Transfer function: $G(s) = \dfrac{N(s)}{D(s)}$ → poles

State matrix: $\underline{\dot{x}}(t) = [A]\,\underline{x}(t) + \underline{b}\,u$

↓ eigenvalues of $[A]$ ↳ Control input

1 dof damped sys:

$$m\ddot{x} + r\dot{x} + kx = F$$

zero initial condit. ↓ Laplace transform

$$(ms^2 + rs + k)\,x(s) = F(s)$$

$$X(s) = \underbrace{\boxed{\dfrac{1}{ms^2 + rs + k}}}_{G(s)}\,F(s) \quad \Rightarrow \quad \text{poles: } ms^2 + rs + k = 0$$

$$P_{1,2} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \frac{k}{m}}$$

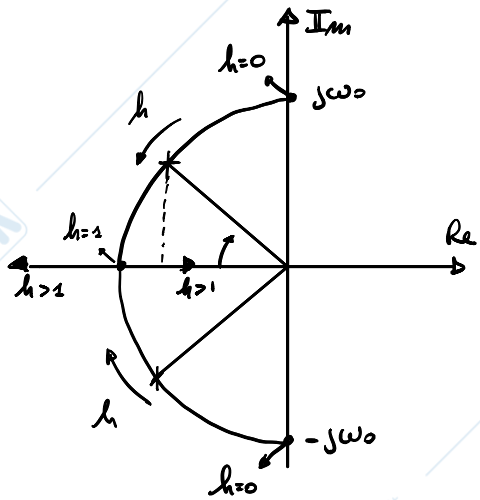
$$\frac{r}{2m}\,\frac{\omega_0}{\omega_0} = \frac{r}{r_c}\,\omega_0 = \underbrace{h}_{\text{damping ratio}}\,\omega_0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$P_{1,2} = -h\omega_0 \pm \sqrt{(h\omega_0)^2 - \omega_0^2}$$

$$P_{1,2} = -h\omega_0 \pm \omega_0\sqrt{h^2 - 1}$$

$h > 1$: $P_{1,2} = -h\omega_0 \pm \omega_0\sqrt{h^2-1}$ → $Re < 0$

$0 < h < 1$: $P_{1,2} = -h\omega_0 \pm j\omega_0\sqrt{1-h^2}$ → $C\ Re(<0)$



If the vector is near to the vertical Imaginary axes we have a small damping if it is horizontal we have a big damping

CONSIDERING STATE MATRIX FORM:

$$\begin{cases} m\ddot{x} + r\dot{x} + kx = F \\ \dot{x} = \dot{x} \quad \text{(matematical trick)} \end{cases} \longrightarrow \begin{cases} \ddot{x} = -\frac{r}{m}\dot{x} - \frac{k}{m}x + \frac{F}{m} \\ \dot{x} = \dot{x} \end{cases}$$

State vector : $\underline{x}(t) = \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix}$; $\quad \dot{\underline{x}}(t) = \begin{Bmatrix} \ddot{x} \\ \dot{x} \end{Bmatrix}$

So:

$$\begin{Bmatrix} \ddot{x} \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} -\frac{r}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ x \end{Bmatrix} + \begin{Bmatrix} \frac{1}{m} \\ 0 \end{Bmatrix} F$$

$$\dot{\underline{x}}(t) = [A]\,\underline{x}(t) + \underline{b}\,u$$

u ↳ Control input variable

NOTE: we can define the state matrix only if sys is linear

Apply Laplace transform

$$\mathcal{L}: \quad s\underline{X}(s) = [A]\,\underline{X}(s) + \underline{b}\,U(s)$$

(collecting $\underline{X}(s)$ state vector)

$$\left(s[I] - [A]\right)\underline{X}(s) = \underline{b}\,U(s)$$

$$\underline{X}(s) = \underbrace{\left(s[I] - [A]\right)^{-1}}_{(n \times n)} \underbrace{\underline{b}}_{(n \times 1)} \underbrace{U(s)}_{(1 \times 1)}$$

$$\underline{X}(s) = \frac{\overbrace{\left(\left[s[I] - [A]\right]^{*}\right)^{T}\underline{b}}^{\text{matrix} \cdot \text{vector} = \text{Colon vector}}}{\underbrace{\left|s[I] - [A]\right|}_{\text{Scalar number obtained from calculate the det } (s[I]-[A])}} \cdot U(s)$$

