04-Convex_programming

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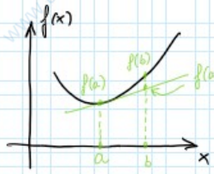
CONVEX PROGRAMMING

How to formulate a problem as a "CONVEX PROBLEM"?

• CHECK CONVEXITY of functions

Th. $f: \mathbb{R}^n \to \mathbb{R}$ is convex $\iff$ $f(y) \geq \underbrace{f(x) + \nabla_x f(x)^T (y-x)}_{\text{TAYLOR EXPANSION truncated at 1ST ORDER}}$, $\forall a,b \in \mathbb{R}^n$

(The tangent must be above the convex plane)



NOTATIONS: JACOBIAN

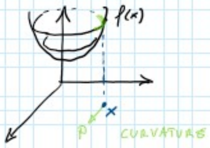
$$\nabla_x f(x) = \frac{\partial f(x)}{\partial x}^T = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}$$

GRADIENT

GRADIENT = JACOBIAN^T (Transpose of Jacobian)

Example: $f(x) = c^T x + d$ all affine functions are convex

Th. $f: \mathbb{R}^n \to \mathbb{R}$ is convex $\iff$ $\nabla_x^2 f(x) \geq 0$ $\forall x \in \mathbb{R}^n$



Example: • $c^T x + x^T H x$, $H \geq 0$ is convex

$(\nabla_x^2 f(x) = H)$

• $e^{x^2} = f(x)$

$\nabla_x f(x) = 2x e^{x^2}$

$\nabla_x^2 f(x) = 2x e^{x^2} + 4x^2 e^{x^2} > 0$ $\forall x$ (STRICTLY CONVEX)

How to preserve convexity of the function?

OPERATIONS THAT PRESERVE CONVEXITY

1) $f^1, f^2: \mathbb{R}^n \to \mathbb{R}$ convex $\Rightarrow$ $f^3 = f^2 + f^1$ is convex (LINEAR COMBINATION of convex functions with POSITIVE sign leads to another convex function)

2) AFFINE INPUT TRANSFORMATION

f is convex $\Rightarrow$ $\bar{f}(x) = f(Ax+b)$ is convex (A has not be square)

3) CONCATENATION with convex and MONOTONICALLY increasing func.

$f: \mathbb{R}^n \to \mathbb{R}$ convex, $g: \mathbb{R} \to \mathbb{R}$ convex & MONOTONICALLY increasing

$\Rightarrow g(f(x))$ is convex $(g \circ f)$

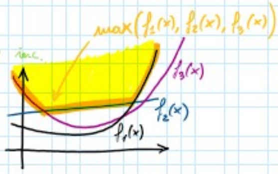
"Dim."

$$\nabla_x^2 g(f(x)) = \underbrace{\frac{\partial^2 g(f(x))}{\partial f(x)}}_{\geq 0} \cdot \underbrace{\nabla_x f(x) \nabla_x f(x)^T}_{\geq 0} + \underbrace{\frac{dg(f(x))}{d f(x)}}_{\geq 0} \underbrace{\nabla_x^2 f(x)}_{\geq 0} \geq 0$$

$\Rightarrow$ We can say that e^{x^2} is convex: e^x: monotonically inc. x^2: convex

4) SUPREMUM of CONVEX FUNCTIONS is CONVEX

$f^1, f^2, \dots, f^N$ convex $\Rightarrow$ $\max_{i=1\dots N} f^i(x)$ is convex



5) ALL NORMS ARE CONVEX FUNCTIONS

If the triangular inequality holds $\Rightarrow$ CONVEX

$\|a+b\| \leq \|a\| + \|b\|$

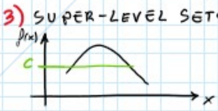
OPERATIONS THAT PRESERVE CONVEXITY OF SETS

1) INTERSECTIONS of convex sets is convex

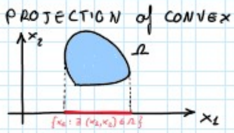
2) SUB-LEVELS of a convex function are convex

