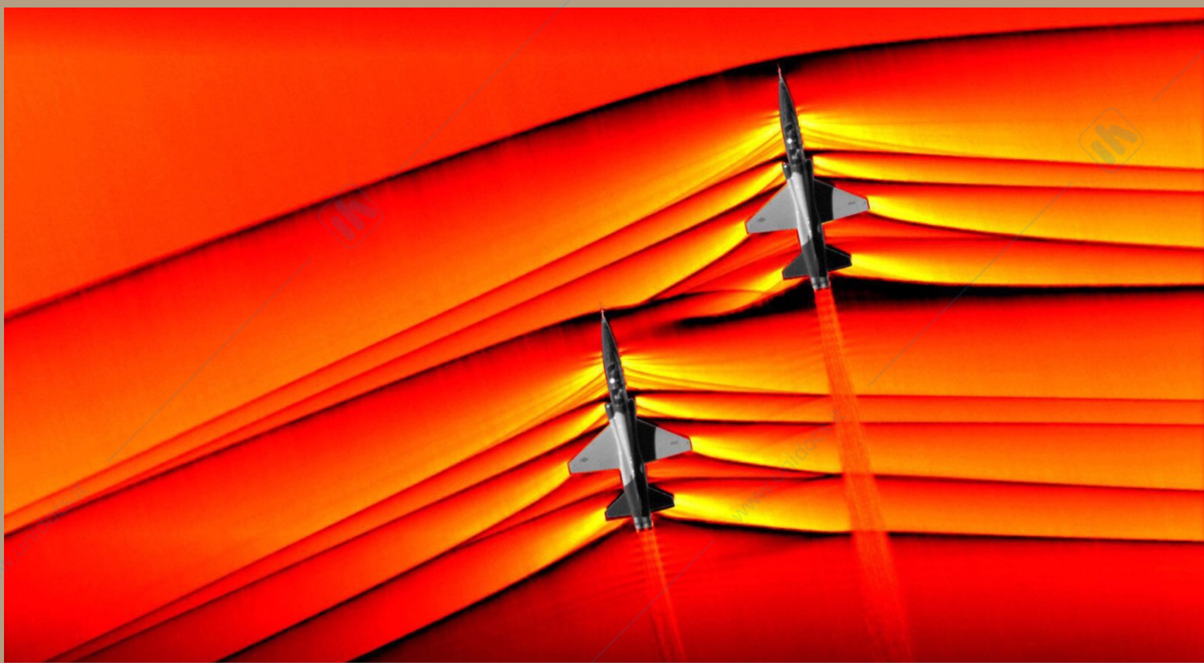
Esercitazioni COMPRESSIBLE FLUID DYNAMICS

A. GUARDONE



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CONSERVATION LAWS

19/03/2020

1.2 Find the value of the divergence $\nabla \cdot \mathbf{u}$ and the vorticity $\nabla \times \mathbf{u}$ for each of the following velocity fields. The factors A and B are constants.

(a) $\mathbf{u} = \mathbf{e}_1 A$
(b) $\mathbf{u} = \mathbf{e}_1 A x_1$
(c) $\mathbf{u} = \mathbf{e}_r f(r,t)$, where r is a spherical coordinate
(d) $\mathbf{u} = \mathbf{e}_1 A x_2 + \mathbf{e}_2 B x_1$
(e) $\mathbf{u} = A(\mathbf{e}_1 \times \mathbf{x})$
(f) $\mathbf{u} = A\left(\frac{\mathbf{e}_1}{x_1} + \frac{\mathbf{e}_2}{x_2} + \frac{\mathbf{e}_3}{x_3}\right)$

$$\underline{u} = u_1 \cdot \underline{e_1} + u_2 \cdot \underline{e_2} + u_3 \cdot \underline{e_3}$$

$$\nabla \cdot \underline{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

$$\nabla \times \underline{u} = \begin{bmatrix} \underline{e_1} & \underline{e_2} & \underline{e_3} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{bmatrix} =$$

$$= \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\underline{e_1} - \left(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right)\underline{e_2} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\underline{e_3}$$

a) $\nabla \cdot \underline{u} = 0$, $\nabla \times \underline{u} = 0$

.....

1.3 Suggest physical means for producing a flow with $\nabla \cdot \mathbf{u} > 0$.

Withdrawing the piston of a syringe we create a positive divergence flow (the volume increases) plugging the opening on the other side.

1.7 Show formally that the velocity field $\mathbf{u}(\mathbf{x},t)$ derived from a potential $\phi(\mathbf{x},t)$

$u = \nabla\phi$

is irrotational.

$$\underline{u}(x,t) = \nabla\phi(x,t) = \frac{\partial\phi}{\partial x}\underline{e_1} + \frac{\partial\phi}{\partial y}\underline{e_2} + \frac{\partial\phi}{\partial z}\underline{e_3}$$

$$\nabla \times \underline{u} = \cancel{\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)}\underline{e_1} + \cancel{\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial y}\right)}\underline{e_2} + \cancel{\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial x_2}\right)}\underline{e_3} = 0$$

Scwartz

$$\frac{\partial^2\phi}{\partial z\partial y} - \frac{\partial^2\phi}{\partial y\partial z}$$

1.10 Find the material acceleration $D\mathbf{u}/Dt$ for the velocity field

$$\mathbf{u} = \mathbf{e}_1 \alpha \frac{x_1}{t}$$

where α is a constant, and interpret the result for the special case $\alpha = 1$.

Answer $\frac{D\mathbf{u}}{Dt} = \mathbf{e}_1 \alpha(\alpha - 1)\frac{x_1}{t^2}$

$$\underline{u} = \alpha \frac{x}{t} \underline{e_1}$$

$$\frac{D\underline{u}}{Dt} = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = \left[-\alpha \frac{x}{t^2} + \frac{\alpha^2}{t^2} x\right]\underline{e_1} = \left[\alpha - 1\right]\alpha \frac{x}{t^2}\underline{e_1}$$

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1.12 Show that the Euler momentum equation

$$\rho \frac{Du_i}{Dt} + P_{,i} = 0$$

can be put in the form

$$\frac{\partial}{\partial t} \rho u_i + (\rho u_i u_k + P\delta_{ik})_{,k} = 0$$

The quantity $\rho u_i u_k$ is the *momentum tensor*.

- $\rho \frac{Du_i}{Dt} + P_{,i} = 0 \qquad \left(\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \underline{\nabla}\right)$

$$\rho \frac{\partial u_1}{\partial t} + \rho u_1 \frac{\partial u_1}{\partial x} + \rho u_2 \frac{\partial u_1}{\partial y} + \rho u_3 \frac{\partial u_1}{\partial z} + \frac{\partial P}{\partial x} = 0 \quad (1)$$

$$\rho \frac{\partial u_2}{\partial t} + \rho u_1 \frac{\partial u_2}{\partial x} + \rho u_2 \frac{\partial u_2}{\partial y} + \rho u_3 \frac{\partial u_2}{\partial z} + \frac{\partial P}{\partial y} = 0$$

$$\rho \frac{\partial u_3}{\partial t} + \rho u_1 \frac{\partial u_3}{\partial x} + \rho u_2 \frac{\partial u_3}{\partial y} + \rho u_3 \frac{\partial u_3}{\partial z} + \frac{\partial P}{\partial z} = 0$$

- $\frac{\partial}{\partial t} \rho u_i + (\rho u_i u_k + P\delta_{ik})_{,k} = 0$

$$\delta_{ik} = \begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i\neq k \end{cases}$$

$i = 1 \cdot (\sum_k)$

$$\frac{\partial \rho u_1}{\partial t} + \frac{\partial}{\partial x}(\rho u_1^2) + \frac{\partial}{\partial y}(\rho u_1 u_2) + \frac{\partial}{\partial z}(\rho u_1 u_3) + \frac{\partial P}{\partial x} = 0 \quad (2)$$

$$\frac{\partial \rho u_2}{\partial t} + \frac{\partial}{\partial x}(\rho u_1 u_2) + \frac{\partial}{\partial y}(\rho u_2^2) + \frac{\partial}{\partial z}(\rho u_2 u_3) + \frac{\partial P}{\partial y} = 0$$

$$\frac{\partial \rho u_3}{\partial t} + \frac{\partial}{\partial x}(\rho u_1 u_3) + \frac{\partial}{\partial y}(\rho u_2 u_3) + \frac{\partial}{\partial z}(\rho u_3^2) + \frac{\partial P}{\partial z} = 0$$

Let's demonstrate (1) = (2) ...

$$-\left[\begin{array}{l} (1) \quad \cancel{\rho \frac{\partial u_1}{\partial t}} + \cancel{\rho u_1 \frac{\partial u_1}{\partial x}} + \cancel{\rho u_2 \frac{\partial u_1}{\partial y}} + \cancel{\rho u_3 \frac{\partial u_1}{\partial z}} + \cancel{\frac{\partial P}{\partial x}} = 0 \\ (2) \quad \frac{\partial \rho u_1}{\partial t} + \frac{\partial}{\partial x}(\rho u_1^2) + \frac{\partial}{\partial y}(\rho u_1 u_2) + \frac{\partial}{\partial z}(\rho u_1 u_3) + \frac{\partial P}{\partial x} = 0 \end{array}\right.$$

$$\cancel{\rho \frac{\partial u_1}{\partial t}} + u_1 \frac{\partial \rho}{\partial t} + \cancel{\rho u_1 \frac{\partial u_1}{\partial x}} + u_1 \frac{\partial (\rho u_1)}{\partial x} + \cancel{\rho u_2 \frac{\partial u_1}{\partial y}} + u_1 \frac{\partial}{\partial y}(\rho u_2) +$$

$$+ \cancel{\rho u_3 \frac{\partial u_1}{\partial z}} + u_1 \frac{\partial (\rho u_3)}{\partial z} + \cancel{\frac{\partial P}{\partial x}} = 0$$

because of the mass conservation

$$(2) - (1) = u_1 \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{u})\right] = 0$$

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1.13 For an incompressible fluid at rest, with $e = c_v T + \text{const}$, show that the energy equation (1.71) reduces to the *heat equation*

$$\rho c_v \frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

where the thermal conductivity κ has been assumed constant.

$$\rho \frac{D}{Dt}\left(e + \cancel{\frac{|\underline{v}|^2}{2}}\right) + \underline{\nabla}\cdot(\cancel{p\underline{v}}) = \underline{\nabla}\cdot(\cancel{\underline{v}\pi}) - \underline{\nabla}\cdot\underline{q} - \cancel{\rho\underline{g}\cdot\underline{v}}$$

$$e = c_v T + \text{const}$$

$$\underline{q} = -\kappa \underline{\nabla} T$$

$$\rho c_v \frac{\partial T}{\partial t} = \rho \frac{\partial e}{\partial t} = -\underline{\nabla}\cdot\underline{q} = \underline{\nabla}\cdot(\kappa \underline{\nabla} T) = \kappa \nabla^2 T \longrightarrow \boxed{\rho c_v \frac{\partial T}{\partial t} = \kappa \nabla^2 T}$$

DIFFUSION EQUATION

Alternative way ...

Energy balance in non-conservative form...

$$\rho \frac{De}{Dt} = -\cancel{P\,\underline{\nabla}\cdot\underline{u}} + \cancel{\pi\,\underline{\nabla}\cdot\underline{u}} - \underline{\nabla}\cdot\underline{q}$$

$$\rho \frac{\partial e}{\partial t} + \cancel{\rho\underline{u}\cdot\underline{\nabla} e} = \underline{\nabla}\cdot(\kappa\underline{\nabla} T) \longrightarrow \rho c_v \frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

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THERMODYNAMICS OF MOTION

26/03/2020

2.1 Write explicit expressions for the dissipation function Υ for the velocity fields of Examples 1.2 and 1.3 (pages 21 and 24).

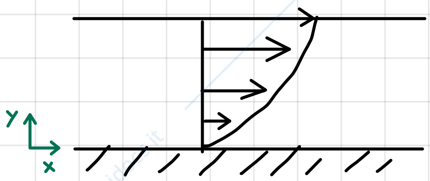
$$\boxed{\Upsilon = \Sigma_{ik} D_{ik}}$$

where : $\boxed{\Sigma_{ik} = 2\eta\left(D_{ik} - \frac{1}{3}\delta_{ik} D_{jj}\right) + \mu_v \delta_{ik} D_{jj}}$ (viscous part of the stress tensor)

$\boxed{D_{ik} = \frac{1}{2}\left(u_{i,k} + u_{k,i}\right)}$ (symmetric part of the gradient)

$u(y)$

•



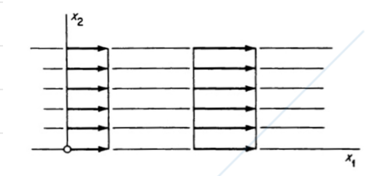
$$\underline{u} = (u,0,0)^T \rightarrow \nabla \underline{u} = \begin{bmatrix} 0 & \overset{\frac{\partial u}{\partial y}}{A} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\nabla \underline{u}$ has 0 on the diagonal, so $\nabla \cdot \underline{u} = 0$
(incompressible field $\rightarrow$ $D_{jj} = 0$)

$$D_{ij} = \begin{bmatrix} 0 & A/2 & 0 \\ A/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow D_{ij} D_{ij} = \begin{bmatrix} 0 & A/2 & 0 \\ A/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & A/2 & 0 \\ A/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A^2/4 & 0 & 0 \\ 0 & A^2/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Upsilon = 2\mu D_{ik} D_{ik} = 2\mu \frac{A^2}{4} 2 = \eta A^2$$

•



$$\underline{u} = (u,0,0)^T \rightarrow \nabla \underline{u} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\nabla \underline{u}$ has not 0 on the diagonal, so $\nabla \cdot \underline{u} \neq 0$
(compressible field)

$$\Upsilon = \left[2\mu\left(A - \frac{1}{3}A\right) + \mu_v A\right] A = \left(\frac{4}{3}\mu + \mu_v\right) A^2$$

2.2 Show that $c_p = T(\partial s/\partial T)_p$.

$$c_p = \left.\frac{\partial h}{\partial T}\right|_p \qquad \text{where} \qquad dh = T ds + v dP$$

$$c_p = T \left.\frac{\partial s}{\partial T}\right|_p + \cancel{v \left.\frac{\partial P}{\partial T}\right|_p} = T \left.\frac{\partial s}{\partial T}\right|_p$$

$$\left[c_p = \left(\frac{\partial h}{\partial s}\right)_p \left(\frac{\partial s}{\partial T}\right)_p = T\left(\frac{\partial s}{\partial T}\right)_p \right]$$

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2.4 Given that

$$dh = T\,ds + v\,dP$$

$$c_p \equiv \left(\frac{\partial h}{\partial T}\right)_p$$

$$\left(\frac{\partial s}{\partial P}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p$$

find, for the variation of c_p with pressure,

$$\left(\frac{\partial c_p}{\partial P}\right)_T = -T\left(\frac{\partial^2 v}{\partial T^2}\right)_p$$

Like before we obtain: $c_p = T \left.\frac{\partial s}{\partial T}\right|_P$

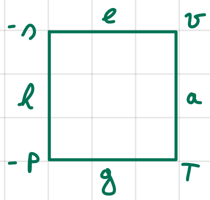
$$\left.\frac{\partial c_p}{\partial P}\right|_T = \left.\frac{\partial}{\partial P}\right|_T \left(T \left.\frac{\partial s}{\partial T}\right|_P\right) = T \left.\frac{\partial}{\partial P}\right|_T \left.\frac{\partial s}{\partial T}\right|_P =$$

Schwartz

$$= T \left.\frac{\partial}{\partial T}\right|_P \left.\frac{\partial s}{\partial P}\right|_T = T \left.\frac{\partial}{\partial T}\right|_P \boxed{\left.\frac{\partial s}{\partial P}\right|_T} =$$

$$\boxed{\left.\frac{\partial s}{\partial P}\right|_T} = -\left.\frac{\partial v}{\partial T}\right|_P$$

$$= -T \left.\frac{\partial^2 v}{\partial T^2}\right|_P$$



$$e = e(s, v)$$

$$de = \underbrace{\boxed{\left.\frac{\partial e}{\partial s}\right|_v}}_{T} ds + \underbrace{\boxed{\left.\frac{\partial e}{\partial v}\right|_s}}_{-P} dv = T\,ds - P\,dv$$

$$\frac{\partial e^2}{\partial s \partial v} = \frac{\partial e^2}{\partial v \partial s} \longrightarrow -\left.\frac{\partial P}{\partial s}\right|_v = \left.\frac{\partial T}{\partial v}\right|_s$$

2.5 If the ratio of specific heats γ for an ideal gas varies according to

$$\frac{\gamma}{\gamma - 1} = a_0 + a_1 T$$

where a_0 and a_1 are constants, find an equation relating P and T for an isentropic process analogous to the perfect gas relation

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}$$

Answer $\quad \frac{P_2}{P_1} = e^{a_1(T_2 - T_1)}\left(\frac{T_2}{T_1}\right)^{a_0}$

Hp: • $\frac{\gamma}{\gamma-1} = a_0 + T a_1$ • IG

• $ds = 0$

$$c_p - c_v = R \longrightarrow c_p\left(1 - \frac{1}{\gamma}\right) = R$$

$$c_p\left(\frac{\gamma - 1}{\gamma}\right) = R$$

$$T\,ds = dh - v\,dP$$

$$c_p = \left.\frac{\partial h}{\partial T}\right|_P \overset{IG}{=} \frac{dh}{dT} \longrightarrow dh = c_p\,dT$$

$$\cancel{T\,ds} = c_p\,dT - v\,dP \qquad \text{Where for IG } v = \frac{RT}{P}$$

$$c_p\,dT = \frac{RT}{P}\,dP$$

So: $\begin{cases} c_p\,dT = \frac{RT}{P}\,dP \\ c_p = R\frac{\gamma}{\gamma-1} \end{cases} \longrightarrow \frac{\gamma}{\gamma-1}\frac{dT}{T} = \frac{dP}{P} \longrightarrow \int_{T_0}^{T}\left(\frac{a_0}{T} + a_1\right)dT = \int_{P_0}^{P}\frac{dP}{P} \longrightarrow$

$$\longrightarrow a_0 \ln\frac{T}{T_0} + a_1(T - T_0) = \ln\frac{P}{P_0} \longrightarrow \frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{a_0} \exp\left(a_1(T - T_0)\right)$$

2.6 Find the molecular weight and ratio of specific heats for an equimolar mixture of helium and oxygen at room temperature.