$*$: matrix of cofactor

T: transposition

Where:

$$\underline{X}(s) = \begin{Bmatrix} G_{11} \\ G_{21} \end{Bmatrix} U(s) \qquad G_{ij} = \frac{N_{ij}(s)}{D_{ij}(s)} \rightarrow$$ poles of each TF_{ij} that are equal

$$\left| s[I] - [A] \right| =$$ characteristic polynomial of $[A]$

$\Downarrow$

solution of charact. polyn. correspond to the <u>eigenvalues</u> of $[A]$

Solving:

$$s[I] - [A] = \begin{bmatrix} s + \frac{r}{m} & \frac{k}{m} \\ -1 & s \end{bmatrix}$$

cofactor: $C_{ij} = (-1)^{(i+j)} M_{ij}$

$$\left(s[I] - [A]\right)^{*} = \begin{bmatrix} s & 1 \\ -\frac{k}{m} & s + \frac{r}{m} \end{bmatrix}$$

$$\left(\left[s[I] - [A]\right]^{*}\right)^{T} = \begin{bmatrix} s & -\frac{k}{m} \\ 1 & s + \frac{r}{m} \end{bmatrix}$$

$$\left(\left[s[I] - [A]\right]^{*}\right)^{T} \underline{b} = \begin{bmatrix} s & -\frac{k}{m} \\ 1 & s + \frac{r}{m} \end{bmatrix} \begin{Bmatrix} \frac{1}{m} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{s}{m} \\ \frac{1}{m} \end{Bmatrix}$$

$$\det \left| S[I]-[A] \right| = \begin{vmatrix} s+\frac{r}{m} & \frac{k}{m} \\ -1 & s \end{vmatrix} = s^2 + \frac{r}{m}s + \frac{k}{m}$$

$$\frac{1}{\left| S[I]-[A] \right|} \begin{Bmatrix} s/m \\ 1/m \end{Bmatrix} = \begin{Bmatrix} G_{11} \\ G_{21} \end{Bmatrix} = \begin{Bmatrix} \dfrac{s/m}{s^2+\frac{rs}{m}+\frac{k}{m}} \\ \dfrac{1/m}{s^2+\frac{rs}{m}+\frac{k}{m}} \end{Bmatrix}$$

So:

$$X(s) = \overset{\text{Poles}}{G_{21}}\; F(s) \rightarrow \mathcal{L}(\text{Force})$$

$\hookrightarrow \mathcal{L}$ displacement

$$s^2 + \frac{rs}{m} + \frac{k}{m} = 0 \Rightarrow \text{Poles: } p_{1,2} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \frac{k}{m}}$$

Exception: Case of multiple inputs

$$\dot{\underline{x}}(t) = [A]\,\underline{x}(t) + [B]\,\underline{U} \xrightarrow{\mathcal{L}} \underline{X}(s) = \frac{\left([S[I]-[A]]^*\right)^T}{\left| S[I]-[A] \right|}[B]\,\underline{U}$$

$$\underline{X}(s) = \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ \vdots & \ddots & \vdots \\ G_{n1} & \cdots & G_{nm} \end{bmatrix} \underline{U}$$

G_{ij}

Polar Plot

$G(s) \longrightarrow j\omega \longrightarrow G(j\omega) = G_R + jG_I = |G|e^{j\varphi}$

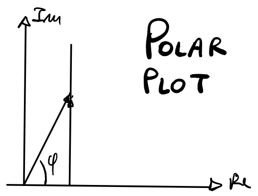
$G(s)$: complex number of a complex variable