$\{x: f(x) \leq c,\ f \text{ convex}\}$

3) SUPER-LEVEL SETS of a CONCAVE function are CONVEX



4) PROJECTION of convex set is convex



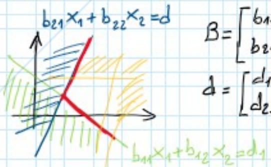
5) POLYHEDRA ARE CONVEX

$\Omega = \{x: Bx \geq d,\ B \in \mathbb{R}^{m \times n},\ d \in \mathbb{R}^m\}$

"Dim." $B(x + t(y-x)) = Bx + tBy - tBx$

$= (1-t)Bx + tBy$

$\geq (1-t)d + td = d$



$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

adding more constraints we still end up with a convex plane

6) AFFINE IMAGE:

if Ω is convex

THEN $A\Omega + b = \{x \in \mathbb{R}^m : x = Ay + b, y \in \Omega\}$

NOTE: $A \in \mathbb{R}^{m \times n}$, $m \neq n$

7) AFFINE PRE-IMAGE

if Ω is convex

NOTE: $A \in \mathbb{R}^{m \times n}$, $m \neq n$

7) AFFINE PRE-IMAGE

if Ω is CONVEX

THEN $\{y \in \mathbb{R}^n : Ay + b \in \Omega\}$ is convex

Convexity is important bc. • EASY TO SOLVE

• LOCAL MINIMIZER is also GLOBAL MINIMIZER

Example / Exercise

$$\min_{x \in \mathbb{R}^2} \cos(x_1)\ell(x_2) + \cos(x_1)^2 + 5\ell(x_2)^2$$

s.t.

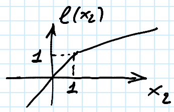
$x_1 \leq \pi$

$x_1 \geq 0$

$x_2 \leq 2$

$x_2 \geq -1$

• $\ell(x_2) = \begin{cases} x_2 & \text{if } x_2 \leq 1 \\ \frac{1}{2} + \frac{x_2}{2} & \text{if } x_2 > 1 \end{cases}$



1) Is the FEASIBLE SET Ω CONVEX ?

2) Is f CONVEX ?

1) we can rewrite :

$$\begin{matrix} -x_1 \geq -\pi \\ x_1 \geq 0 \\ -x_2 \geq -2 \\ x_2 \geq -1 \end{matrix} \Rightarrow \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}}_{B} x \geq \underbrace{\begin{bmatrix} -\pi \\ 0 \\ -2 \\ -1 \end{bmatrix}}_{d} \Rightarrow \text{CONVEX SET !}$$

2)

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow f(x) = \cos(x_1)x_2 + \cos(x_1)^2 + 5x_2^2$$

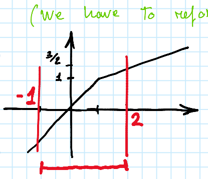
$$\nabla_x f(x) = \begin{bmatrix} -\sin(x_1)x_2 + 2\cos(x_1)\cdot(-\sin(x_1)) \\ \cos(x_1) + 10x_2 \end{bmatrix}$$

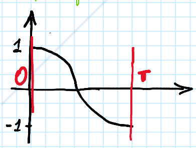
$$\nabla_x^2 f(x) = \begin{bmatrix} -\cos(x_1)x_2 - 2\sin(x_1)^2 - 2\cos(x_1)^2 & -\sin(x_1) \\ -\sin(x_1) & 10 \end{bmatrix}$$

with $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 \\ 0 & 10 \end{bmatrix} \not\succeq 0 \Rightarrow f(x)$ is NOT CONVEX

Can we solve this problem with CONVEX OPTIMIZATION ?

(we have to reformulate the problem to be a convex problem)





• The 2 functions are INVERIBLE in the FEASIBLE SET

$\Rightarrow$ we can think to a change of variables to make $f(x)$ convex

• $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos(x_1) \\ \ell(x_2) \end{bmatrix} \Rightarrow \min z_1 z_2 + z_1^2 + 5z_2^2 = [z_1 \; z_2] \underbrace{\begin{bmatrix} 1 & 0.5 \\ 0.5 & 5 \end{bmatrix}}_{H} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

s.t.

$-1 \leq z_2 \leq \frac{3}{2}$

$-1 \leq z_1 \leq 1$

// CONVEX OPT.