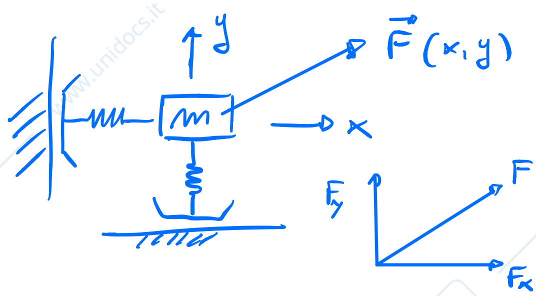
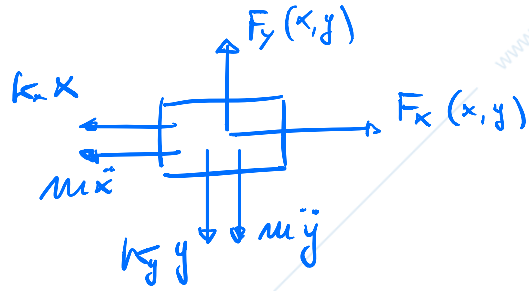
Two dofs sys subjected to a positional force field





$$\begin{cases} m\ddot{x} + k_x x = F_x(x,y) \\ m\ddot{y} + k_y y = F_y(x,y) \end{cases} \qquad \underline{z} = \begin{Bmatrix} x \\ y \end{Bmatrix}$$

STEPS:

1) Static eq. position

$$\begin{cases} k_x x_0 = F_x(x_0,y_0) \\ k_y y_0 = F_y(x_0,y_0) \end{cases} \qquad \underbrace{\underline{z}_0 = \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix}}_{\text{one of the possible equilibrium position}} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \begin{Bmatrix} x_0 \\ y_0 \end{Bmatrix} = \begin{Bmatrix} F_{x_0} \\ F_{y_0} \end{Bmatrix} \Rightarrow [K]\,\underline{z}_0 = \underline{F}_0$$

2) Linearization

$$[M]\,\underline{\ddot{z}} + [K]\,\underline{z} = \underline{F}(\underline{z})$$

we impose $\underline{z}_D = \underline{z} - \underline{z}_0 \rightarrow \underline{z} = \underline{z}_D + \underline{z}_0$

$\underline{\ddot{z}}_D = \underline{\ddot{z}}$

$$[M]\,\underline{\ddot{z}}_D + [K]\,\underline{z}_D + \cancel{[K]\,\underline{z}_0} = \cancel{\underline{F}_0} + \left.\frac{\partial \underline{F}}{\partial \underline{z}}\right|_0 \underline{z}_D + \cancel{\text{hoterms}} \Rightarrow$$

$$\Rightarrow \boxed{[M]\,\underline{\ddot{z}}_D + [K]\,\underline{z}_D = \left[\left.\frac{\partial \underline{F}}{\partial \underline{z}}\right|_0\right] \underline{z}_D}$$

where:

$$\left.\frac{\partial \underline{F}}{\partial \underline{z}}\right|_0 = \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{bmatrix}$$

so:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_D \\ \ddot{y}_D \end{Bmatrix} + \begin{bmatrix} k_x - \frac{\partial F_x}{\partial x} & -\frac{\partial F_x}{\partial y} \\ -\frac{\partial F_y}{\partial x} & k_y - \frac{\partial F_y}{\partial y} \end{bmatrix} \begin{Bmatrix} x_D \\ y_D \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Now simplify all by the general mass m:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\underline{z}}_D + \underbrace{\begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix}}_{[k_T]} \underline{z}_D = \underline{0}$$

$$k_{xx} = \frac{1}{m}\left(k_x - \left.\frac{\partial F_x}{\partial x}\right|_0\right)$$

$$k_{xy} = \frac{1}{m}\left(-\left.\frac{\partial F_x}{\partial y}\right|_0\right)$$

$$k_{yx} = \frac{1}{m}\left(-\left.\frac{\partial F_y}{\partial x}\right|_0\right)$$

$$k_{yy} = \frac{1}{m}\left(k_y - \left.\frac{\partial F_y}{\partial y}\right|_0\right)$$

$$\boxed{[I]\,\ddot{\underline{z}}_D + [k_T]\,\underline{z}_D = \underline{0}} \quad (*)$$

Identity matrix

3) free motion = solution of the eq. of motion

$$\underline{z}_D(t) = \underline{Z}\, e^{\lambda t}$$

$$\ddot{\underline{z}}_D(t) = \lambda^2\, \underline{Z}\, e^{\lambda t}$$

Substituting in ⊛

$$\left([I]\lambda^2 + [k_T]\right) \underset{(2\times1)}{\underline{Z}} e^{\lambda t} = \underline{0} \qquad [I]_{(2\times2)}$$

$$\left([I]\lambda^2 + [k_T]\right) \underline{Z} = \underline{0}$$

algebric linear system
<u>Homogeneus</u>

We can find other solution, not only the trivial solution (soluzione banale), we can let the $\det\left([I]\lambda^2 + [k_T]\right)$ be zero:

So set Det of matrix coeff $= 0$ to get not trivial solution:

$$\det\left|[I]\lambda^2 + [k_T]\right| = 0$$

$$\det\begin{bmatrix} \lambda^2 + k_{xx} & k_{xy} \\ k_{yx} & \lambda^2 k_{yy} \end{bmatrix} = \lambda^4 + \lambda^2\left(k_{xx} + k_{yy}\right) + k_{xx}k_{yy} - k_{xy}k_{yx} = 0$$

So

$$\underbrace{\lambda^2_{I,II} = -\frac{k_{xx}+k_{yy}}{2} \pm \sqrt{\left(\frac{k_{xx}-k_{yy}}{2}\right)^2 - \left(k_{xx}k_{yy} - k_{xy}k_{yx}\right)}}_{\text{one expretion of the solution}}$$

- Alternative expretion of the λ^2 solution

first analise the expretion under square:

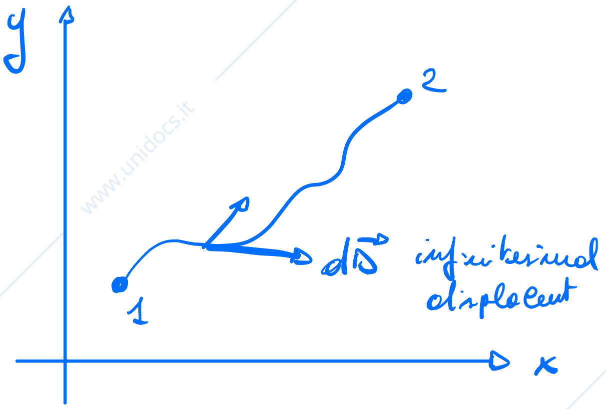
$$\frac{k_{xx}^2 + k_{yy}^2 + 2k_{xx}k_{yy} - 4k_{xx}k_{yy} + 4k_{xy}k_{yx}}{4} = \left(\frac{k_{xx}-k_{yy}}{2}\right)^2 + k_{xy}k_{yx}$$

So:

$$\lambda^2_{I,II} = -\frac{k_{xx}+k_{yy}}{2} \pm \sqrt{\left(\frac{k_{xx}+k_{yy}}{2}\right)^2 - Det(k_T)} \quad 1^*$$

$$\lambda^2_{I,II} = -\frac{k_{xx}+k_{yy}}{2} \pm \sqrt{\left(\frac{k_{xx}-k_{yy}}{2}\right)^2 + k_{yx}k_{xy}} \quad 2^*$$

REVIEW ON CONSERVATIVE FORCE FIELD



$\Gamma_{1,2}$ = path of the point

$$\int_{\Gamma_{1,2}} \vec{F} d\vec{P} = \int_{\Gamma_{1,2}} F_x\,dx + F_y\,dy = W_{1,2} \quad (*)$$

$$\underbrace{F_x dx + F_y dy}_{} = dU(x,y) = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

↳ if is an exact first order infinitesimum ⇒ Conservative

So $F_x = \dfrac{\partial U}{\partial x} \Rightarrow \dfrac{\partial F_x}{\partial y} = \dfrac{\partial^2 U}{\partial x \partial y}$

$F_y = \dfrac{\partial U}{\partial y} \Rightarrow \dfrac{\partial F_y}{\partial x} = \dfrac{\partial^2 U}{\partial y \partial x}$

this 2 expretion are equal

So $(*)$: $$\int_{\Gamma_{1,2}} \vec{F} d\vec{P} = W_{1,2} = \int_{\Gamma_{1,2}} dU = U_2 - U_1$$

So in general: "Ascoltare audio a 1:00:00"

Conservative force field:

$k_{xy} = k_{yx}$ $\longrightarrow$ $\lambda^2_{I,II} = -\gamma \pm \sqrt{\gamma^2 - Det} = -\gamma \pm \sqrt{\beta} = -\gamma \pm \Delta$

in other words K_P is simmetric!

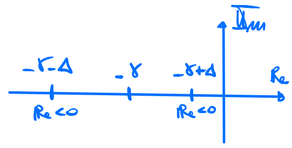
$\beta > 0 \Rightarrow \pm\sqrt{\beta} = \pm\Delta \in Re$

We have to distinguish:

A) $[K_T]^+$ = positive definite = means that all principal minors are >0 ($det, k_{xy}, k_{yy} > 0$)

We assumpt that K_T is positive definite $\Rightarrow$

$$\Rightarrow \lambda^2_{I,II} = -\gamma \pm \sqrt{\gamma^2 - Det} = -\gamma \pm \Delta$$



$\gamma > 0$

$Det > 0 \Rightarrow \sqrt{\gamma^2 - Det} = \Delta \Rightarrow \Delta < \gamma$

The 4 pure and imaginary solution of the caracteristic equation are:

$$\left.\begin{array}{l} \lambda_{1,2} = \pm\sqrt{-\gamma-\Delta} = \pm j\omega_1 \\ \lambda_{3,4} = \pm\sqrt{-\gamma+\Delta} = \pm j\omega_2 \end{array}\right| \Rightarrow \text{Stable}$$