$j\omega$: in order to consider the armonic transfer function

1) $G(s) = 1 + Ts$

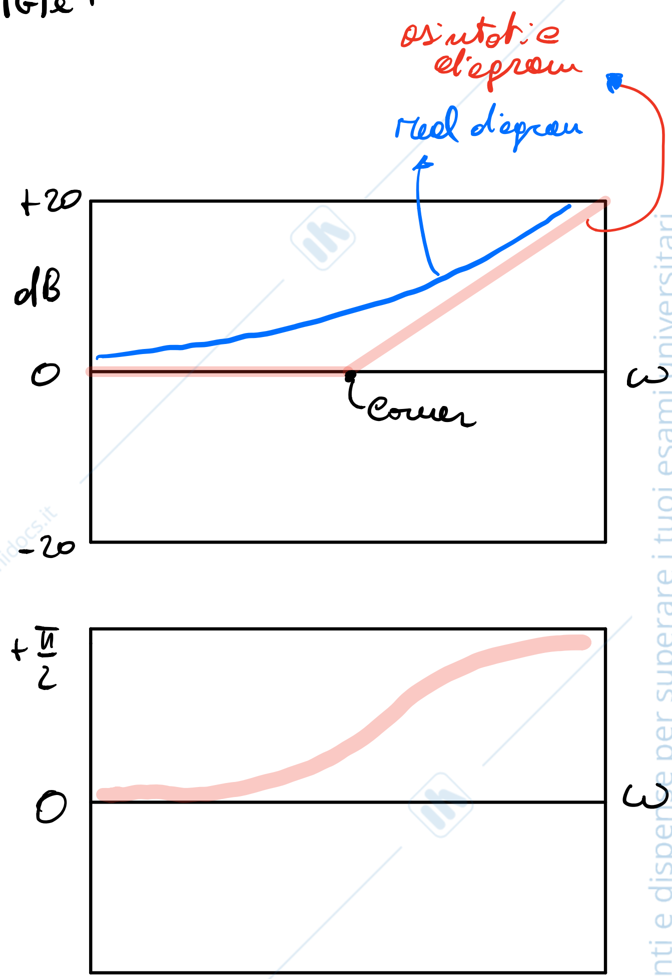
$G(j\omega) = 1 + j\omega T$

$\omega \quad 0 \longrightarrow \infty$



Polar Plot

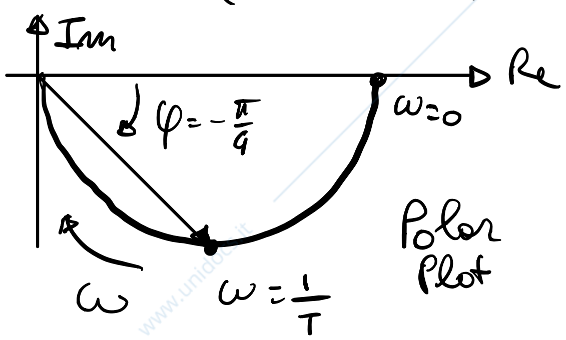
BODE DIAGRAM $\longrightarrow$



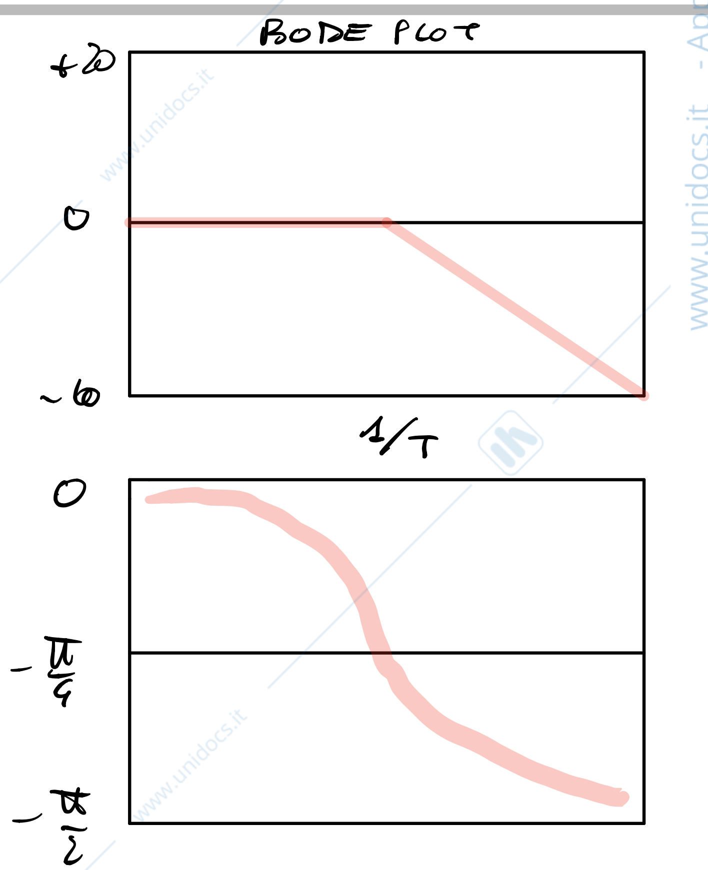
2) $G(s) = \dfrac{1}{1+Ts}$

$$G(j\omega) = \frac{1}{1+j\omega T} \cdot \frac{1-j\omega T}{1-j\omega T}$$