Answer $\quad$ 18.001; 1.500

$$M = \frac{1}{2}M_{O_2} + \frac{1}{2}M_{He}$$

$$M_{O_2} = 32 \text{ g/mol}$$
$$M_{He} = 4 \text{ g/mol}$$

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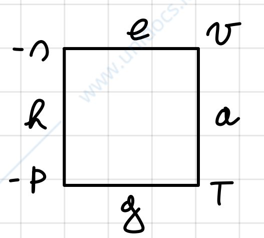
$$\boxed{\gamma = \frac{x_i \frac{\gamma_i}{\gamma_i - 1}}{x_k \frac{1}{\gamma_k - 1}}} \qquad \left(\text{where : } \gamma_{O_2} = \frac{7}{5} \;,\; \gamma_{He} = \frac{5}{3}\right) \qquad x \text{ molar fraction}$$

$$= \frac{\frac{1}{2}\frac{\gamma_O}{\gamma_O - 1} + \frac{1}{2}\frac{\gamma_H}{\gamma_H - 1}}{\frac{1}{2}\frac{1}{\gamma_O - 1} + \frac{1}{2}\frac{1}{\gamma_H - 1}} = \dots$$

2.8 The identity $(\partial P/\partial v)_s = \gamma(\partial P/\partial v)_T$ is not useful at the critical point because $(\partial P/\partial v)_T \to 0$ and $\gamma \to \infty$. Find the alternative identity,

$$\left(\frac{\partial P}{\partial v}\right)_s = \left(\frac{\partial P}{\partial v}\right)_T - \frac{T}{c_v}\left(\frac{\partial P}{\partial T}\right)_v^2$$

by manipulation of fundamental relations.



$$\left.\frac{\partial P}{\partial v}\right|_s \to \frac{\partial}{\partial v} P(s,v) = \frac{\partial}{\partial v} P(T(s,v), v)$$

$$\left.\frac{\partial P}{\partial v}\right|_s = \left.\frac{\partial P}{\partial T}\right|_v \boxed{\left.\frac{\partial T}{\partial v}\right|_s} + \left.\frac{\partial P}{\partial v}\right|_T =$$

$$\left.\frac{\partial T}{\partial v}\right|_s = -\left.\frac{\partial P}{\partial s}\right|_v = -\left.\frac{\partial P}{\partial T}\right|_v \boxed{\left.\frac{\partial T}{\partial s}\right|_v}$$

$$\left(\text{But } \; T = \left.\frac{\partial e}{\partial s}\right|_v = \underbrace{\boxed{\left.\frac{\partial e}{\partial T}\right|_v}}_{c_v} \left.\frac{\partial T}{\partial s}\right|_v \longrightarrow \left.\frac{\partial T}{\partial s}\right|_v = \frac{T}{c_v}\right)$$

$$\left.\frac{\partial T}{\partial v}\right|_s = -\left.\frac{\partial P}{\partial T}\right|_v \frac{T}{c_v}$$

$$= \left.\frac{\partial P}{\partial v}\right|_T - \frac{T}{c_v}\left(\left.\frac{\partial P}{\partial T}\right|_v\right)^2$$

2.10 Evaluate the derivatives $(\partial v/\partial P)_s$, $(\partial^2 v/\partial P^2)_s$, $(\partial^3 v/\partial P^3)_s$ in terms of v, P, and γ for a perfect gas.

- $\left.\dfrac{\partial v}{\partial P}\right|_s$

$$T\,ds = c_p\,dT - v\,dP = c_p\,dT - \frac{RT}{P}\,dP$$

$$ds = c_p \frac{dT}{T} - R\frac{dP}{P} \xrightarrow{\int} \cancel{s - s_0} = \frac{c_p}{c_p} \ln\frac{T}{T_0} - \underbrace{\boxed{\frac{R}{c_p}}}_{\frac{\gamma-1}{\gamma}} \ln\frac{P}{P_0}$$

isentropic process because we are evaluating $\left.\dfrac{\partial v}{\partial P}\right|_s$

$$\frac{T}{T_0} = \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\frac{Pv}{P_0 v_0} = \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} \to \frac{v}{v_0} = \left(\frac{P}{P_0}\right)^{-1/\gamma}$$

- $v = v(P)$

$$v' = -\frac{v_0}{\gamma}\overbrace{\left(\frac{P}{P_0}\right)^{-\frac{1}{\gamma}-1}}^{v\frac{P_0}{P}}\frac{1}{P_0} = -\frac{1}{\gamma}\frac{v}{P}$$

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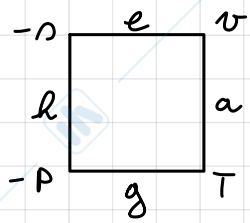
$$v'' = \left(-\frac{v_0}{\gamma p_0} \left(\frac{p}{p_0}\right)^{-\frac{1}{\gamma}-1} \right)' = \frac{v_0}{\gamma p_0^2} \frac{\gamma+1}{\gamma} \left(\frac{p}{p_0}\right)^{-\frac{1}{\gamma}-2} = \frac{\gamma+1}{\gamma} \frac{v}{p^2}$$

$$v''' = \left(\frac{v_0}{p_0^2} \frac{\gamma+1}{\gamma^2} \left(\frac{p}{p_0}\right)^{-\frac{1}{\gamma}-2} \right)' = -\frac{v_0}{p_0^3} \frac{(\gamma+1)(2\gamma+1)}{\gamma^3} \left(\frac{p}{p_0}\right)^{-\frac{1}{\gamma}-3} =$$

$$= -\frac{(\gamma+1)(2\gamma+1)}{\gamma^3} \frac{v}{p^3}$$



2.17 A certain fluid is compressed isentropically. Will the temperature increase?

We basically want to find the sign of $\left.\frac{\partial T}{\partial p}\right|_s$

Using the chain rule...

$$\left.\frac{\partial T}{\partial p}\right|_s \left.\frac{\partial p}{\partial s}\right|_T \left.\frac{\partial s}{\partial T}\right|_p = -1 \longrightarrow \left.\frac{\partial T}{\partial p}\right|_s = -\left.\frac{\partial s}{\partial p}\right|_T \boxed{\left.\frac{\partial T}{\partial s}\right|_p} = -\frac{T}{c_p} \boxed{\left.\frac{\partial s}{\partial p}\right|_T} = \frac{T}{c_p} \left.\frac{\partial v}{\partial T}\right|_p$$

$$\left(\left.\frac{\partial T}{\partial s}\right|_p = \underbrace{\boxed{\left.\frac{\partial T}{\partial h}\right|_p}}_{1/c_p} \underbrace{\boxed{\left.\frac{\partial h}{\partial s}\right|_p}}_{T} = \frac{T}{c_p} \right)$$

c_p is >0 for thermodynamic stability and $T>0$. If $\left.\frac{\partial v}{\partial T}\right|_p$ is >0 then the temperature will increase compressing a fluid isentropically.

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ACOUSTICS

2/04/2020

4.1 Given that the condensation S satisfies the wave equation; which of the quantities P, ρ, T, and s satisfy the wave equation?

$$S \rightarrow \frac{\partial^2 S}{\partial t^2} = c_0^2 \nabla^2 S \qquad P, \rho, T, s$$

- $S = \frac{\rho - \rho_0}{\rho_0} \rightarrow \rho = \rho_0 + \rho_0 S \longrightarrow \boxed{\rho' = \rho_0 S}$
- $P = P(\rho) \xrightarrow{\text{Taylor}} P = P_0 + \left(\frac{\partial P}{\partial \rho}\Big|_s\right)_0 (\rho - \rho_0) \rightarrow P - P_0 = P' = c_0^2 \rho' \rightarrow \boxed{P' = c_0^2 \rho_0 S}$
- $\boxed{s = s_0}$
- $T = T_0 + \left(\frac{\partial T}{\partial \rho}\Big|_s\right)_0 (\rho - \rho_0) \rightarrow$

$$\left[\frac{\partial T}{\partial \rho}\Big|_s = -v^2 \underbrace{\frac{\partial T}{\partial v}\Big|_s}_{\text{From Thompson } *} = -v^2 \sqrt{\frac{(\gamma-1)c^2 T}{v^2 c_p}} = -(\gamma-1)\, c \frac{1}{\rho} \sqrt{\frac{T^2}{\underbrace{\partial R \cdot T}_{c^2}}} = -\frac{(\gamma-1)}{\rho} T \right]$$

* APPENDIX B

$$\rightarrow \boxed{\Delta T = (1-\gamma)\, T_0\, S}$$

4.2 Show formally that a function $F(t - x/c_0)$ satisfies the one-dimensional wave equation.

1D wave equation: $\frac{\partial^2 F}{\partial t^2} = c_0^2 \frac{\partial^2 F}{\partial x^2}$

We define: $y = t - \frac{x}{c_0} \longrightarrow \frac{\partial y}{\partial t} = 1 \;,\; \frac{\partial y}{\partial x} = -\frac{1}{c_0}$

- $\frac{\partial}{\partial x} = \frac{\partial}{\partial y}\frac{\partial y}{\partial x} = -\frac{1}{c_0}\frac{\partial}{\partial y} \longrightarrow \frac{\partial^2}{\partial x^2} = \frac{1}{c_0^2}\frac{d^2}{dy^2} \longrightarrow \frac{\partial^2 F}{\partial x^2} = \frac{1}{c_0^2}\frac{d^2 F}{dy^2} = \frac{F''}{c_0^2}$
- $\frac{\partial}{\partial t} = \frac{\partial}{\partial y}\frac{\partial y}{\partial t} = \frac{d}{dy} \longrightarrow \frac{\partial^2}{\partial t^2} = \frac{d^2}{dy^2} \longrightarrow \frac{\partial^2 F}{\partial t^2} = \frac{d^2 F}{dy^2} = F''$

So: $F'' = c_0^2 \frac{1}{c_0^2} F'' \rightarrow F'' = F''$ ok!

4.3 For a hypothetical substance which has an equation of state $P = a^2\rho$, where a is a constant, find an expression for the sound speed.

Equation of state (EOF): $P(\rho) = a^2 \rho$

$$c^2 = \frac{\partial P}{\partial \rho}\Big|_s = \frac{dP}{d\rho} = a^2$$

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4.5 The equation of state for a Clausius gas is $P(v - b) = RT$, where b is a constant representing the molecular volume and R is the gas constant. Find an expression for the sound speed in such a gas in terms of γ, R, T, v, and b. If possible, interpret the result physically.

Answer $\quad c^2 = \gamma RT\left(\dfrac{v}{v-b}\right)^2$

OEF : $\quad P(v-b) = RT$

$$c^2 = \frac{\partial P}{\partial \rho}\Big|_s = -v^2 \underbrace{\frac{\partial P}{\partial v}\Big|_s} =$$

$$\frac{\partial}{\partial v}\Big|_s P(T(v,s),v) = \frac{\partial P}{\partial T}\Big|_v \underbrace{\frac{\partial T}{\partial v}\Big|_s} + \frac{\partial P}{\partial v}\Big|_T =$$

$$\frac{\partial T}{\partial v}\Big|_s = -\frac{\partial P}{\partial s}\Big|_v = -\frac{\partial P}{\partial T}\Big|_v \underbrace{\frac{\partial T}{\partial s}\Big|_v} =$$

$$\left[T = \frac{\partial e}{\partial s}\Big|_v = \underbrace{\frac{\partial e}{\partial T}\Big|_v}_{c_v} \frac{\partial T}{\partial s}\Big|_v \rightarrow \frac{\partial T}{\partial s}\Big|_v = \frac{T}{c_v}\right]$$

$$\rightarrow \frac{\partial T}{\partial v}\Big|_s = -\frac{T}{c_v}\frac{\partial P}{\partial T}\Big|_v$$

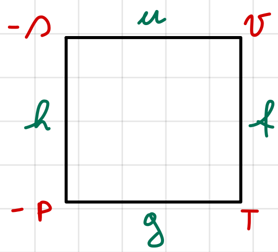
$$\rightarrow \frac{\partial P}{\partial v}\Big|_s = \frac{\partial P}{\partial T}\Big|_v\left(-\frac{T}{c_v}\frac{\partial P}{\partial T}\Big|_v\right) + \frac{\partial P}{\partial v}\Big|_T$$

$$\rightarrow \boxed{c^2 = -v^2\left[\underbrace{\frac{\partial P}{\partial v}\Big|_T} - \frac{T}{c_v}\Big(\underbrace{\frac{\partial P}{\partial T}\Big|_v}\Big)^2\right]}$$

$$\left[\frac{\partial P}{\partial v}\Big|_T = -\frac{RT}{(v-b)^2} \qquad \frac{\partial P}{\partial T}\Big|_v = \frac{R}{v-b}\right]$$

$$\rightarrow c^2 = -v^2\left[-\frac{RT}{(v-b)^2} - \frac{T}{c_v}\frac{R^2}{(v-b)^2}\right] = \frac{v^2}{(v-b)^2}\left(\cancel{RT} + \frac{R^{\cancel{2}}T}{\cancel{R}}(\gamma - \cancel{1})\right) = \gamma RT\frac{v^2}{(v-b)^2}$$

$c_v \rightarrow \dfrac{R}{\gamma-1}$



4.6 For acoustic motion in an ideal gas with the local value of condensation S, find the corresponding value of the relative sound-speed change, $(c - c_0)/c_0$.

Answer $\quad \dfrac{c - c_0}{c_0} = \dfrac{\gamma - 1}{2} S$

$$S = \frac{\rho - \rho_0}{\rho_0}$$

$$c = c_0 + \left(\frac{\partial c}{\partial \rho}\Big|_s\right)_0 (\rho - \rho_0)$$

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$$\left.\frac{\partial c}{\partial \rho}\right|_s = -v^2 \left.\frac{\partial c}{\partial v}\right|_s = -v^2 \left.\frac{\partial}{\partial v}\right|_s c(P(s,v),v) = -v^2\left[\left.\frac{\partial c}{\partial P}\right|_v \left.\frac{\partial P}{\partial v}\right|_s + \left.\frac{\partial c}{\partial v}\right|_P\right]$$

As in the previous exercise we obtain $\left.\frac{\partial P}{\partial v}\right|_s = -\frac{T}{c_v}\left(\left.\frac{\partial P}{\partial T}\right|_v\right)^2 + \left.\frac{\partial P}{\partial v}\right|_T =$

For PIG: $P = \frac{RT}{v}$ $\quad c = \sqrt{\gamma R T}$

$$\left.\frac{\partial P}{\partial v}\right|_T = -\frac{RT}{v^2} \qquad \left.\frac{\partial P}{\partial T}\right|_v = \frac{R}{v}$$

$$\left.\frac{\partial c}{\partial P}\right|_v = \frac{\gamma v}{2c} \qquad \left.\frac{\partial c}{\partial v}\right|_P = \frac{\gamma P}{2c}$$

$$\longrightarrow \left.\frac{\partial P}{\partial v}\right|_s = -\frac{RT}{v^2} - \frac{T}{c_v}\frac{R^2}{v^2} = -\frac{c^2}{v^2}$$

$$\left.\frac{\partial c}{\partial \rho}\right|_s = -v^2\left[-\frac{\gamma v}{2c}\frac{c^2}{v^2} + \frac{\gamma P}{2c}\right] = \frac{\gamma v c^2}{2c} - \frac{\gamma P v^2}{2c} = \frac{cv}{2}(\gamma - 1)$$

$$C = C_0 + \frac{\gamma-1}{2}\frac{C_0}{\rho_0}(\rho - \rho_0) \rightarrow \frac{C - C_0}{C_0} = \frac{\gamma - 1}{2} s$$

4.8 Calculate the particle displacement amplitude for one-dimensional sinusoidal progressive waves in air at 300 K at the following intensity levels and frequencies:

(a) 20 dB, 10 kHz

(b) 60 dB, 50 Hz

These levels are roughly the lowest at which the sound can be clearly heard at the particular frequency.