Considering $\lambda = \lambda_{1,..4} \longrightarrow$ det $\left| [I]\lambda^2 + [K_T] \right| = 0$

the two rowes are linear combination one of the other

$$\begin{cases} (\lambda^2 + k_{xx}) x_D + k_{xy} y_D = 0 \\ \cancel{k_{yx} x_D + (\lambda^2 + k_{yy}) y_D = 0} \end{cases}$$

2 linear algebric equation but the second one is not valid

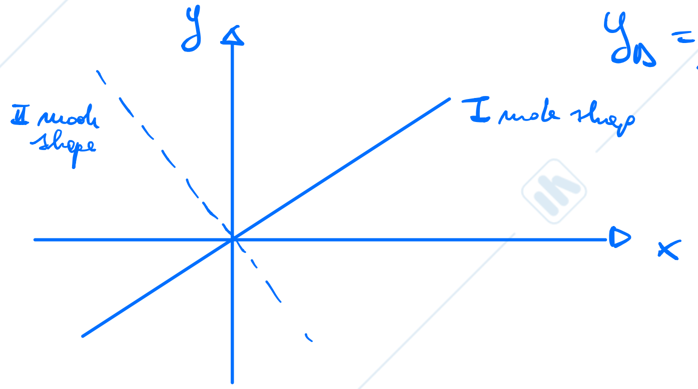
$(\lambda^2_{I,II} + k_{xx}) x_D + k_{xy} y_D$

We have 1 valid eq. but 2 unknown so we can just find the value of $\frac{y_D}{x_D}$

So we can calculate the mode shape

$$\left(\frac{y_D}{x_D}\right)_I = \frac{\lambda_{I,II}^2 + k_{xx}}{k_{xy}} = \mu_I \longrightarrow \underline{\phi}_I = \begin{Bmatrix} 1 \\ \mu_I \end{Bmatrix} = \begin{Bmatrix} x_{D1} \\ y_{D1} \end{Bmatrix} \Rightarrow \text{first mode shape}$$

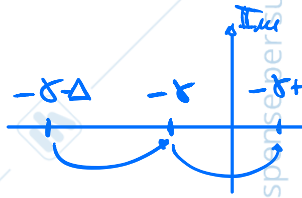
$y_D = \mu_{II} x_D$ $\qquad$ $y_D = \mu_I x_D$



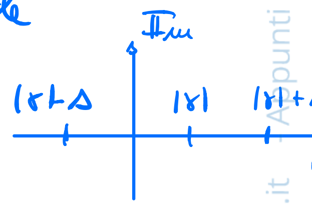
B) $[K_T]$ is <u>not</u> positive definite (it's negative definite)

- Det < 0 → see (1*)

$$\Delta > |\gamma| \quad \gamma > 0 \nearrow \begin{aligned} &\lambda_I^2 = -\gamma + \Delta \; ; \quad \lambda_{II}^2 = -\gamma - \Delta \\ &\lambda_{1,2} = \sqrt{-\gamma - \Delta} = \pm j\omega \\ &\lambda_{3,4} = \sqrt{\underbrace{-\gamma + \Delta}_{>0}} = \pm \alpha \rightarrow \text{instable} \end{aligned}$$



$$\searrow \begin{aligned} &\lambda_I^2 = |\gamma| + \Delta \; ; \quad \lambda_{II}^2 = |\gamma| - \Delta \\ &\lambda_{1,2} = \sqrt{|\gamma| + \Delta} = I\alpha \rightarrow \text{instable} \\ &\lambda_{3,4} = \sqrt{|\gamma| - \Delta} = I j\omega \end{aligned}$$



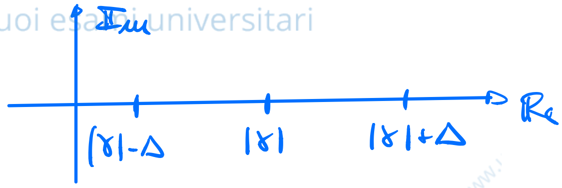
- Det > 0 → see (1*) $\quad \Delta < |\gamma|$

$$Det = k_{xx} k_{yy} - k_{xy} k_{yx} > 0 \Rightarrow k_{xx} k_{yy} > \underbrace{k_{xy} k_{yx}}_{\text{this term is positive}} > 0$$

So k_{xx} and k_{yy} must have the same sign

$$\gamma = \frac{k_{xx} + k_{yy}}{2} < 0$$

$\lambda_I = |\gamma| + \Delta$ $\qquad$ $\lambda_{II} = |\gamma| - \Delta$



$\lambda_{1,2} = \pm\sqrt{|\gamma|+\Delta} = \pm\,\alpha_1 \rightarrow$ instable

$\lambda_{3,4} = \pm\sqrt{|\gamma|-\Delta} = \pm\,\alpha_2 \rightarrow$ instable