LAPLACE TRANSFORM

Complex

$$s \doteq \sigma + j\omega$$

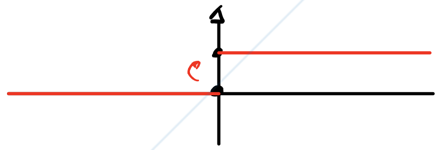
Def: given $y = f(t) \rightarrow \mathcal{L}(f(t)) = \int_0^{\infty} f(t)\, e^{-st}\, dt$;

Condition of existence : $\lim_{t \to \infty} f(t)\, e^{-\underbrace{a}_{\text{real number}} t} = \text{finite value}$

Example 1 :

$\mathcal{L}(\text{step}(t)) = ?$

$\hookrightarrow \text{step}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$



$$\mathcal{L}(\text{step}(t)) = \int_0^{\infty} \text{step}(t)\, e^{-st}\, dt = \int_0^{\infty} 1\, e^{-st}\, dt = -\frac{1}{s} \left| e^{-st} \right|_0^{\infty}$$

$$\underbrace{\mathcal{L}(\text{step}(t)) = -\frac{1}{s} \left| 0 - 1 \right| = \frac{1}{s}}$$

due to laplace transfor we are able to eliminate the discontinuity of the function step(t)

Example 2

$$\mathcal{L}(\underset{f(t)}{e^{-at}}) = \int_0^{\infty} e^{-at} e^{-st}\, dt = \int_0^{\infty} e^{-(a+s)t}\, dt$$

$$\mathcal{L}(e^{-at}) = \frac{1}{a+s}$$

negative time domain $\xrightarrow{\mathcal{L}}$ shift of "a" is explain

- PROPERTY 1 (LINEAR OPERATOR)

$$\mathcal{L}(a f(t) + b g(t)) = \int_0^{\infty} (a f(t) + b g(t))\, e^{-st}\, dt$$

$$= a \int_0^{\infty} f(t)\, e^{-st}\, dt + b \int_0^{\infty} g(t)\, e^{-st}\, dt$$

$$= a F(s) + b G(s)$$

- PROPERTY 2 (FIRST DERIVATIVE OPERATION)

$$\mathcal{L}(\dot{f}(t)) = \int_0^\infty \frac{df}{dt} e^{-st} dt = \int_0^\infty e^{-st} \left(\frac{df}{dt} dt \right)$$

$$= \left| e^{-st} (f(t)) \right|_0^\infty - \int_0^\infty f(t)(-s) e^{-st} dt$$

$$= 0 - f(0) + s \underbrace{\int_0^\infty f(t) e^{-st} dt}_{\text{definition}} = s F(s) - f(0)$$

initial condition

- PROPERTY 3 (SECOND DERIVATIVE)

$$\mathcal{L}(\ddot{f}(t)) = \mathcal{L}\left(\frac{d\dot{f}}{dt} \right) = s \mathcal{L}(\dot{f}(t)) - \dot{f}(0)$$

apply the second property

$$= s \left[s F(s) - f(0) \right] - \dot{f}(0) = s^2 F(s) - s f(0) - \dot{f}(0)$$

(NOTE: so the order of derivation becomes the "power" of s)

FINAL VALUE THEOREM (FVT)

statement: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)$

def: $\mathcal{L}(\dot{f}(t)) = s F(s) - f(0)$

$$\lim_{s \to 0} \left(\int_0^\infty \frac{df}{dt} e^{-st} dt \right) = \lim_{s \to 0} \left(s F(s) - f(0) \right)$$

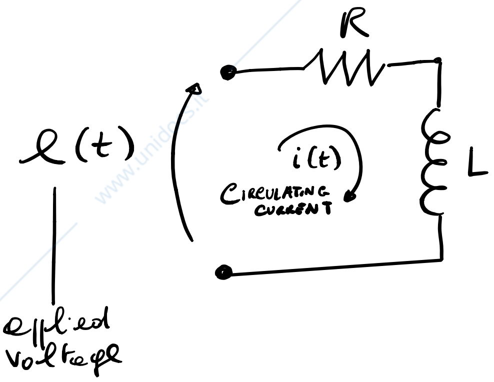
we have applied the same operation to the left and to the right definition

So :

$$\int_0^{\infty} \lim_{s\to 0} \frac{df}{dt} \underbrace{e^{-st}}_{\to 1 \text{ because of the } \lim_{s\to0} e^{-st}=1} dt = \int_0^{\infty} \left(\frac{df}{dt}\right) dt = \int_0^{\infty} df = f(\infty) - f(0)$$

$$= \lim_{t\to\infty} f(t) - \cancel{f(0)} = \lim_{s\to 0} sF(s) - \cancel{f(0)}$$