Answer $\quad 1.11 \times 10^{-11}$ m; 2.23×10^{-7} m

sinusoidal wave 1D

progressive wave $(g=0)$

$T_0 = 300$ K

$P_0 = 1$ atm

$\Delta = 10 \log_{10} \frac{\Phi_E}{10^{-12}}$

The energy flow is $\quad \Phi_E = \frac{1}{T}\int_0^T p' u' \, dt = \rho_0 c_0^3 \frac{1}{T}\int_0^T s^2 \, dt = \rho_0 c_0^3 \boxed{\overline{s^2}} = \rho_0 c_0^3 \boxed{\frac{s_0^2}{2}}$

sinusoidal wave

We know that:
- $\phi = F(x - c_0 t)$
- $s = -\frac{1}{c_0^2}\frac{\partial \phi}{\partial t} = \frac{F'}{c_0} = f$
- $u = \nabla\phi = F' = c_0 f$

$$\left(s = -s_0 \sin \frac{2\pi}{\lambda}(x - c_0 t) \qquad \boxed{\lambda = \frac{c_0}{\nu}} \quad \text{and} \quad T = \frac{1}{\nu}\right)$$

period

frequency

Displacement $\quad d\xi = u \, dt \longrightarrow \xi = \int_0^t \overbrace{c_0 f(x_0 - c_0 t)}^{u'(x_0 - c_0 t)} \, dt = -s_0 c_0 \int_0^t \sin \frac{2\pi}{\lambda}(x_0 - c_0 t) =$

AMPLITUDE (it's our request!) $\longleftarrow \xi_0$

$$= \frac{s_0 \lambda}{2\pi}\left[1 - \cos\frac{2\pi}{\lambda}(x_0 - c_0 t)\right] \rightarrow \text{time evolution of displacement}$$

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$$\Delta = 10\log_{10}\frac{\phi_E}{\phi_{E ref}} \rightarrow \phi_E = \phi_{E ref}\, 10^{\frac{\Delta}{10}} = 10^{-12}\, 10^{\Delta/10} = \rho_0 c_0^3 \frac{S_0^2}{2} \longrightarrow S_0^2 = \frac{2\cdot 10^{-12}}{\rho_0 c_0^3}\, 10^{\Delta/10}$$

$$\xi_0 = \frac{S_0 \lambda}{2\pi} = \frac{S_0 c_0}{2\pi \nu}$$

4.9 A loudspeaker advertisement claims a total (peak-to-peak) diaphragm displacement of 5/8 in. for bass notes. Assuming one-dimensional sinusoidal motion at frequency $\nu = 35$ Hz, calculate the one-dimensional energy flux and find the corresponding decibel level at the speaker.

Answer 148 dB

$$\left.\begin{array}{l}\xi_M = \frac{5}{8} \text{ in} \\ \nu = 35 \text{ Hz} \\ \left.\begin{array}{l} T_0 = 300 \text{ K} \\ P_0 = 1 \text{ atm}\end{array}\right\} c_0 = \dots \end{array}\right\} \lambda = \frac{c_0}{\nu} = \dots$$

Exactly as before we obtain $\xi(t, x_0) = \frac{S_0 \lambda}{2\pi}\left[1 - \cos\frac{2\pi}{\lambda}(x_0 - c_0 t)\right]$

At the speaker we have the maximum displacement ...

$$\xi_M = 2\xi_0 = \frac{S_0 \lambda}{\pi} \longrightarrow S_0 = \frac{\xi_M \pi \nu}{c_0}$$

$$\Phi_E = \rho_0 c_0^3 \frac{S_0^2}{2} \qquad \Delta = 10 \log_{10} \frac{\Phi_E}{\Phi_{ref}} \qquad \text{where } \Phi_{ref} = 10^{-12}$$

4.11 For acoustic radiation with spherical symmetry, the velocity potential is $\phi = (1/r)F(r - ct)$. If F is sinusoidal, with wavelength λ, find a criterion for the negligibility of the $1/r^2$ term in the expression for particle velocity; i.e., find the condition on r such that this term is negligible.

Answer $kr \gg 1$

$$\phi = \frac{1}{r} F(r - ct) \qquad \boxed{F = F_0 \sin\left[\frac{2\pi}{\lambda}(r - ct)\right]} \longrightarrow F' = \frac{F_0 2\pi}{\lambda} \cos\left[\frac{2\pi}{\lambda}(r - ct)\right]$$

$$\underline{u} = \underline{\nabla}\phi \longrightarrow \overset{\text{radial velocity}}{u_r} = \frac{\partial \phi}{\partial r} = -\frac{1}{r^2} F + \frac{1}{r}\overbrace{\frac{\partial F}{\partial r}}^{F'} = \boxed{-\frac{F_0}{r^2}} \sin\frac{2\pi}{\lambda}(r - c_0 t) + \boxed{\frac{2\pi}{r}\frac{F_0}{\lambda}} \cos\frac{2\pi}{\lambda}(r - c_0 t)$$

$$\frac{\cancel{F_0}}{r^{\cancel{2}}} \ll \frac{2\pi}{\cancel{r}}\frac{\cancel{F_0}}{\lambda} \longrightarrow \underbrace{\frac{2\pi}{\lambda}}_{K = \text{wave number}} r \gg 1 \longrightarrow Kr \gg 1$$

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MOC AND BURGERS EQUATIONS

3/04/2020

Exercise 2.1. *Consider the first order partial differential equation*

$$\frac{\partial u}{\partial x} + 2x\left(\frac{\partial u}{\partial y} - \frac{3}{2}x\right) = 0$$

The initial condition for $u(x, y)$ is given by

$$u(0, y) = 5y + 10$$

Using the Method of Characteristics, determine:

(a) *The conservative and quasi-linear form of the PDE*

(b) *The characteristic and compatibility equations*

(c) *The equation for the characteristic passing throught the point (2, 4)*

(d) *The compatibility equation valid along the characteristic passing throught the point (2, 4)*

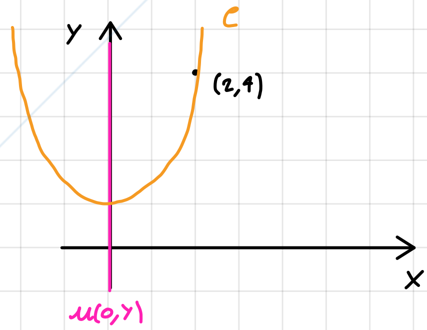
(e) *The value of $u(2, 4)$*

PDE: $u_{/x} + 2x\left(u_{/y} - \frac{3}{2}x\right) = 0$

I.C.: $u(0,y) = 5y + 10 = U_0(y)$

a) Conservative form: $u_{/x} + (2xu)_{/y} - 3x^2 = 0$

Quasi-linear form: $u_{/x} + 2xu_{/y} - 3x^2 = 0$



b) $\boxed{\begin{array}{ll} a f_{/x} + b f_{/y} + c = 0 & \\ \frac{dy}{dx} = \frac{b}{a} & \text{CHARACTERISTIC} \\ \frac{dF}{dx} = -\frac{c}{a} & \text{COMPATIBILITY} \end{array}}$ (where $f = u$, $F = U$)

Characteristic equation: $\frac{dy}{dx} = 2x$ ①

Compatibility equation: $\frac{dU}{dx} = 3x^2$ ②

c) ① $dy = 2x\,dx \xrightarrow{\int} y = x^2 + C \xrightarrow{(2,4)} 4 = 4 + C \rightarrow C = 0$

$y_{C_{(2,4)}} = x^2$ (characteristic passing through (2,4))

d) ② $dU = 3x^2 dx \xrightarrow{\int} U = x^3 + D$

$U(0) = u(0,0) = 10 = D \rightarrow D = 10$

$U_{C(2,4)} = x^3 + 10$

e) $u(2,4) = U_{C(2,4)}(2) = 18$

Exercise 2.2. *Consider the first order partial differential equation*

$$\frac{\partial u}{\partial x} + 2x\frac{\partial u}{\partial y} - 5y^2 = 0$$

Compute the value of the solution at point (2, 4) assuming the initial condition

$$u(0, y) = 5y + 10$$

PDE: $u_{/x} + 2xu_{/y} - 5y^2 = 0$

I.C.: $u(0,y) = 5y + 10$

? $u(2,4)$

Characteristic: $\frac{dy}{dx} = 2x$ ①

Compatibility: $\frac{dU}{dx} = 5y^2$ ②

① $dy = 2x\,dx \xrightarrow{\int} y = x^2 + C \xrightarrow{(2,4)} 4 = 4 + C \rightarrow C = 0$

$y_{C(2,4)} = x^2$

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② $dU = 5y^2 dx = 5x^4 dx \xrightarrow{\int} U = x^5 + D$

$U_0(0) = u(0,0) = 10 = D \rightarrow D = 10$

$U_{e(2,4)} = x^5 + 10$

$u(2,4) = U_{e(2,4)}(2) = 42$

Exercise 2.3. *Consider the first order partial differential equation*

$$\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} - 3y^2 = 0$$

The initial condition for $u(x, y)$ is given by

$$u(0, y) = 5y + 10$$

Compute the expression $u(x, y)$.

PDE: $u_{/x} - 2y u_{/y} - 3y^2 = 0$
I.C.: $u(0,y) = 5y + 10$

? $u(x,y)$

Characteristic equation: $\left\{ \begin{array}{l} \frac{dy}{dx} = -2y \quad ① \\ \end{array} \right.$
Compatibility equation: $\quad \frac{dU}{dx} = 3y^2 \quad ②$

① $\frac{dy}{y} = -2dx \xrightarrow{\int} \ln y = -2x + C_1 \rightarrow \boxed{y = C_1 e^{-2x}}$

② $dU = 3y^2 dx = 3C_1^2 e^{-4x} dx \xrightarrow{\int} \boxed{U = -\frac{3}{4} C_1^2 e^{-4x} + D}$

Imposing I.C. ...

$y(0) = C_1 e^{-2\cdot 0} = C_1$

$U_0(y_0) = u(0, y_0) = 5C_1 + 10 = -\frac{3}{4}C_1^2 + D \rightarrow D = \frac{3}{4}C_1^2 + 5C_1 + 10$

$y_e = C_1 e^{-2x} \rightarrow C_1 = y e^{2x}$

$U = -\frac{3}{4}C_1^2 e^{-4x} + \frac{3}{4}C_1^2 + 5C_1 + 10$

$u(x,y) = -\frac{3}{4}y^2 + \frac{3}{4}y^2 e^{4x} + 5y e^{2x} + 10$

Exercise 2.4. *Consider the first order partial differential equation*

$$\frac{\partial u}{\partial x} + 3x^2\frac{\partial u}{\partial y} + y = 0$$

The initial condition for $u(x, y)$ is given by

$$u(x, 0) = x^2 - 3$$

Compute the value of the solution at point $(2, 9)$.

PDE: $u_{/x} + 3x^2 u_{/y} + y = 0$
I.C.: $u(x,0) = x^2 - 3$

? solution at $(2,9)$

Characteristic equation: $\left\{ \frac{dy}{dx} = 3x^2 \quad ① \right.$

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Compatibility equation : $\left\{ \frac{dU}{dx} = -y \right.$ ②

① $dy = 3x^2 dx \xrightarrow{\int} y = x^3 + C \xrightarrow{(2,9)} 9 = 8 + C \rightarrow C = 1$

$y_{e_{(2,9)}} = x^3 + 1$

② $dU = -y\,dx = -(x^3+1)\,dx \xrightarrow{\int} U = -\frac{x^4}{4} - x + D$

At I.C. $y_0 = 0 \xrightarrow{\text{substituting}} x_0 = \sqrt[3]{y_0 - 1} = -1$

$U_0(x_0) = u(x_0, 0) = u(-1,0) = (-1)^2 - 3 = -\frac{1}{4} + 1 + D \longrightarrow D = -\frac{3}{4} - 2 = -\frac{11}{4}$

$U_{e(2,9)} = -\frac{1}{4}x^4 - x - \frac{11}{4}$

$u(2,9) = U(2) = -\frac{(2)^4}{4} - 2 - \frac{11}{4} = -\frac{35}{4}$

Exercise 3.1. *Consider the following inviscid Burgers's equation*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

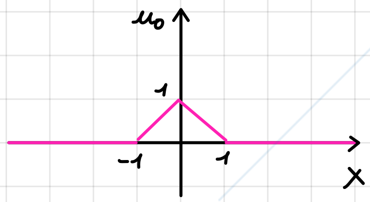
where x is the spatial coordinates and t is the time. For the scalar unknown u, draw the solution at time $t = \frac{1}{2}$, $t = 1$, and $t = 2$. The initial data is:

$$u_0(x_0) = \begin{cases} 0 & x_0 \le -1 \\ x_0 + 1 & -1 < x_0 < 0 \\ 1 - x_0 & 0 \le x_0 \le 1 \\ 0 & x_0 > 1 \end{cases}$$

Burgers eq. : $u_{/t} + u\, u_{/x} = 0$

I.C. : $u_0 = \begin{cases} 0 & x < -1 \\ x+1 & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

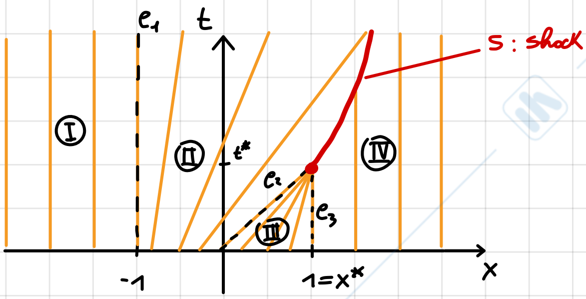
u for $t = \frac{1}{2}, 1, 2$?



Characteristic equation : $\left\{ \frac{dX}{dt} = U \right. \longrightarrow \frac{dX}{dt} = U_0 \rightarrow$ value of the solution at the point where characteristic lines intersect the initial data line

Compatibility equation : $\frac{dU}{dt} = 0 \rightarrow U = \text{const.} = U_0$

①



from compatibility U is constant along every line on the characteristic

① $\boxed{u_I = 0}$

② $u_{II}(x,t) = u_0(x_0) = x_0 + 1$

① $x(t) = u_0(x_0)\,t + x_0 = x_0 t + t + x_0 \rightarrow$

$\rightarrow x_0 = \frac{x-t}{1+t}$

$\boxed{u_{II} = \frac{x-t}{1+t} + 1 = \frac{x+1}{1+t}}$ $\quad -1 < x < t \quad t < t^*$ → time of shock

$-1 < x < x_s \quad t > t^*$ → position of the shock

③ $u_{III}(x,t) = u_0(x_0) = 1 - x_0$

① $x(t) = u_0(x_0)\,t + x_0 = (1-x_0)\,t + x_0 \longrightarrow x_0 = \frac{x-t}{1-t}$

$\boxed{u_{III}(x,t) = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}}$ $\quad t < x < 1 \quad t < t^*$

④ $\boxed{u_{IV} = 0}$ $\quad x > 1 \quad t < t^*$

$x > x_s \quad t > t^*$

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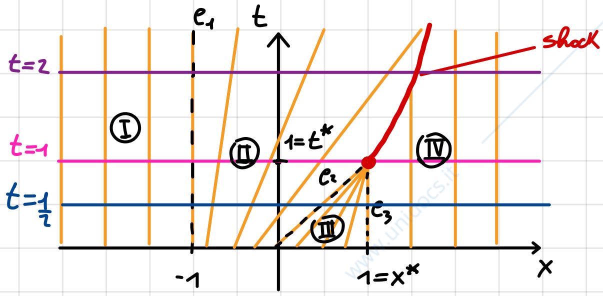
In order to find the different positions we still need to compute t^* and x_s...

$$C_2 : x = \overbrace{u_0(x_0)}^{1}t + \overbrace{x_0}^{0} = t \qquad C_3 : x = \underbrace{u_0(x_0)}_{0}t + \underbrace{x_0}_{1} = 1 \rightarrow x_{C_2}(t^*) = x_{C_3}(t^*) \rightarrow \boxed{t^* = 1} \xrightarrow{\text{Substituting}} x^* = 1$$

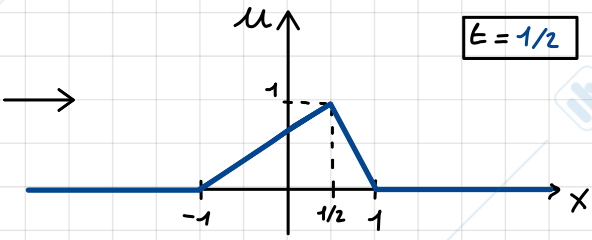
Velocity of the shock : $$S = \frac{f(u_L) + f(u_R)}{u_L - u_R} = \frac{\overset{u_{II}}{u_L} + \overset{u_{IV}}{u_R}}{2} = \frac{x+1}{2(t+1)} = \frac{dx_s}{dt} \longrightarrow \frac{dx_s}{x+1} = \frac{dt}{2(t+1)} \longrightarrow$$

$$\longrightarrow \ln(x_s+1) = \frac{1}{2}\ln(1+t) + \underbrace{C}_{\ln C} \longrightarrow x_s + 1 = C\sqrt{1+t} \xrightarrow{x^*, t^*} C = \sqrt{2}$$

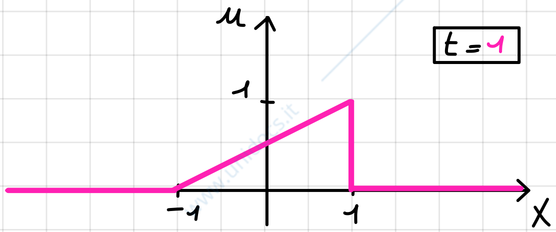
$$x_s + 1 = \sqrt{1+t}\,\sqrt{2} \longrightarrow \boxed{x_s = \sqrt{2}\sqrt{1+t} - 1}$$