$$G(j\omega) = \frac{1}{1+(\omega T)^2} - j\,\frac{\omega T}{1+(\omega T)^2}$$



Polar Plot

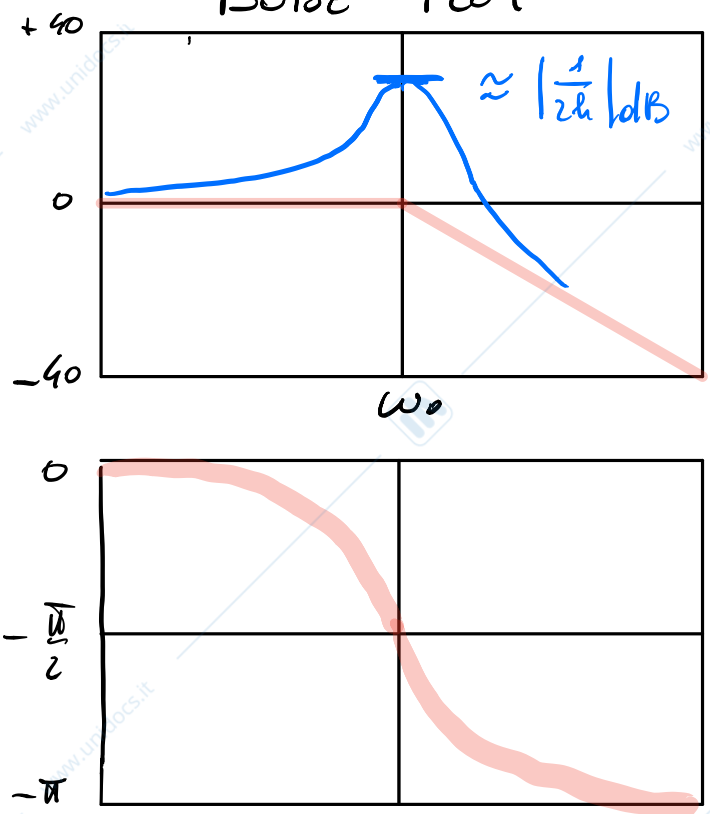


Second order:

3) $$G(s) = \frac{1}{1 + \frac{2h s}{\omega_0} + \frac{s^2}{\omega_0^2}}$$

$$G(j\omega) = \frac{1}{1 + \frac{j2h\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2}}$$

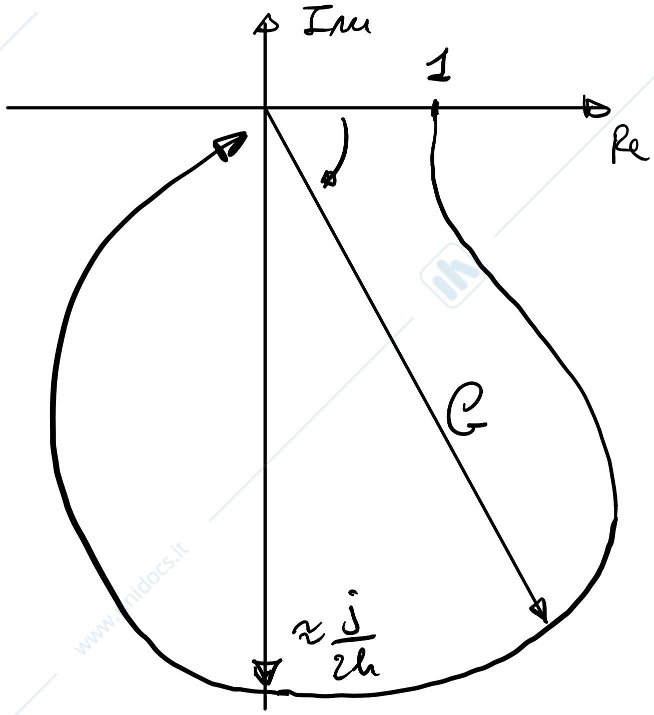
BODE PLOT



Polar plot:

if $\omega = \omega_0$:

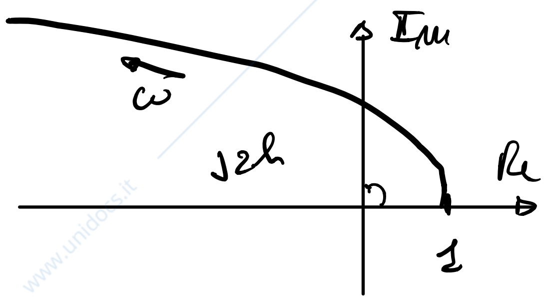
$$G(j\omega) = -\frac{j}{2h}$$



4) $$G(s) = 1 + \frac{2hs}{\omega_0} + \frac{s^2}{\omega_0^2}$$

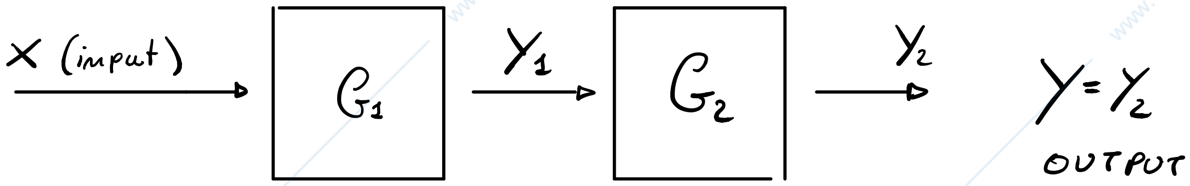
$$G(j\omega) = 1 + \frac{j2h\omega}{\omega_0} - \frac{\omega^2}{\omega_0^2}$$

Polar plot



Block Diagrams (elementary combinations of TFs)

1) Series



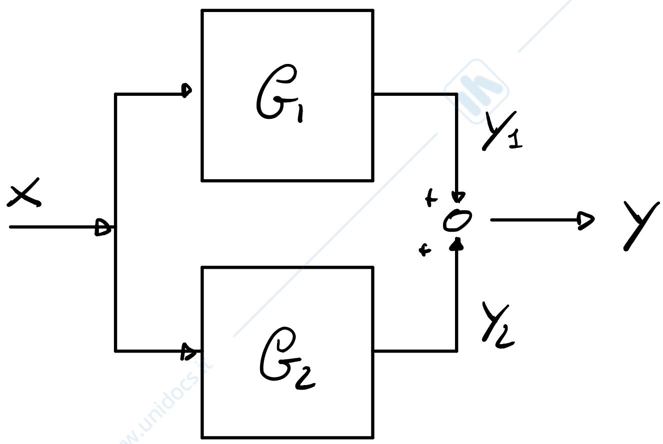
analytical point of view:

$$Y_1 = G_1 X$$

$$Y_2 = G_2 Y_1 = G_2 G_1 X$$

$$Y = G X \quad \text{where} \quad \begin{cases} Y = Y_2 \\ G = G_1 G_2 \end{cases}$$

2) Parallel

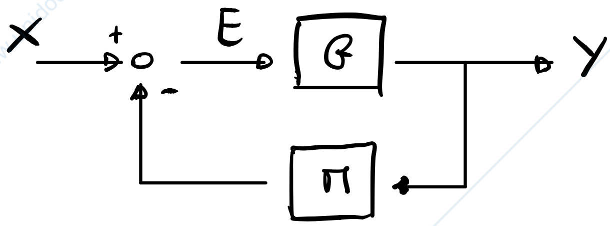


$$Y = Y_1 + Y_2 = G_1 X + G_2 X =$$

$$Y = (G_1 + G_2) X = G X$$

$$G = G_1 + G_2$$

3) Feedback



$$Y = GE = GX - GMY$$

$$\rightarrow Y(1+GM) = GX$$

$$Y = \frac{G}{1+GM} X = LX$$

where:

L = closed-loop function

GM = loop function (open-loop funct)

G = Forward function

$E \rightarrow$ feedback

Imposing $G = \frac{N_G}{D_G}$ and $M = \frac{N_M}{D_M}$

Substituting:

$$L = \frac{N_G}{D_G} \frac{1}{1 + \frac{N_G}{D_G} \frac{N_M}{D_M}} \cdot \frac{N_G}{\cancel{D_G}} \frac{\cancel{D_G} D_M}{D_G D_M + N_G N_M}$$

$$L = \frac{N_G D_M}{N_G N_M + D_G D_M}$$