Example



$$Ri + L\frac{di}{dt} = e \xrightarrow[\text{(Zero initial condition)}]{\mathcal{L}} R\underbrace{I(s)}_{\text{Laplace transform of the current}} + LsI(s) = \underbrace{E(s)}_{\text{Laplace transform of the voltage}}$$

$$(sL + R)\, I(s) = E(s)$$

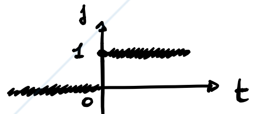
$$\boxed{I(s) = \frac{1}{sL+R} E(s)}$$

OUTPUT — TRANSFER FUNCTION $G(s)$ — INPUT

$$I(s) = G(s)\, E(s)$$

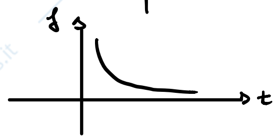
Remember : we have seen :

Step(t)



$\Rightarrow \frac{1}{s}$

e^{-at}



$\Rightarrow \frac{1}{s+a}$

$$I(s) = \frac{1}{sL+R} \underbrace{\frac{e_0}{s}}_{e_0 \,\text{step}(t)} = \frac{1/L}{s+\frac{R}{L}} \frac{e_0}{s} = \left(\frac{A}{s+\frac{R}{L}} + \frac{B}{s}\right) e_0 = \frac{As + B(s+\frac{R}{L})}{s(s+\frac{R}{L})} e_0$$

$$I(s) = \frac{s(A+B) + B\frac{R}{L}}{s(s+\frac{R}{L})} e_0$$

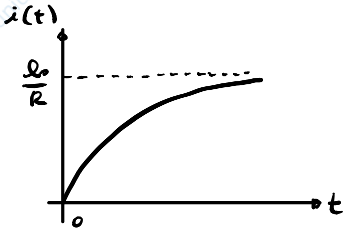
$$A + B = 0 \longrightarrow A = -\frac{1}{R}$$

$$B\frac{R}{L} = \frac{1}{L} \longrightarrow B = \frac{1}{R}$$

$$I(s) = \frac{e_0}{R}\left(-\frac{1}{s+\frac{R}{L}} + \frac{1}{s}\right) \underset{\mathcal{L}^{-1}}{\Longrightarrow} i(t) = \frac{e_0}{R}\left(-e^{-\frac{R}{L}t} + 1\right)$$

$$I(s) = \frac{e_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

time behaveour

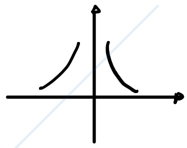


Application of FVT : $\lim_{t \to \infty} i(t) = \lim_{s \to 0} s\,I(s)$

$$= \lim_{s \to 0} \cancel{s}\left(\frac{1}{sL+R}\right)\cdot\frac{e_0}{\cancel{s}} = \frac{e_0}{R}$$

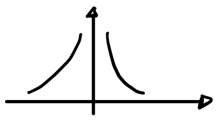
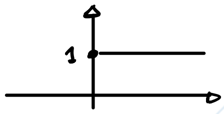
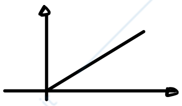
Transfer Function

INP(t) (unit impulse) $\begin{cases} = 0 & t \neq 0 \\ \int_{-\infty}^{\infty} imp(t)\,dt = 1 \end{cases}$



$$\int_{-\infty}^{+\infty} imp(t)\, f(t)\, dt = f(0)$$

$$\mathcal{L}(imp(t)) = \int_0^{\infty} imp(t)\, e^{-st}\, dt = e^{-s(0)} = 1$$

	$f(t)$	$F(s)$
	$imp(t)$	1
	$step(t)$	$1/s$
	$ramp(t)$	$1/s^2$

↑ going up we multiply by s (derivate)

↓ going down we divide by s (integrate)

Input $\quad$ Output

$$X(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

$$Y(s) = G(s)\,X(s)$$

let $X(s)$ be a unit impulse

$$\boxed{Y(s) = G(s)\cdot 1}$$

TF of $\mathcal{L}$ is the time domain response to

What we are changing is just the dynamic of the system! eliminating the discontinuity

$$G(s) = T.F.(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots b_0}{a_0 s^n + a_1 s^{n-1} + \dots a_n} = \frac{N(s)}{D(s)}$$

→ solution of $N(s)$ are called ZEROS

→ solution of $D(s)$ are called POLES

$$G(s) = \mu \frac{\overbrace{\prod_m (1+\tau_m s)}^{\text{real zeros}} \overbrace{\prod_l \left(1 + \frac{2\zeta_l s}{\omega_{0l}} + \frac{s^2}{\omega_{0l}^2}\right)}^{\text{complex zeros}}}{s^N \underbrace{\prod_k (1 + T_k s)}_{\text{real poles}} \underbrace{\prod_m \left(1 + \frac{2\zeta_m s}{\omega_{0m}} + \frac{s^2}{\omega_{0m}^2}\right)}_{\text{complex poles}}}$$