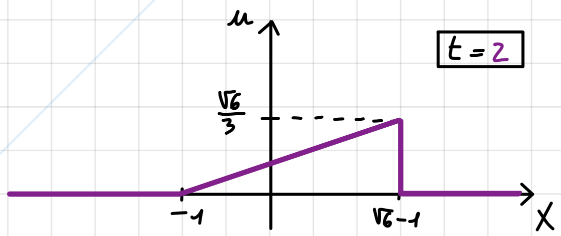
$$C_2 : x(t=\tfrac{1}{2}) = \overbrace{u_0(x_0)}^{1}\cdot\frac{1}{2} + \overbrace{x_0}^{0} = \frac{1}{2} \longrightarrow$$



$$x_s = \sqrt{2}\sqrt{t+1} - 1 = \sqrt{2}\cdot\sqrt{2} - 1 = 1$$
$$u_{II_s} = \frac{x_s+1}{1+t} = \frac{1+1}{1+1} = 1 \longrightarrow$$



$$x_s = \sqrt{2}\sqrt{t+1} - 1 = \sqrt{2}\sqrt{3} - 1 = \sqrt{6} - 1$$
$$u_{II_s} = \frac{x_s+1}{1+t} = \frac{\sqrt{6}-1+1}{1+2} = \frac{\sqrt{6}}{3} \longrightarrow$$



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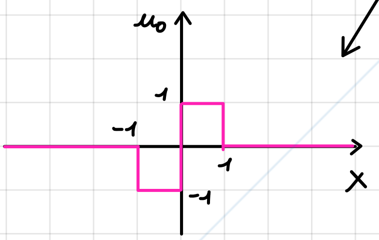
16/04/2020

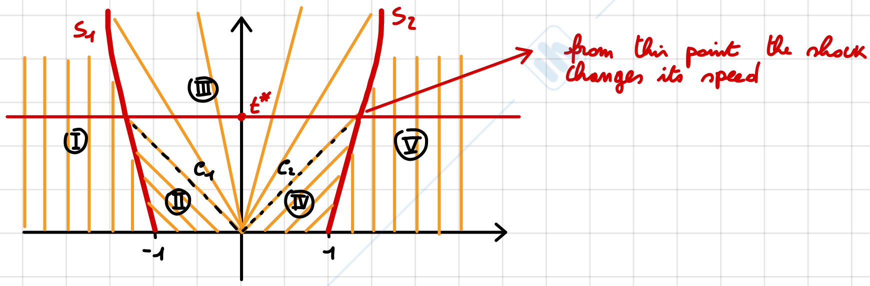
Exercise 3.2. *Consider the following inviscid Burgers's equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

where x is the spatial coordinates and t is the time. For the scalar unknown u, draw the solution at time $t = \frac{1}{2}$, $t = 1$, and $t = 2$. The initial data is:

$$u_0(x_0) = \begin{cases} 0 & x_0 \le -1 \\ -1 & -1 < x_0 < 0 \\ 1 & 0 \le x_0 \le 1 \\ 0 & x_0 > 1 \end{cases}$$





Characteristic equation : $\dfrac{d\chi}{dt} = U \longrightarrow \chi = x_0 + U_0 t$

Compatibility equation : $\dfrac{dU}{dt} = 0 \longrightarrow U = U_0 = \text{const}$

I $u = 0$

II $u = -1$

IV $u = 1$

V $u = 0$

III $u(x,t) = \hat{u}(\eta) \qquad \eta = \dfrac{x}{t} \longrightarrow$ solution in III is a similarity sol. to Burgers's equation

$$\frac{\partial \hat{u}}{\partial \eta}\frac{\partial \eta}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial \eta}\frac{\partial \eta}{\partial x} = 0 \rightarrow -\frac{x}{t^2}\hat{u}' + \frac{1}{t}\hat{u}\hat{u}' = 0 \longrightarrow \hat{u}'\left(-\frac{x}{t^2} + \frac{1}{t}\hat{u}\right) = 0$$

$$\hat{u} = \frac{x}{t} \rightarrow \boxed{u(x,t) = \frac{x}{t}}$$

$$c_2 : \chi(t) = \overbrace{U_0(X_0)}^{1} t + \overbrace{X_0}^{0} = t$$

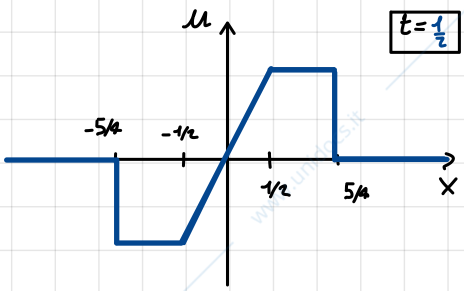
$$c_1 : \chi(t) = \overbrace{U_0(X_0)}^{-1} t + \overbrace{X_0}^{0} = -t$$

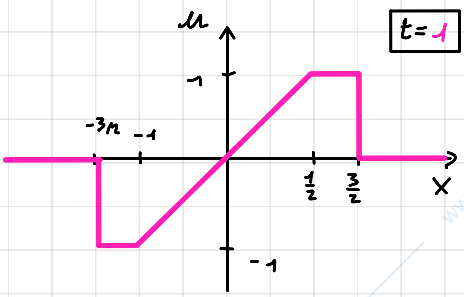
$$S_1 = \frac{u_{II} + u_I}{2} = -\frac{1}{2}, \quad S_2 = \frac{u_{IV} + u_V}{2} = \frac{1}{2}$$

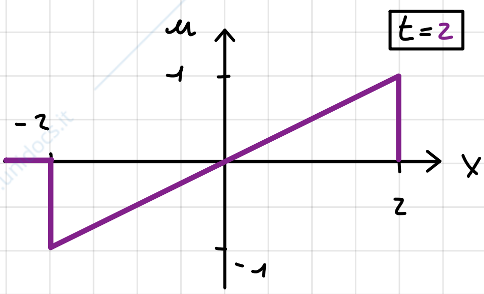
We can find the position of the shock ... $S = \dfrac{dx_s}{dt}$

$$x_{s_1}(t) = -\frac{1}{2}t - 1, \quad x_{s_2}(t) = \frac{1}{2}t + 1 \qquad t \le t^*$$

- $x_{s_1}(t^*) = \chi_{c_1}(t^*) \longrightarrow -\frac{1}{2}t^* - 1 = -t^* \rightarrow t^* = 2 \xrightarrow{\text{substituting}}$
  - $x_1^* = -2$
  - $x_2^* = 2$







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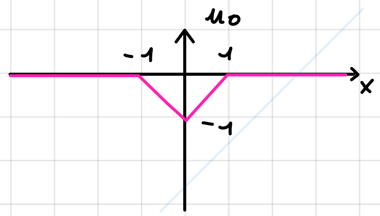
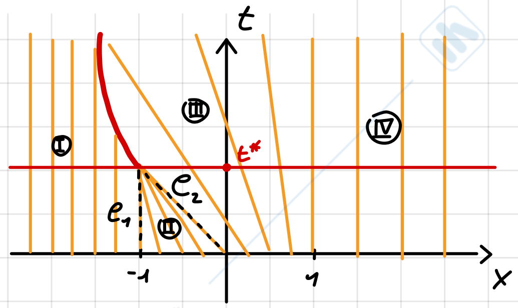
Exercise 3.3. *Consider the following inviscid Burgers's equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

where x is the spatial coordinates and t is the time. For the scalar unknown u, draw the solution at time $t = \frac{1}{2}$, $t = 1$, and $t = 2$. The initial data is:

$$u_0(x_0) = \begin{cases} 0 & x_0 \le -1 \\ -x_0 - 1 & -1 < x_0 < 0 \\ x_0 - 1 & 0 \le x_0 \le 1 \\ 0 & x_0 > 1 \end{cases}$$





Characteristic equation : $\begin{cases} \dfrac{dX}{dt} = U \longrightarrow X(t) = x_0 + U_0 t \\ \end{cases}$

Compatibility equation : $\dfrac{dU}{dt} = 0 \rightarrow U = U_0 = \text{const}$

I $u = 0$

IV $u = 0$

II $u = U_0(X_0) = -X_0 - 1$

$X(t) = U_0(x_0)t + X_0 = -(1+X_0)t + X_0 \longrightarrow X_0 = \dfrac{X+t}{1-t}$

$u = -1 - \dfrac{X+t}{1-t} = \dfrac{X+1}{t-1}$

III $u = U_0(X_0) = X_0 - 1$

$X = (X_0 - 1)t + X_0 \longrightarrow X_0 = \dfrac{X+t}{1+t}$

$u = \dfrac{X-1}{t+1}$

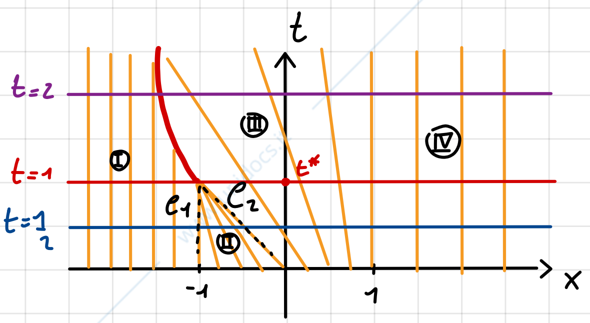
$\left.\begin{array}{l} e_1 : X = -1 \\ e_2 : X = -t \end{array}\right\} \; X_{e_1}(t^*) = X_{e_2}(t^*) \longrightarrow t^* = 1 \xrightarrow{\text{Substituting}} x^* = -1$

$S_1 = \dfrac{u_L + u_R}{2} = \dfrac{u_I + u_{III}}{2} = \dfrac{x-1}{2(t+1)} = \dfrac{dx_s}{dt} \longrightarrow \dfrac{dx_s}{x_s - 1} = \dfrac{dt}{2(t+1)} \xrightarrow{\int} \ln(x_s - 1) = \frac{1}{2}\ln(t+1) + \underset{\sim}{\overset{\ln c}{c}}$

NB occhio a c!

$\longrightarrow x_s - 1 = c\sqrt{1+t} \xrightarrow{x^*, t^*} -2 = c\sqrt{2} \longrightarrow c = -\sqrt{2}$

$x_s - 1 = -\sqrt{2}\sqrt{t+1} \longrightarrow \boxed{x_s(t) = 1 - \sqrt{2}\sqrt{t+1}} \quad t \ge 1$



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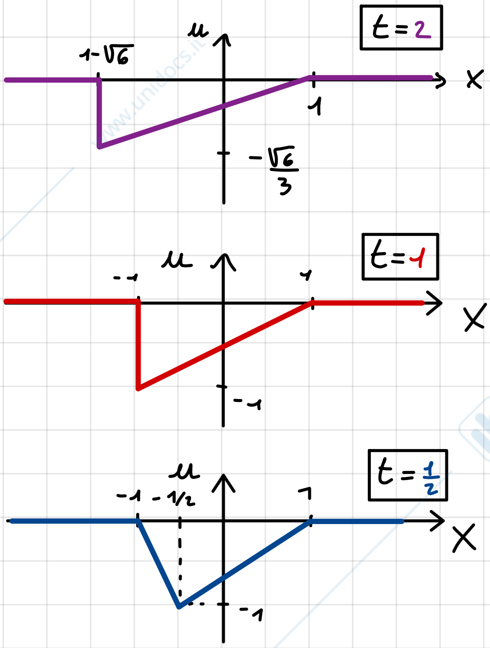
$x_S(t=2) = 1 - \sqrt{6}$

$u_{II}(1-\sqrt{6}, 2) = \frac{1-\sqrt{6}-1}{3} = -\frac{\sqrt{6}}{3}$



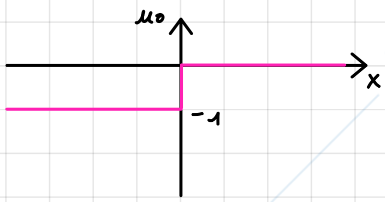
$c_2: x = -t \rightarrow x = -\frac{1}{2} \leftarrow$

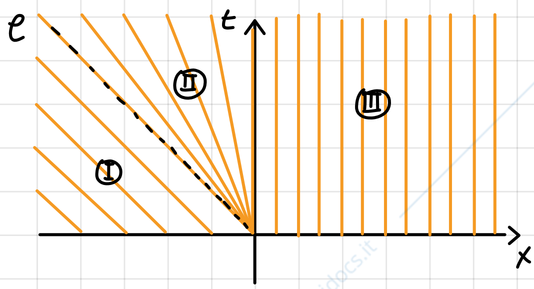
Exercise 3.4. *Consider the following inviscid Burgers's equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

where x is the spatial coordinates and t is the time. For the scalar unknown u, draw the solution at time $t = \frac{1}{2}$, t = 1, and t = 2. The initial data is:

$$u_0(x_0) = \begin{cases} -1 & x_0 < 0 \\ 0 & x \geq 0 \end{cases}$$





I $u = 0$

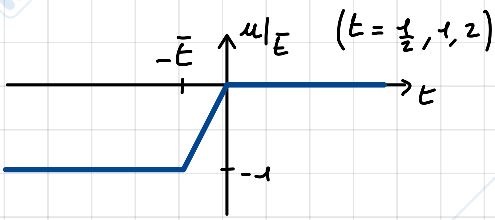
III $u = 0$

II $u(x,t) = \hat{u}(\eta) \qquad \eta = \frac{x}{t} \longrightarrow$ solution in II is a similarity sol. to Burgers's equation

$$\frac{\partial \hat{u}}{\partial \eta}\frac{\partial \eta}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial \eta}\frac{\partial \eta}{\partial x} = 0 \rightarrow -\frac{x}{t^2}\hat{u}' + \frac{1}{t}\hat{u}\hat{u}' = 0 \longrightarrow \hat{u}'\left(-\frac{x}{t^2} + \frac{1}{t}\hat{u}\right) = 0$$

$$\hat{u} = \frac{x}{t} \rightarrow \boxed{u(x,t) = \frac{x}{t}}$$

$c: \ x = -t$



4.1 - 4.3 Leveque

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4.1 Traffic flow

Consider the flow of cars on a highway. Let ρ denote the density of cars (in vehicles per mile, say) and u the velocity. In this application ρ is restricted to a certain range, $0 \le \rho \le \rho_{max}$, where ρ_{max} is the value at which cars are bumper to bumper.

Since cars are conserved, the density and velocity must be related by the continuity equation derived in Section 1,

$$\boxed{\rho_t + (\rho u)_x = 0.} \tag{4.1}$$

In order to obtain a scalar conservation law for ρ alone, we now assume that u is a given function of ρ. This makes sense: on a highway we would optimally like to drive at some speed u_{max} (the speed limit perhaps), but in heavy traffic we slow down, with velocity decreasing as density increases. The simplest model is the linear relation

$$\boxed{u(\rho) = u_{max}(1 - \rho/\rho_{max}).} \tag{4.2}$$

At zero density (empty road) the speed is u_{max}, but decreases to zero as ρ approaches ρ_{max}. Using this in (4.1) gives

$$\boxed{\rho_t + f(\rho)_x = 0} \tag{4.3}$$

where

$$f(\rho) = \rho u_{max}(1 - \rho/\rho_{max}). \tag{4.4}$$

Whitham notes that a good fit to data for the Lincoln tunnel was found by Greenberg in 1959 by

$$f(\rho) = a\rho \log(\rho_{max}/\rho),$$

a function shaped similar to (4.4).

The characteristic speeds for (4.3) with flux (4.4) are

$$f'(\rho) = u_{max}(1 - 2\rho/\rho_{max}), \tag{4.5}$$

while the shock speed for a jump from ρ_l to ρ_r is

$$s = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} = u_{max}(1 - (\rho_l + \rho_r)/\rho_{max}). \tag{4.6}$$

The entropy condition (3.45) says that a propagating shock must satisfy $f'(\rho_l) > f'(\rho_r)$ which requires $\rho_l < \rho_r$. Note this is the opposite inequality as in Burgers' equation since here f is concave rather than convex.

We now consider a few examples of solutions to this equation and their physical interpretation.

EXAMPLE 4.1. Take initial data

$$\rho(x,0) = \begin{cases} \rho_l & x < 0 \\ \rho_r & x > 0 \end{cases} \tag{4.7}$$

where $0 < \rho_l < \rho_r < \rho_{max}$. Then the solution is a shock wave traveling with speed s given by (4.6). Note that although $u(\rho) \ge 0$ the shock speed s can be either positive or negative depending on ρ_l and ρ_r.

Consider the case $\rho_r = \rho_{max}$ and $\rho_l < \rho_{max}$. Then $s < 0$ and the shock propagates to the left. This models the situation in which cars moving at speed $u_l > 0$ unexpectedly encounter a bumper-to-bumper traffic jam and slam on their brakes, instantaneously reducing their velocity to 0 while the density jumps from ρ_l to ρ_{max}. This discontinuity occurs at the shock, and clearly the shock location moves to the left as more cars join the traffic jam. This is illustrated in Figure 4.1, where the vehicle trajectories ("particle paths") are sketched. Note that the distance between vehicles is inversely proportional to density. (In gas dynamics, $1/\rho$ is called the *specific volume*.)

The particle paths should not be confused with the characteristics, which are shown in Figure 4.2 for the case $\rho_l = \frac{1}{2}\rho_{max}$ (so $u_l = \frac{1}{2}u_{max}$), as is the case in Figure 4.1 also. In this case, $f'(\rho_l) = 0$. If $\rho_l > \frac{1}{2}\rho_{max}$ then $f'(\rho_l) < 0$ and all characteristics go to the left, while if $\rho_l < \frac{1}{2}\rho_{max}$ then $f'(\rho_l) > 0$ and characteristics to the left of the shock are rightward going.

EXERCISE 4.1. *Sketch the particle paths and characteristics for a case with $\rho_l + \rho_r < \rho_{max}$.*

Traffic flow equation : $\rho_{/t} + \underbrace{(\rho u)}_{f(\rho)}{}_{/x} = 0$ (conservative form)

$f(\rho) \rightarrow$ flux

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$u(\rho) = \bar{u}\left(1 - \frac{\rho}{\bar{\rho}}\right)$ $\bar{u}$ = maximum velocity , $\bar{\rho}$ = maximum density

The flux is...

$f(\rho) = \bar{u}\,(\rho - \rho^2/\bar{\rho}) \longrightarrow f'(\rho) = \bar{u}\,(1 - 2\,\rho/\bar{\rho})$

The velocity of propagating shocks is...

$S = \frac{f_R - f_L}{\rho_R - \rho_L} = \bar{u}\left(1 - \frac{\rho_L + \rho_R}{\bar{\rho}}\right) \longrightarrow$ velocity of the shock can be >0 or <0

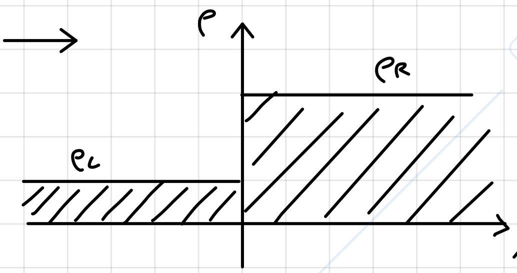
Entropy condition says that a propagating shock must satisfy...

$f'(\rho_L) > f'(\rho_R) \xrightarrow{\text{for traffic eq.}} \rho_L < \rho_R$ (It is the opposite of Burgers equation)

Equation of characteristic curves: $\frac{dX}{dt} = f'(\rho) = \bar{u}\,(1 - 2\,\rho/\bar{\rho})$ (characteristic equation)

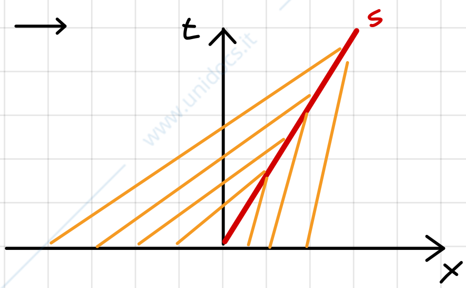
$\frac{dR}{dt} = 0$ (Compatibility equation)

<u>Exercise</u> The initial condition are : $\longrightarrow$

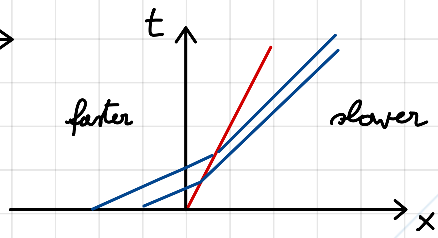


$\rho_L + \rho_R < \bar{\rho}$

The characteristic lines are : $\longrightarrow$



The particle path is : $\longrightarrow$



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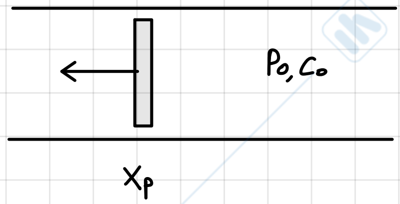
1D UNSTEADY FLOWS - EXPANSION WAVES

23/04/2020

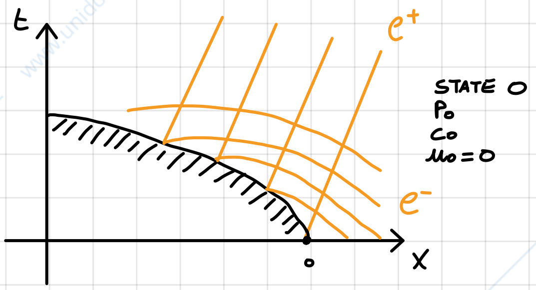
8.7 A very long tube of constant area contains uniform stationary gas at P_0, c_0 and is closed by a piston. Starting at time zero, the piston is withdrawn with displacement $X = -\frac{1}{2}(at^2)$, where a is a constant. Find the normalized pressure P/P_0 on the face of the piston as a function of time. Assume a perfect gas.



$$x_P = -\frac{1}{2} a t^2 \longrightarrow V_P = -at$$

$$\frac{P}{P_0} = \tilde{P}(t) \; ?$$

Let's draw the characteristics ...



$$e^-: \; J^- = J_0^- \rightarrow u - f = u_0 - f_0 \rightarrow u - \frac{2c}{\gamma-1} = -\frac{2c_0}{\gamma-1}$$

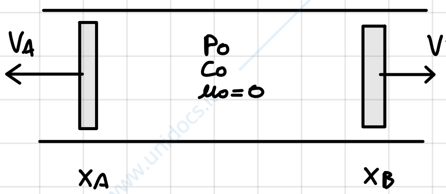
$$c_P = c_0 + \frac{\gamma-1}{2} V_P > 0 \longrightarrow c_0 - \frac{\gamma-1}{2} a t > 0 \rightarrow t < t^* = \frac{2c_0}{a(\gamma-1)}$$

Above this time we have vacuum $\xrightarrow{\text{therefore}}$ $\tilde{P}(t \geq t^*) = 0$

$$c_P = c_0 + \frac{\gamma-1}{2} V_P = c_0 - \frac{\gamma-1}{2} a t$$

From isentropic relations : $\tilde{P}(t<t^*) = \boxed{\frac{P_P}{P_0} = \left(\frac{c_P}{c_0}\right)^{\frac{2\gamma}{\gamma-1}}} = \left(1 - \frac{\gamma-1}{2c_0} a t\right)^{\frac{2\gamma}{\gamma-1}}$

Exercise 4.1. *A tube is filled with air (R = 287 kJ/kgK, γ = 1.4) at rest. In the initial state, c_0 = 300 m/s and P_0 = 2 atm. The tube is closed at both ends by two moving pistons. At time t = 0, the pistons start moving outwards with two different velocities V_{pA} and V_{pB}. The velocity of the piston at the right end of the tube is V_{pB} = 100 m/s. Compute the velocity of the piston at the left end of the tube V_{pA} so that the pressure P_2 in the uniform region after the first wave interaction is P_2 = 1 atm. Draw the flow field in the (x,t) plane.*



$C_0 = 300$ m/s
$P_0 = 2$ atm
$V_B = 100$ m/s
$P_2 = 1$ atm

V_A ?

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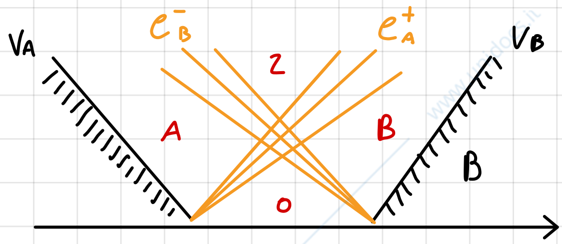
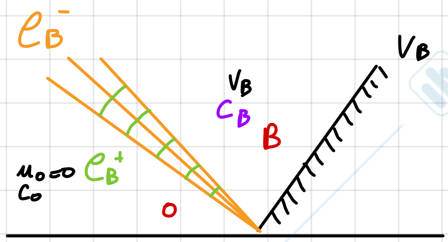


$e^+ : \frac{dx^+}{dt} = u + c$

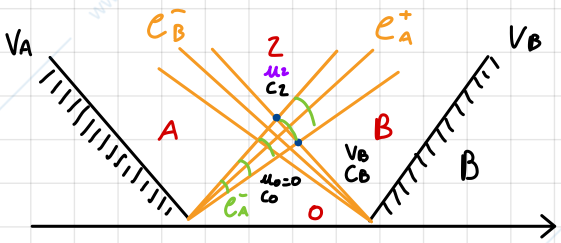
$e^- : \frac{dx^-}{dt} = u - c$



e_B^+: $J_B^+ = J_0^+ \rightarrow u_B + f_B = u_0 + f_0 \rightarrow u_B + \frac{2c_B}{\gamma-1} = \cancel{u_0} + \frac{2c_0}{\gamma-1}$

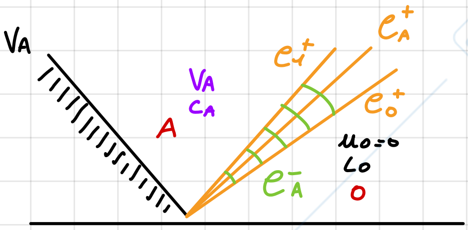
$c_B = c_0 - \frac{\gamma-1}{2}\overset{V_B}{\boxed{u_B}} = 280 \ m/s$

From isentropic relation $\boxed{\frac{p_2}{p_0} = \left(\frac{c_2}{c_0}\right)^{\frac{2\gamma}{\gamma-1}}} \rightarrow c_2 = c_0 \left(\frac{p_2}{p_0}\right)^{\frac{\gamma-1}{2\gamma}} = 271.22 \ m/s$



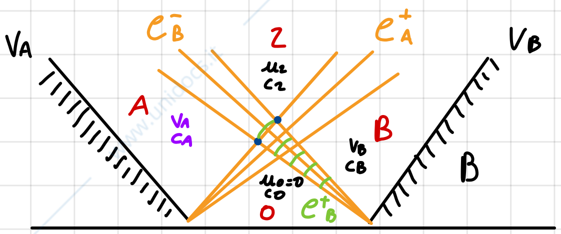
e_A^-: $J_2^- = J_B^- \rightarrow u_2 - f_2 = u_B - f_B \rightarrow u_2 - \frac{2c_2}{\gamma-1} = u_B - \frac{2c_B}{\gamma-1}$

$u_2 = u_B + \frac{2}{\gamma-1}(c_2 - c_B) = 58.98 \ m/s$



e_A^-: $J_A^- = J_0^- \rightarrow u_A - f_A = u_0 - f_0 \rightarrow u_A - \frac{2c_A}{\gamma-1} = \cancel{u_0} - \frac{2c_0}{\gamma-1}$

$c_A = c_0 + \frac{\gamma-1}{2} u_A$



e_B^+: $J_A^+ = J_2^+ \rightarrow u_A + f_A = u_2 + f_2 \rightarrow u_A + \frac{2c_A}{\gamma-1} = u_2 + \frac{2c_2}{\gamma-1}$

$c_A = c_2 + \frac{\gamma-1}{2}(u_2 - u_A)$

So we get: $c_0 + \frac{\gamma-1}{2} u_A = c_2 + \frac{\gamma-1}{2}(u_2 - u_A)$

$$u_A = \frac{1}{\gamma-1}\left(c_2 - c_0 + \frac{\gamma-1}{2} u_2\right) = -42.66 \ m/s$$

We can check the condition:
- $|V_A| < |u_{escape}| = \frac{2c_0}{\gamma-1} = 1500 \ m/s$ ok!
- $|V_B| < |u_{escape}| = \frac{2c_0}{\gamma-1} = 1500 \ m/s$ ok!

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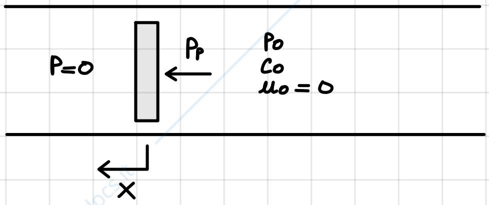
8.4 A lightweight frictionless piston of length L and density ρ is held in a long constant-area tube and released at $t = 0$. The perfect gas within the tube is initially stationary at pressure P_0 with sound speed c_0. The exterior of the tube is at zero pressure. Find the approximate ordinary differential equation for the free-piston velocity $\dot{X} = V(t)$ in terms of the given quantities. If possible, solve the differential equation. Assume that $V < c_0$, and use the binomial expansion if required.

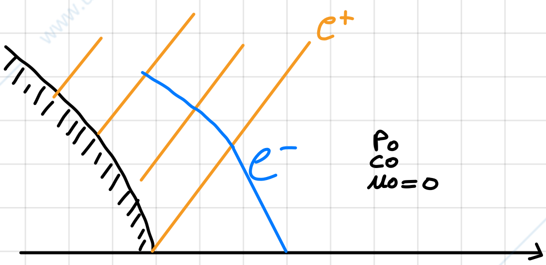
Answer $V \approx \frac{c_0}{\gamma}(1 - e^{-\gamma P_0 t/\rho L c_0})$

Pressure on the piston



Equation of motion. $\ddot{X}\, m = \Sigma F = A P_p$

$\ddot{X}(t) = \frac{A_p}{m} P(t)$



$J^- = J_0^- \rightarrow u - \frac{2c}{\gamma-1} = -\frac{2c_0}{\gamma-1}$

$u_p - \frac{2c_p}{\gamma-1} = -\frac{2c_0}{\gamma-1} \rightarrow c_p = c_0 + \frac{\gamma-1}{2} u_p$

From the isentropic relation. $\boxed{\frac{P_p}{P_0} = \left(\frac{c_p}{c_0}\right)^{\frac{2\gamma}{\gamma-1}}} \longrightarrow P_p = P_0\left(1 + \frac{\gamma-1}{2c_0} \boxed{u_p}\right)^{\frac{2\gamma}{\gamma-1}}$

$\dot{x}(t)$

Substituting in $\ddot{x}(t)$...

$\ddot{x}(t) = \frac{A}{m} P_0 \left(1 + \frac{\gamma-1}{2c_0}\dot{x}(t)\right)^{\frac{2\gamma}{\gamma-1}} \rightarrow$ equation of motion of the piston

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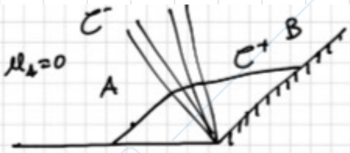

DOWNWARD CONCAVITY

$$\left.\frac{dX^+}{dt}\right|_B > \left.\frac{dX^+}{dt}\right|_A$$

$$\left.\frac{dX^+}{dt}\right|_A = c_A$$

$$C^+: \quad u + \frac{2c}{\gamma-1} = \frac{2c_A}{\gamma-1} \rightarrow c = c_A - \frac{\gamma-1}{2}u$$

$$\left.\frac{dX^+}{dt}\right|_B = u_B + c_B = u_B + c_A - \frac{\gamma-1}{2}u_B$$

$$= c_A + \frac{3-\gamma}{2}u$$

$$\left.\frac{dX^+}{dt}\right|_A < \left.\frac{dX^+}{dt}\right|_B$$

$$\Rightarrow c_A < c_A + \frac{3-\gamma}{2}u_A$$

$$\Rightarrow \boxed{\gamma < 3}$$

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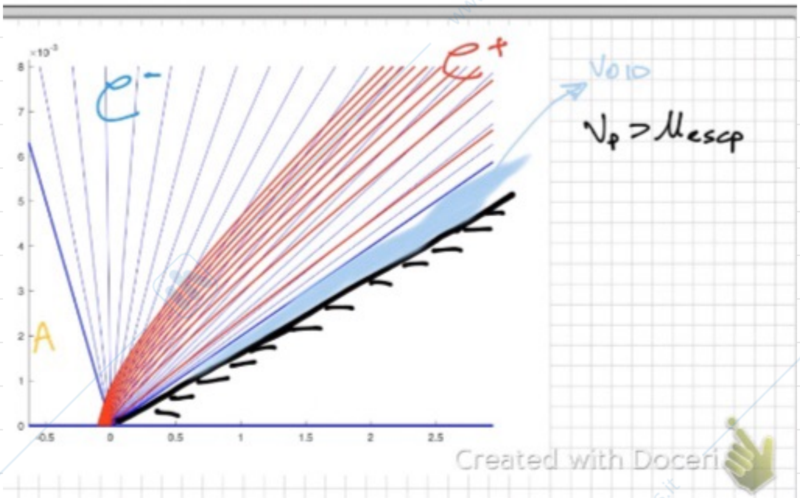

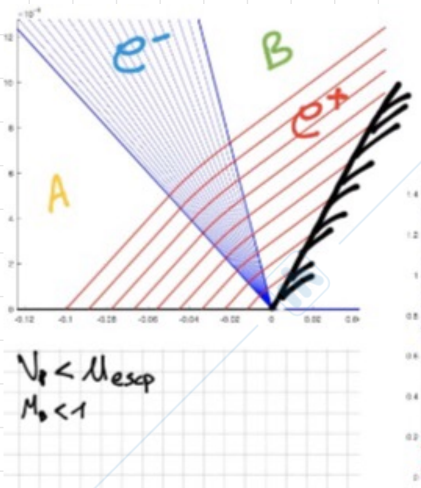

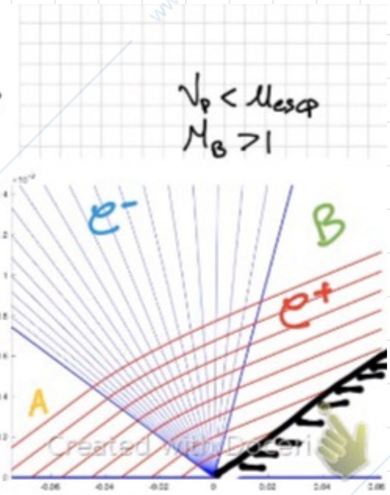

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1D UNSTEADY FLOWS - SHOCK WAVES

30/04/2020

Exercise 5.1 *A tube is filled with air (R = 287 kJ/kgK, γ = 1.4) at rest. In the initial state, ρ_1 = 1.225 kg/m^3 and P_1 = 1 atm. A shock wave moves from left to right, and reaches the right end of the tube at time $t = t^*$. Before the shock reaches the right end of the tube, the pressure downstream of the shock is P_2 = 4.5 atm. The open end of the tube can be:*

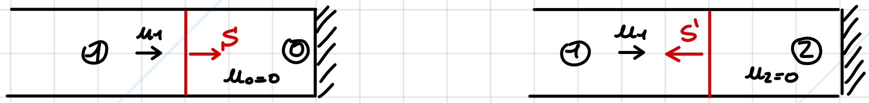
(a) A closed wall

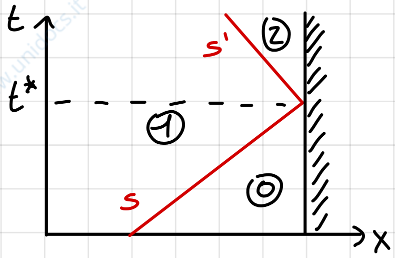
(b) An open end *at constant pressure* P_1

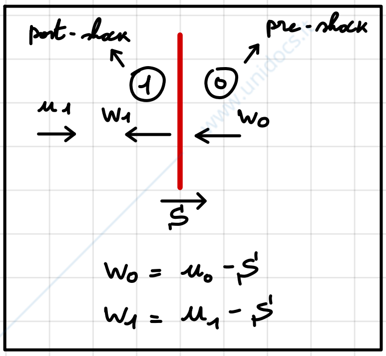
For both configurations, plot the velocity and pressure profiles for time $t_1 < t^*$ and time $t_2 > t^*$.

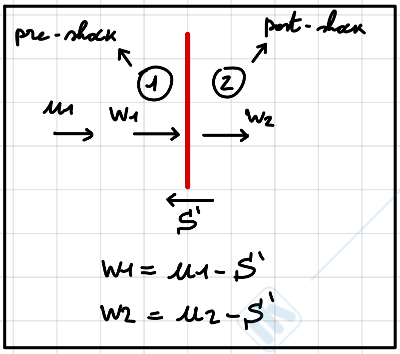
$R = 287$ KJ/kgK
$\gamma = 1,4$
$\rho_0 = 1,225$ kg/m^3
$P_0 = $ atm
$P_1 = 4,5$ atm

a)









S. $u_0 = 0 \longrightarrow W_0 = -S$
$W_1 = u_1 - S$

$$\frac{P_1}{P_0} = 4,5$$

From normal shock tables (entering $\frac{P_1}{P_0}$) : • $M_0 = 2$ (Mach of relative velocity coming into shock)
• $M_1 = 0,5776$ (" going out ")
• $C_1/C_0 = 1,299$

$$C_0 = \sqrt{\gamma R T_0} = \sqrt{\gamma \frac{P_0}{\rho_0}} = 340 \text{ m/s}$$

$$M_0 = \frac{|W_0|}{C_0} \longrightarrow S = |W_0| = M_0 C_0 = 680,6 \text{ m/s}$$

$$C_1 = 1,299\, C_0 = 442 \text{ m/s}$$

$$M_1 = \frac{|W_1|}{C_1} \longrightarrow |W_1| = M_1 C_1 = 225,23 \text{ m/s}$$

Since u_1 and W_1 have opposite direction ...

$$\boxed{u_1} = (-W_1) + S = \boxed{425,35 \text{ m/s}}$$

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s' : $u_2 = 0 \;\rightarrow\; w_1' = u_1 - s'$
$\qquad\qquad\qquad\quad w_2' = -s'$

$$\frac{|[\![w']\!]|}{c_1} = \frac{|\cancel{-s'} - u_1 + \cancel{s'}|}{c_1} = \frac{u_1}{c_1} = 0.9623$$

$w_2' - w_1'$

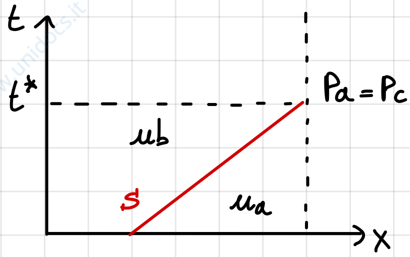
From normal shock tables (entering $\frac{[\![w']\!]}{c_1}$) :
- $M_1 = 1.732$
- $\frac{P_2}{P_1} = 3{,}325$

$$M_1 = \frac{|w_1'|}{c_1} \rightarrow |w_1'| = M_1 c_1 = 765.6 \text{ m/s}$$

$$s' = u_1 - w_1' = 340.25 \text{ m/s} \quad \text{(velocity of the rebounding shock)}$$

$$\boxed{P_2} = \frac{P_2}{P_1} P_1 = \boxed{14{,}99 \text{ atm}}$$

b)



$P_a = P_c \longrightarrow$ we have 3 different outcomes..

$$M_b = \frac{u_1}{c_1} = \frac{425{,}35}{442} = 0.962 < 1$$

We cannot be in case ①! We use the MOC to see if we are in the case of a subsonic or supersonic outflow...

Across the rarefaction fan...

$$C^+ : J_b^+ = J_c^+ \rightarrow u_b + f_b = u_c + f_c$$

$$u_b + \frac{2 c_b}{\gamma - 1} = u_c + \frac{2 c_c}{\gamma - 1}$$

If a <u>subsonic outflow is assumed</u>, $P_c = P_a$. If these conditions are attained, the isentropic expansion from b to c is such that:

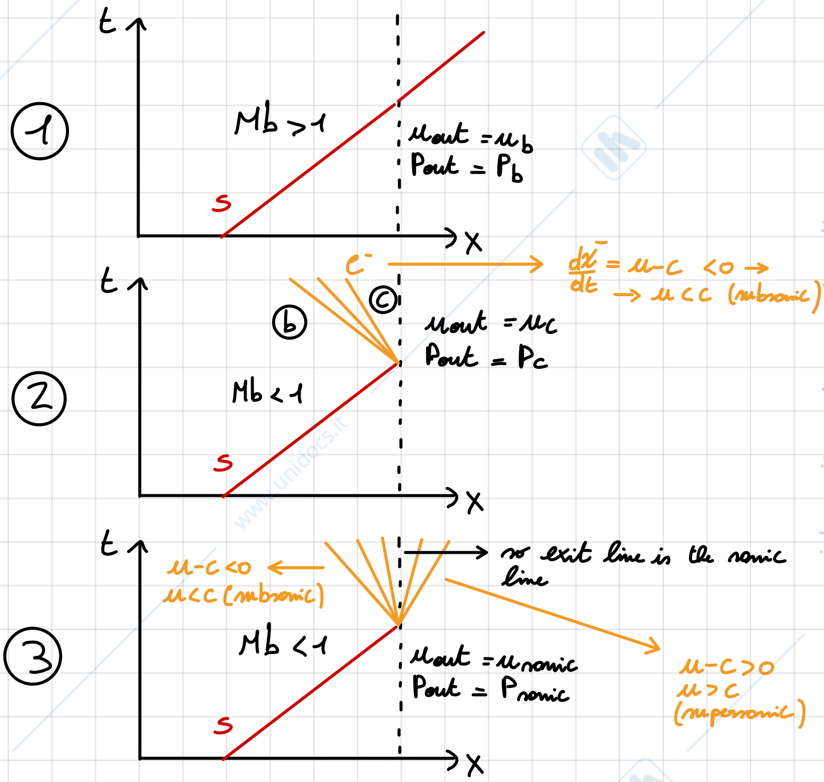
$$\frac{c_c}{c_b} = \left(\frac{P_c}{P_b}\right)^{\frac{\gamma-1}{2\gamma}} = \left(\frac{P_a}{P_b}\right)^{\frac{\gamma-1}{2\gamma}} \rightarrow c_c = 356.6 \text{ m/s}$$

Substituting...

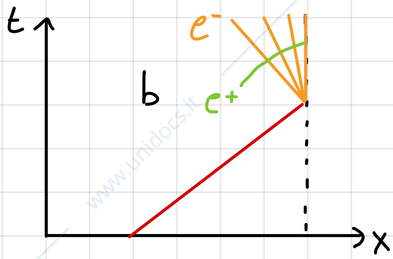
$$u_c = u_b + \frac{2}{\gamma - 1}(c_b - c_c) = 853 \text{ m/s}$$

$$M_c = \frac{u_c}{c_c} = 2.39 \rightarrow \text{it is not subsonic!}$$

So we have a SONIC outflow...



In b the flow is subsonic and then it accelerates till sonic



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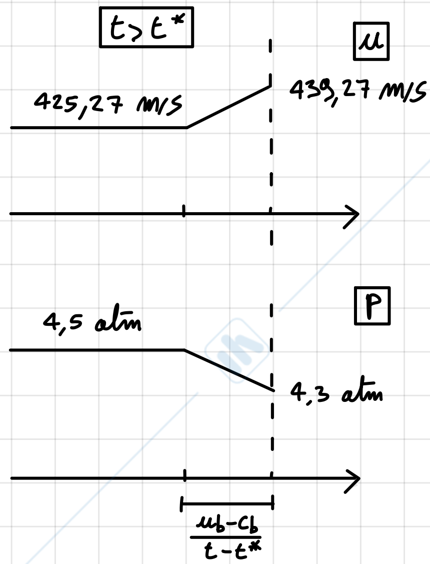
C^-_{out} : $\frac{x}{t} = 0 = u_s - c_s \longrightarrow u_s = c_s$

C^+_{out} : $J^+_s = J^+_b \rightarrow u_b + \frac{2c_b}{\gamma - 1} = u_s + \frac{2c_s}{\gamma-1}$ ($c_s = u_s$)

$\boxed{u_s} = \boxed{439.27 \ m/s}$

$\boxed{P_s} = P_b \left(\frac{c_s}{c_b}\right)^{\frac{2\gamma}{\gamma-1}} = \boxed{4,3 \ atm}$



7/05/2020

② $\gamma = 1,4$

$R = 286,84$

$V_p = u_1 = 104 \ m/s$

$L = 1 \ m$

$P_0 = 1 \ atm$

$T_0 = 288 \ K$

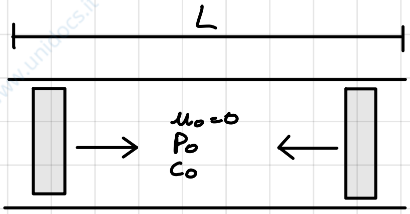
$c_0 = \sqrt{\gamma R T_0} = 348 \ m/s$

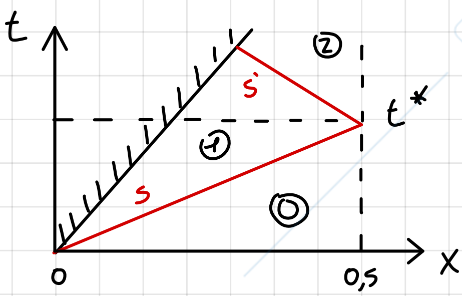
$c(t = 2.1 \ ms)$?

Pistons' velocities are the same

$\downarrow$

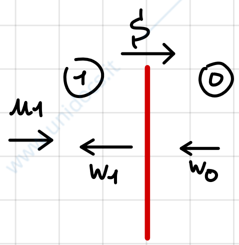
The problem is symmetric ...





We compute S to see if $t \gtrless 2.1$ ms ...

S :



$W_0 = \cancel{u_0} - S = -S$

$W_1 = u_1 - S$

$$\frac{|[\![W]\!]|}{c_0} = \frac{|w_1 - w_0|}{c_0} = \frac{|-\cancel{S} - u_1 + \cancel{S}|}{c_0} = \frac{u_1}{c_0} = 0,3006$$

From normal shock tables (entering $\frac{|W|}{c_0}$) :
- $M_0 = 1,2$
- $c_1/c_0 = 1,062$

$c_1 = \frac{c_1}{c_0} \cdot c_0 = 367,45 \ m/s$

$M_0 = \frac{|W_0|}{c_0} = \frac{S}{c_0} \longrightarrow S = M_0 \cdot c_0 = 416,4 \ m/s$

To compute t^* : $S = \frac{x}{t} \xrightarrow{t^*, x^*} t^* = \frac{L/2}{S} = 1,2 ms < 2.1 \ m/s$

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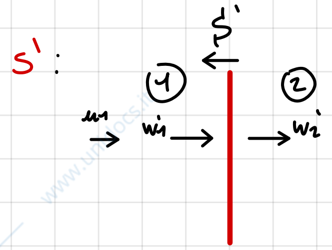
S' :



$w_1' = u_1 - S'$
$w_2' = \cancel{u_2} - S'$

because the 2 pistons have the same velocities !

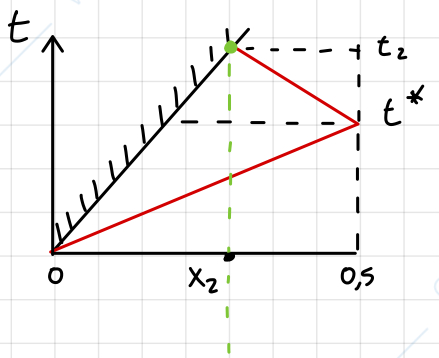
$$\frac{[\![w']\!]}{c_1} = \frac{|w_2' - w_1'|}{c_1} = \frac{|\cancel{-S'} - u_1 + \cancel{S'}|}{c_1} = \frac{u_1}{c_1} = 0.283$$

From normal shock tables (entering $\frac{[\![w']\!]}{c_1}$) : • $M_1' = 1{,}185$
• $c_2/c_1 = 1{,}057$

$$c_2 = \frac{c_2}{c_1} \cdot c_1 = 388{,}39 \ m/s$$

$$M_1' = \frac{|w_1'|}{c_1} \rightarrow |w_1'| = M_1' c_1 = 435.43 \ m/s$$

$$S' = u_1 - |w_1'| = -331.43 \ m/s$$



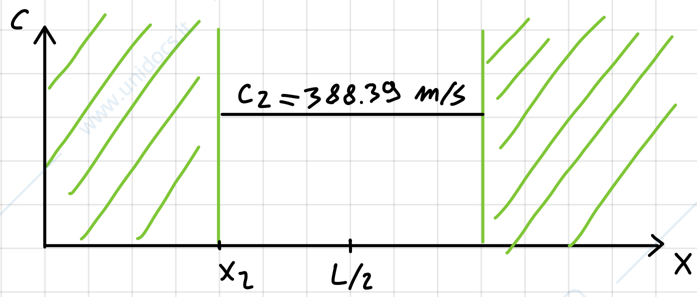
$x_p(t) = V_p t$

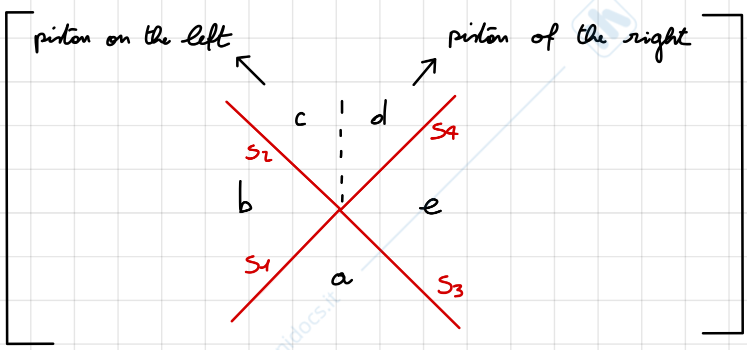
$x_{S'}(t) = \frac{L}{2} + S' \cdot (t - t^*)$

$x_p(t_2) = x_{S'}(t_2) \rightarrow V_p \cdot t_2 = \frac{L}{2} + S'(t_2 - t^*)$

$$t_2 = \frac{L/2 - S' t^*}{V_p - S'} = 0.0021 \ s = 2.1 \ ms \rightarrow x_2 = \dots$$

So we have to draw the solution at t_2 ...





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FANNO, RAYLEIGH AND QUASI 1D FLOWS

FANNO

Exercise 6.1. *Consider the flow of air through a pipe of inside diameter D = 120 mm and length L = 1500 mm. The inlet flow conditions are M_1 = 3, P_1 = 1 bar, and T_1 = 300 K. Assuming f = constant = 0.005, calculate:*

(a) The length of the duct required to choke the flow

(b) The flow conditions at exit: M_2, P_2, T_2, and P_2^t

$D = 120$ mm $\quad T_1 = 300$ K

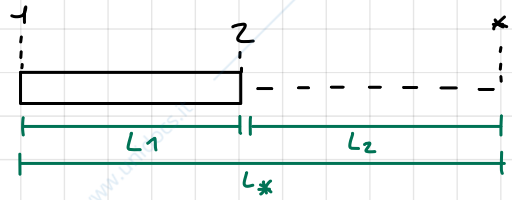
$L_1 = 1500$ mm $\quad f = 0{,}005$

$M_1 = 3$

$P_1 = 1$ bar $\quad L_*, M_2, P_2, T_2, P_2^t$?

From FANNO table (entering with M_1):
- $\frac{4fL_*}{D} = 0{,}5222 \longrightarrow L_* = 3133$ mm

  length at which the flow reach sonic conditions
- $T_1/T_* = \ldots \longrightarrow T_* =$
- $P_1/P_* = \ldots \longrightarrow P_* =$
- $P_1^t/P_*^t = \ldots$



$$\frac{4f}{D}L_* = \frac{4f}{D}(L_1+L_2) \longrightarrow \frac{4f}{D}L_2 = \frac{4f}{D}(L_* - L_1) = 0{,}2722$$

From FANNO table (entering $\frac{4fL_2}{D}$):
- $M_2 = 1{,}9$
- $T_2/T_* = \ldots$
- $P_2/P_* = \ldots$
- $P_2^t/P_*^t = \ldots$

$$T_2 = \frac{T_2}{T_*}\frac{T_*}{T_1}T_1 = 487{,}8 \text{ K}$$

$$P_2 = \frac{P_2}{P_*}\frac{P_*}{P_1}P_1 = 2{,}014 \text{ bar}$$

From isentropic tables (entering M_1): $\frac{P_1}{P_1^t} = 0.0272 \longrightarrow P_1^t = 36.76$ bar

$$P_2^t = \frac{P_2^t}{P_*^t}\frac{P_*^t}{P_1^t}P_1^t = 13{,}49 \text{ bar}$$

RAYLEIGH

Exercise 6.2. *Air enters a constant-area duct at M_1 = 3, P_1 = 1 bar, and T_1 = 300 K. Inside the duct, the heat added per unit mass is $q = 3\cdot10^5$ J/kg. How much heat per unit mass must be added in order to choke the flow?*

$M_1 = 3$ $\qquad$ q such that the flow is chocked (q^*)?

$P_1 = 1$ bar $\qquad$ heat per unit mass

$T_1 = 300$ K

$q = 3\cdot10^5$ J/kg

$$\boxed{q^* = c_p\left(T_*^t - T_1^t\right)}$$

From isentropic relations: $T_1^t = T_1\left(1 + \frac{\gamma-1}{2}M_1^2\right) = 840$ K

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From RAYLEIGH table (entering M_1) : • $\frac{T_1^t}{T_*^t} = 0{,}654$

$T_*^t = \frac{T_*^t}{T_1^t} \cdot T_1^t = 1284 \text{ K}$

$q^* = 4{,}46 \cdot 10^5 \text{ J/Kg}$ $\longrightarrow$ So the heat we provide is not sufficient to choke the flow

QUASI 1D (NOZZLE)

Exercise 6.3. *A conic nozzle is fed by a reservoir containing air (R = 287 J/kgK, γ = 1.4) at P_t = 7 bar and T_t = 500 K. The nozzle is designed to operate at an altitude of 25 000 m, where $P_{amb,d}$ = 26 mbar. The semi-aperture of the diverging portion is ϑ = 15°. The length of the converging portion l_c is $l_c = 0.75 l_d$, where l_d is the length of the diverging portion, and the inlet area A_i is $A_i = 0.75 A_e$, where A_e is the exhaust area. The nozzle discharges a mass flow rate $\dot{m}$ = 9.54 kg/s. Determine:*

(a) *The length of the diverging portion*

(b) *The flow velocity at the exit section when the nozzle operates at sea level ($P_{amb,sea}$ = 1 bar)*

(c) *Plot the $\rho(x)$, $P(x)$, and $M(x)$ profiles corresponding to the limiting solutions, to the solution computed at point (b), and to the solutions computed for ambient pressures P_{amb} = 6.995 bar, P_{amb} = 5 bar, and P_{amb} = 3 bar*

(d) *Plot the flux function $f(\rho)$ and the area function $A(\rho)$ in adapted conditions*

$P^t = 7$ bar

$T^t = 500$ K

$P_{amb,d} = 26$ mbar

$\theta = 15°$

$L_c = 0{,}75\, L_D$

$A_{in} = 0{,}75\, A_e$

$\dot{m} = 9{,}54$ Kg/s

a) We have to find L_D in design conditions

$\beta = \frac{P_{amb,d}}{P^t} = 0{,}0037143$ (design ambient to total pressure ratio)

From isoentropic tables (entering β) :
- $M_e = 4{,}4423$
- $T_e/T^t = 0{,}2022$
- $A_e/A_* = 15{,}7694$

exit section

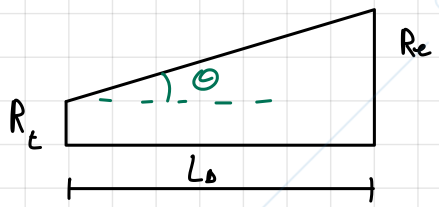
$T_e = \frac{T_e}{T^t} \cdot T^t = 101{,}08$ K

$c_e = \sqrt{\gamma R T_e} = 201{,}53$ m/s

$\rho_e = \frac{\overset{P_{amb,d}}{P_e}}{R T_e} = 0{,}0896 \text{ Kg/m}^3$

$\dot{m} = \rho_e u_e A_e = \rho_e M_e c_e A_e \longrightarrow A_e = 0{,}1189 \text{ m}^2$

$\overset{\text{throat}}{A_t} = A_* = \frac{A_*}{A_e} A_e = 0{,}00754 \text{ m}^2$



$R_e = \sqrt{\frac{A_e}{\pi}} = 0{,}1945$ m

$R_t = \sqrt{\frac{A_t}{\pi}} = 0{,}04899$ m

$L_D = \frac{R_e - R_t}{\tan\theta} = 543$ mm

14/05/2020

b) $P_{amb,sea} = \boxed{1 \text{ bar}}$

We have to compute the limiting solutions in order to identify the functioning regime corresponding to this operating condition...

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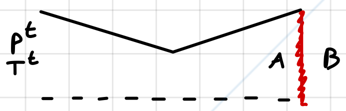
- SUBSONIC - SONIC - SUBSONIC

From isoentropic tables (entering $\frac{A_e}{A_t}$) :
- $M_e = 0{,}0367$
- $P_e / P_t = 0{,}9991$

$$P_e = \frac{P_e}{P_t} \cdot P_t = \boxed{6{,}9933 \text{ bar}}$$

- SUBSONIC - SONIC - SUPERSONIC - SHOCK WAVE (exit)



$P_A^t = P^t$, $M_A = M_{e,d} = 4.4423$

From normal shock tables (entering M_A) :
- $M_B = 0{,}4247$
- $\frac{P_B^t}{P_A^t} = 0{,}0961$

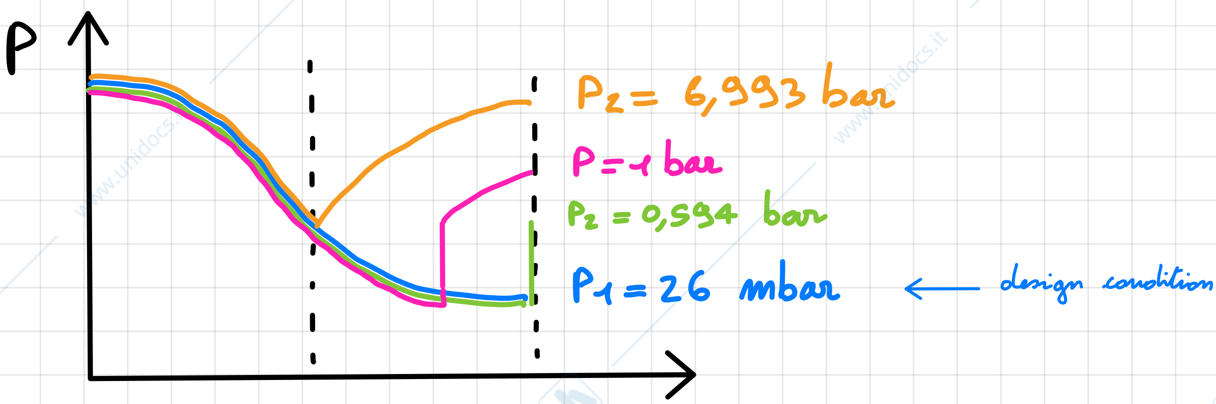
$$\frac{P_B}{P_B^t} = \left(1 + \frac{\gamma-1}{2} M_B^2\right)^{-\frac{\gamma}{\gamma-1}} = 0{,}8834$$

$$P_B = \frac{P_B}{P_B^t} \frac{P_B^t}{P_A^t} P^t = \boxed{0{,}59426}$$

- ADAPTED FLOW (see point a)

$M_e = M_{e,d} = 4.4423$

$P_e = P_{amb,d} = \boxed{26 \text{ mbar}}$



From the formulas for the shock in the divergent section ...

$$\begin{cases} \beta = \frac{P^t}{P_{amb,\,da}} \frac{A_t}{A_e} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} = 0.257 \\ M_e^2 = -\frac{1-\sqrt{1+2\beta^2(\gamma-1)}}{\gamma-1} = 0.06514 \longrightarrow M_e = 0.255 \end{cases}$$

From isoentropic tables (entering M_e) : $\frac{T_e}{T_e^t} = \dots$

T^t (constant through shock)

$$T_e = \frac{T_e}{T_e^t} \cdot \boxed{T_e^t} = 493{,}57 \text{ K}$$

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$c_e = \sqrt{\gamma R T_e} = 445{,}33 \ m/s$

$u_e = M_e \ c_e = 113{,}66 \ m/s$

b.2) If we want to compute the position of the shock (in the divergent)

From isentropic tables (entering M_e) : $\frac{P_e}{P_e^t} = \dots$

$P_e^t = \frac{P_e^t}{P_e} \ P_e^{\nearrow 1\,bar} = \dots$

$\frac{P_e^t}{P_A^t} = \frac{P_e^t}{P^t} = \dots$

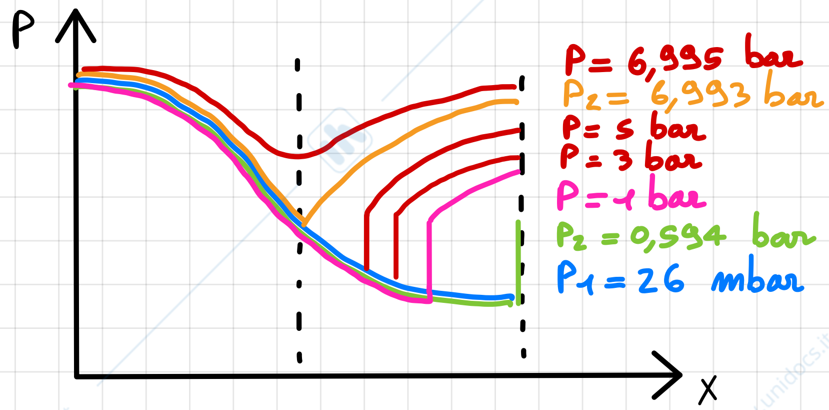
From normal-shock tables (entering $\frac{P_B^t}{P^t}$) : $M_A = \dots$

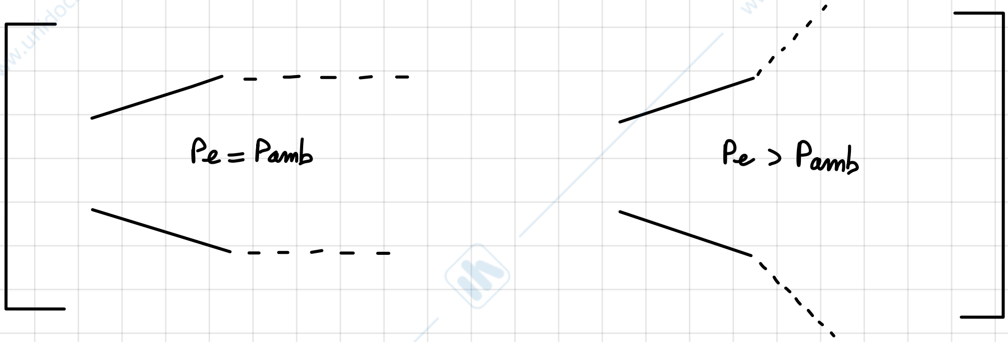
From isentropic tables (entering M_A) : $\frac{A_A}{A_A^*} = \frac{A_A}{A_t} = \dots$

$A_A = \frac{A_A}{A_t} \cdot A_t = \dots \longrightarrow R_A = \sqrt{\frac{A_A}{\pi}}$

$x_s = x_t + \frac{R_A - R_t}{\tan\theta}$

c)





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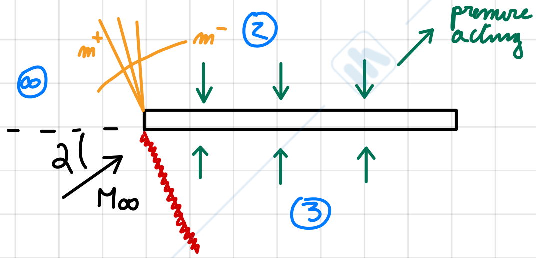
STEADY 2D SUPERSONIC FLOWS

9.7 Find the *lift coefficient* $C_L \equiv 2F_L/\rho_\infty u_\infty{}^2 L$ for a flat plate airfoil with $M_\infty = 2$, $\alpha = 8°$. The fluid is a perfect gas with $\gamma = 1.40$. Compare with the results from linear theory.



$M_\infty = 2$
$\alpha = 8°$
$\gamma = 1,4$

$C_L = \dfrac{2L}{\rho_\infty U_\infty^2 \ell}$?

a) ∞ From isentropic tables (entering M_∞):
- $w_\infty = 26,38$
- $\dfrac{P_\infty}{P_\infty^t} = 0.1278$

2 : m^- : $\underset{\alpha}{\boxed{\theta_\infty}} + w_\infty = \underset{0°}{\boxed{\theta_2}} + w_2$

(m^-: $\theta + w$ slope $\theta - \mu$
m^+: $\theta - w$ slope $\theta + \mu$)

$w_2 = \alpha + w_\infty = 34,38$

From isentropic tables (entering w_2):
- $M_2 = 2,3$
- $\dfrac{P_2}{P^t} = 0,08$

$$\frac{P_2}{P_\infty} = \frac{P_2}{P^t}\frac{P^t}{P_\infty} = 0.625978$$

3 : From oblique shock tables (entering M_∞, α):
- $\beta = 37,21°$
- $M_3 = 1,71$
- $\dfrac{P_3}{P_\infty} = 1,54$

$$\boxed{L = F\cos\alpha} = \Delta P\,\ell\,\cos\alpha = (P_3 - P_2)\,\ell\,\cos\alpha$$

$$C_L = \frac{2(P_3 - P_2)\ell\cos\alpha}{\rho_\infty U_\infty^2 \ell\, \frac{c_\infty^2}{c_\infty^2}} = \frac{2(P_3 - P_2)\cos\alpha}{\rho_\infty M_\infty^2 \gamma R T_\infty} = \frac{2(P_3 - P_2)\cos\alpha}{\gamma M_\infty^2 P_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{P_3}{P_\infty} - \frac{P_2}{P_\infty}\right)\cos\alpha = 0,3233$$

b) From linear theory: $\boxed{C_L = \dfrac{4\alpha}{\sqrt{M_\infty^2 - 1}}} = 0,3225$ (very close!)

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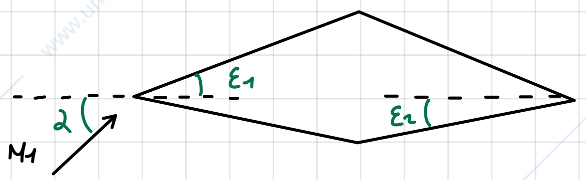
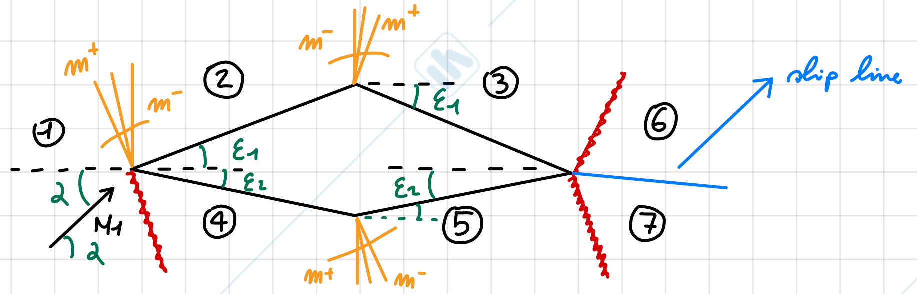
2 Flow field around diamond airfoil

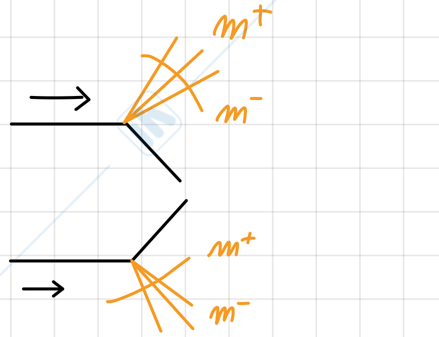
21/05/2020



$\varepsilon_1 = 10°$
$\varepsilon_2 = 6°$
$\lambda = 15°$
$M_1 = 3$

$C_L, C_D, C_{m_{LE}}$? (linear and non linear theory)





1 From isentropic tables (entering M_1):
- $P_1/P^t = 0{,}0272$
- $w_1 = 49{,}76$

2 m^-: $\overset{\lambda}{\theta_1} + w_1 = \overset{\varepsilon_1}{\theta_2} + w_2 \longrightarrow w_2 = w_1 + (\overset{\lambda}{\theta_1} - \overset{\varepsilon_1}{\theta_2}) = 49{,}76 + (15 - 10) = 54{,}76°$

From isentropic tables (entering w_2):
- $M_2 = 3{,}27$
- $P_2/P^t = 0{,}0183$

3 m^-: $\overset{\varepsilon_1}{\theta_2} + w_2 = \overset{-\varepsilon_1}{\theta_3} + w_3 \longrightarrow w_3 = w_2 + (\overset{\varepsilon_1}{\theta_2} - \overset{-\varepsilon_1}{\theta_3}) = 54{,}76° + (10° - (-10°)) = 74{,}76°$

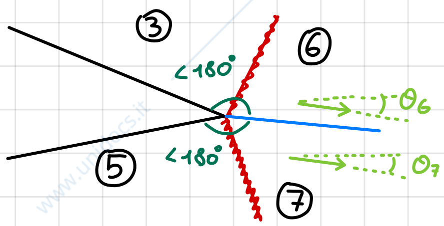
From isentropic tables (entering w_3):
- $M_3 = 4{,}78$
- $P_3/P^t = 0{,}00246$

4 From oblique shock tables (entering $\overset{3}{\boxed{M_1}}$, $\overset{21}{\boxed{\lambda + \varepsilon_2}}$):
- $\beta = 38{,}97°$
- $P_4/P_1 = 3{,}969$
- $M_4 = 1{,}94$

From isentropic tables (entering M_4):
- $P_4/P_4^t = 0{,}1403$
- $w_4 = 24{,}71°$

5 m^+: $\overset{-\varepsilon_2}{\theta_4} - w_4 = \overset{\varepsilon_2}{\theta_5} - w_5 \longrightarrow w_5 = w_4 + (\overset{6}{\theta_5} - \overset{-6}{\theta_4}) = 24{,}71 + (6+6) = 36{,}71°$

From isentropic tables (entering w_5):
- $M_5 = 2{,}4$
- $P_5/P_4^t = 0{,}0684$



$\longrightarrow$ We can use an iterative procedure knowing that P is constant along the slip line and that velocity direction is the same

$P_6 = P_7$

$\theta_6 = \theta_7 = \theta$

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Hp: we start the iteration considering 2 shocks at the TE which means having angles $< 180°$ $(-\varepsilon_1 \leq \tilde{\theta} \leq \varepsilon_2)$

$\uparrow$ ($\tilde{\theta}$: tentative value)

<u>Procedure</u>: We guess an angle $\tilde{\theta} = \theta_6 = \theta_7 \dots$

⑥ From oblique shock tables (entering M_3, $\underbrace{\boxed{\varepsilon_1 + \tilde{\theta}}}_{\beta_1}$): • $\tilde{M}_6$ • $\tilde{P}_6/P_3 \rightarrow \tilde{P}_6/P_1$

⑦ From oblique shock tables (entering M_5, $\underbrace{\boxed{\varepsilon_2 - \tilde{\theta}}}_{\beta_2}$): • $\tilde{M}_7$ • $\tilde{P}_7/P_5 \rightarrow \tilde{P}_7/P_1$

Check $\tilde{P}_6$ and $\tilde{P}_7$ and stop when $\tilde{P}_6 = \tilde{P}_7$

$\boxed{\tilde{\theta} = 0°}$

⑥: From oblique shock tables (entering $M_3 = 4.78$, $\beta_1 = 10°$): • $\frac{\tilde{P}_6}{P_3} = 2.918$

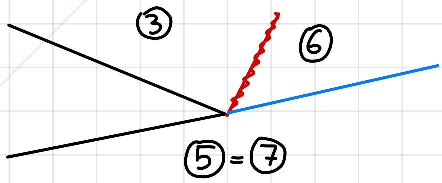
$$\frac{\tilde{P}_6}{P_1} = \frac{\tilde{P}_6}{P_3}\frac{P_3}{P_3^t}\frac{P_3^t}{P_1} = 0.2639$$

⑦ From oblique shock tables (entering $M_5 = 2.4$, $\beta_2 = 6°$): • $\frac{\tilde{P}_7}{P_5} = 1.4504$

• $$\frac{\tilde{P}_7}{P_1} = \frac{\tilde{P}_7}{P_5}\frac{P_5}{P_5^t}\frac{P_4^t}{P_4}\frac{P_4}{P_1} = 2.8207$$

So $\tilde{P}_7 > \tilde{P}_6 \longrightarrow$ increase $\tilde{\theta}$

$\boxed{\tilde{\theta} = 6°}$



⑥: From oblique shock tables (entering $M_3 = 4.78$, $\beta_1 = 16°$): • $\frac{\tilde{P}_6}{P_3} = 4.903$

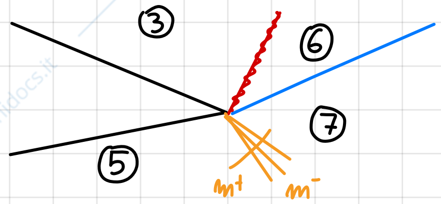
$$\frac{\tilde{P}_6}{P_1} = \frac{\tilde{P}_6}{P_3}\frac{P_3}{P_3^t}\frac{P_3^t}{P_1} = 0.443$$

⑦: $\frac{\tilde{P}_7}{P_5} = 1$

$$\frac{\tilde{P}_7}{P_1} = \frac{\tilde{P}_7}{P_5}\frac{P_5}{P_5^t}\frac{P_4^t}{P_4}\frac{P_4}{P_1} = 1.9447$$

So $\tilde{P}_7 > \tilde{P}_6 \longrightarrow$ increase $\tilde{\theta}$

$\boxed{\tilde{\theta} = 10°}$



⑥. From oblique shock tables (entering $M_3 = 4.78$, $\beta_1 = 20°$): $\frac{\tilde{P}_6}{P_3} = 6.603$

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$$\frac{\tilde{P_6}}{P_1} = \frac{\tilde{P_6}}{P_3}\frac{P_3}{P_3^t}\frac{P_3^t}{P_1} = 0.597$$

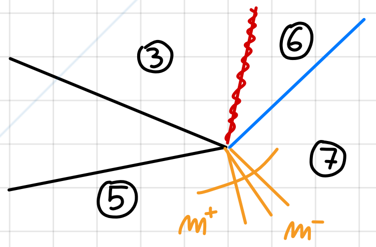
⑦ : m^+ : $\overset{\varepsilon_2}{\theta_5} - w_5 = \overset{\tilde{\theta}}{\theta_7} - w_7 \longrightarrow w_7 = w_5 + (\theta_7 - \theta_5) = 36.71 + (10-6) = 40{,}71°$

From isoentropic tables (entering w_7) : $\frac{P_7}{P_7^t} = 0.0526$

$$\frac{\tilde{P_7}}{P_1} = \frac{\tilde{P_7}}{P_7^t}\frac{P_5^t}{P_4}\frac{P_4}{P_1} = 1.4955$$

So $\tilde{P_7} > \tilde{P_6} \longrightarrow$ increase $\tilde{\theta}$

$\boxed{\tilde{\theta} = 20°}$



⑥ . From oblique shock tables (entering $M_3 = 4.78$, $\beta_1 = 30°$) : $\frac{\tilde{P_6}}{P_3} = 12.15$

$$\frac{\tilde{P_6}}{P_1} = \frac{\tilde{P_6}}{P_3}\frac{P_3}{P_3^t}\frac{P_3^t}{P_1} = 1.0389$$

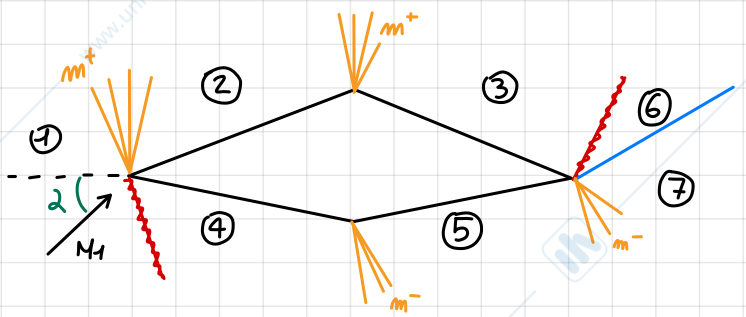
⑦ : m^+ : $\overset{\varepsilon_2}{\theta_5} - w_5 = \overset{\tilde{\theta}}{\theta_7} - w_7 \longrightarrow w_7 = w_5 + (\theta_7 - \theta_5) = 50.71°$

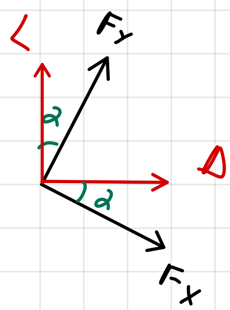
From isoentropic tables (entering w_7) : $\frac{P_7}{P_7^t} = 0.0253$

$$\frac{\tilde{P_7}}{P_1} = \frac{\tilde{P_7}}{P_7^t}\frac{P_5^t}{P_4}\frac{P_4}{P_1} = 0.7193$$

So $\tilde{P_7} < \tilde{P_6} \longrightarrow$ decrease $\tilde{\theta} \longrightarrow$ $\boxed{10° < \theta_S < 20°}$ θ slip line

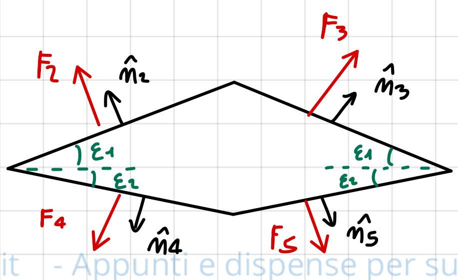
So our situation is...





<u>Non linear theory</u> : $L = F_Y \cos\alpha - F_X \sin\alpha \longrightarrow C_L = \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 \, c \cdot c^2} \overset{c^2 = \gamma RT}{=} \frac{2L}{\gamma P_\infty M_\infty^2 c}$ (c → chord)

$D = F_X \cos\alpha + F_Y \sin\alpha \longrightarrow C_D = \frac{2D}{\gamma P_\infty M_\infty^2 c}$



$$\ell_+ = \frac{c/2}{\cos\varepsilon_1} \qquad \ell_- = \frac{c/2}{\cos\varepsilon_2}$$

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$$F_2 = -P_2\, \ell_+\, \hat{m}_2 = -\frac{P_2}{P_t}\frac{P_t}{P_1} P_1\, \ell_+\, \overset{\begin{Bmatrix} -\sin\varepsilon_1 \\ \cos\varepsilon_1 \end{Bmatrix}}{\hat{m}_2} = c P_1 \begin{bmatrix} 0{,}0593 \\ -0{,}3365 \end{bmatrix} \begin{matrix} \to \hat{x} \\ \to \hat{y} \end{matrix}$$

$$F_3 = -P_3\, \ell_+\, \hat{m}_3 = -\frac{P_3}{P_t}\frac{P_t}{P_1} P_1\, \ell_+\, \hat{m}_3 = c P_1 \begin{bmatrix} -0{,}008 \\ -0{,}0452 \end{bmatrix} \begin{matrix} \to \hat{x} \\ \to \hat{y} \end{matrix}$$

$$F_4 = -\frac{P_4}{P_1} P_1\, \ell_-\, \hat{m}_4 = c P_1 \begin{bmatrix} 0{,}2096 \\ 1{,}9945 \end{bmatrix} \begin{matrix} \to \hat{x} \\ \to \hat{y} \end{matrix}$$

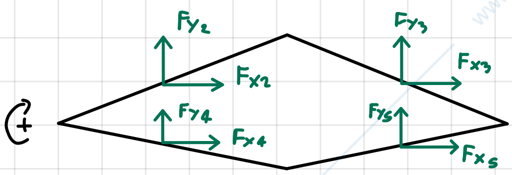
$$F_5 = -\frac{P_5}{P_5^t}\frac{P_4^t}{P_4}\frac{P_4}{P_1} P_1\, \ell_-\, \hat{m}_5 = c P_1 \begin{bmatrix} -0{,}1022 \\ 0{,}9925 \end{bmatrix} \begin{matrix} \to \hat{x} \\ \to \hat{y} \end{matrix}$$

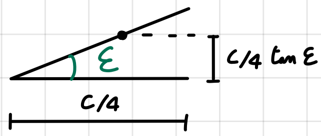
$$L = \sum_i F_{iy} \cos\alpha - \sum_i F_{ix} \sin\alpha$$

$$C_L = \frac{2L}{\gamma P_1 M^2 c} = \boxed{0.3899}$$

$$D = \sum_i F_{ix} \cos\alpha - \sum_i F_{iy} \sin\alpha$$

$$C_D = \frac{2D}{\gamma P_\infty M_\infty^2 c} = \boxed{0.1305}$$

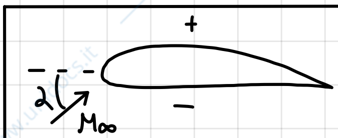




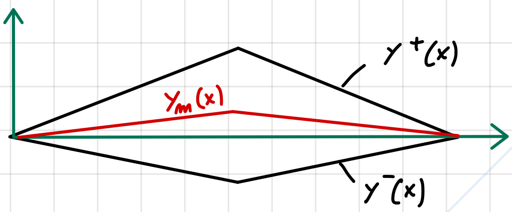
$$C_{m_{LE}} = \frac{M}{\frac{1}{2}\rho_\infty V_\infty^2 c^2} = \frac{2}{P_\infty \gamma M_\infty^2 c^2}\Big[-\frac{c}{4}(F_{y_2} + F_{y_4}) - \frac{3}{4}c\,(F_{3y} + F_{5y}) +$$
$$\frac{c}{4}\tan\varepsilon_1\,(F_{x_2} + F_{x_3}) - \frac{c}{4}\tan\varepsilon_2\,(F_{x_4} + F_{x_5})\Big] = \boxed{-0.176}$$

Linear theory: $c_p = -2\phi_x \to c_p^\pm = \pm \frac{2\theta^\pm}{\sqrt{M_\infty^2 - 1}}$

angle of the flow



$$\theta^+ = \frac{dy^+}{dx} - \alpha$$
$$\theta^- = \frac{dy^-}{dx} - \alpha$$



$$y^+ = y_m + h$$
$$y^- = y_m - h$$

$$c_L = \int_0^c c_p^- \cancel{\cos\theta^-} - c_p^+ \cancel{\cos\theta^+}\, \frac{dx}{c} = \frac{2}{\sqrt{M_\infty^2 - 1}} \int_0^c \left(\left(-\frac{dy^-}{dx} + \alpha - \frac{dy^+}{dx} + \alpha\right)\frac{dx}{c}\right.$$

$$\boxed{C_L} = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} - \frac{2}{\beta c}\underbrace{\int_0^c \underbrace{\frac{dy^-}{dx} + \frac{dy^+}{dx}}_{y_m - \cancel{h} + y_m + \cancel{h} = 2y_m}}_{y_m(c) - y_m(0) = 0} = \boxed{\frac{4\alpha}{\sqrt{M_\infty^2 - 1}}} = \boxed{0.3702}$$

$$\boxed{C_D} = \int_0^c c_p^+ \overbrace{\sin\theta^+}^{\sim\theta^+} - c_p^- \overbrace{\sin\theta^-}^{\sim\theta^-}\, \frac{dx}{c} = \frac{2}{\sqrt{M_\infty^2 - 1}} \int_0^c \left[\left(\frac{dy^+}{dx}\right)^2 + \left(\frac{dy^-}{dx}\right)^2 + 2\alpha^2\right]\frac{dx}{c} =$$

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$$= \boxed{\frac{4\alpha^2}{\sqrt{M_\infty^2-1}} + \frac{2}{\sqrt{M_\infty^2-1}} \int_0^c \left[\left(\frac{dy^+}{dx}\right)^2 + \left(\frac{dy^-}{dx}\right)^2\right] \frac{dx}{c}} =$$

$$= \frac{4\alpha^2}{\sqrt{M_\infty^2-1}} + \frac{2}{\sqrt{M_\infty^2-1}} \left(\tan^2\varepsilon_1 + \tan^2\varepsilon_2\right) = \boxed{0.2167}$$

$$C_{m_{LE}} = C_{m_0} = \frac{4}{\sqrt{M_\infty^2-1}} \int_0^1 \left(\frac{dy_m}{dx} \frac{x}{c}\right) d\left(\frac{x}{c}\right) = \dots$$

$$y_m\left(\frac{1}{2}c\right) = \frac{\frac{c}{2}\tan\varepsilon_1 - \frac{c}{2}\tan\varepsilon_2}{2} = \frac{c}{4}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right)$$

$$y_m = \begin{cases} c/4\,(\tan\varepsilon_1 - \tan\varepsilon_2) + \dfrac{c/4\,(\tan\varepsilon_1-\tan\varepsilon_2)}{c/2}\, x & x < \dfrac{c}{2} \\ c/4\,(\tan\varepsilon_1 - \tan\varepsilon_2) - \dfrac{c/4\,(\tan\varepsilon_1-\tan\varepsilon_2)}{c/2}\, x & x > \dfrac{c}{2} \end{cases}$$

$$\frac{dy_m}{dx} = \begin{cases} \dfrac{1}{2}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right) & \dfrac{x}{c} < 0.5 \\ -\dfrac{1}{2}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right) & \dfrac{x}{c} > 0.5 \end{cases}$$

$$\int_0^{0.5} \frac{dy_m}{dx} \frac{x}{c}\, d\left(\frac{x}{c}\right) + \int_{0.5}^1 \frac{dy_m}{dx} \frac{x}{c}\, d\left(\frac{x}{c}\right) = \left[\frac{1}{2}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right)\frac{1}{2}\left(\frac{x}{c}\right)^2\right]_0^{0.5} - \left[\frac{1}{2}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right)\frac{1}{2}\left(\frac{x}{c}\right)^2\right]_{0.5}^{1}$$

$$= \frac{1}{16}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right) - \frac{1}{4}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right) + \frac{1}{16}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right) = -\frac{1}{8}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right)$$

$$C_{m_{LE}} = -\frac{1}{2}C_L + \frac{4}{\sqrt{M_\infty^2-1}}\left(-\frac{1}{8}\left(\tan\varepsilon_1 - \tan\varepsilon_2\right)\right) = \boxed{-0.19769}$$

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