

## 21. TRANSFORMATIONS OF THREE-PHASE QUANTITIES

Transformations of a three-phase system of quantities are used to simplify the analysis of three-phase electric circuits. Convention for the quantity notations is as follows:

Space phasor/vector and matrix: bold

Complex quantity: point/bar over the symbol

### Remark

Amplitude is a quantity that may have a positive or negative value; magnitude is the module of a quantity and has only positive value.

### 1) $\alpha\beta\gamma$ (Clarke) transformation

Let us consider a three-phase system a,b,c

$$x_a = x_a(t)$$

$$x_b = x_b(t)$$

$$x_c = x_c(t)$$

where  $x_a$ ,  $x_b$  and  $x_c$  are termed phase quantities and are arbitrary functions of time. The three-phase system can be conveniently written in matrix (column) form as follows:

$$\mathbf{x}_{abc} = \begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}$$

The Clarke transformation transforms a three-phase system of quantities in another three-phase system. The Clarke transformation in matrix form is defined as

$$\mathbf{x}_{\alpha\beta\gamma} = \mathbf{C}\mathbf{x}_{abc}$$

where

$$\mathbf{x}_{\alpha\beta\gamma} = \begin{bmatrix} x_\alpha \\ x_\beta \\ x_\gamma \end{bmatrix}, \quad \mathbf{C} = k_1 \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ k_2 & k_2 & k_2 \end{bmatrix}$$

where  $k_1$  and  $k_2$  are two constant coefficients (they are determined later on) and  $x_\alpha$ ,  $x_\beta$  and  $x_\gamma$  are termed transformed quantities. Note that all the coefficients of the transformation matrix  $\mathbf{C}$  are constant, i.e. they do not depend on the time.

In extended form, the expression of the Clarke-transformed quantities is

$$x_\alpha = k_1 \left( x_a - \frac{1}{2}x_b - \frac{1}{2}x_c \right)$$

$$x_\beta = k_1 \left( \frac{\sqrt{3}}{2}x_b - \frac{\sqrt{3}}{2}x_c \right)$$

$$x_\gamma = k_1 k_2 (x_a + x_b + x_c)$$

It can be proved that the Clarke matrix is invertible for any non-zero value of the coefficients  $k_1$  and  $k_2$ . Then it is possible to trace back from  $x_\alpha, x_\beta, x_\gamma$  to  $x_a, x_b, x_c$

$$\mathbf{x}_{abc} = \mathbf{C}^{-1} \mathbf{x}_{\alpha\beta\gamma}$$

The Clark transformation has some useful properties. The most useful one is that in many electric applications the three-phase system has no bias component (in sinusoidal system, the bias component is called zero-sequence component), i.e. it is

$$(x_a + x_b + x_c) = 0$$

Then

$$x_\gamma = k_1 k_2 (x_a + x_b + x_c) = 0$$

and the transformed quantities reduce to two (two-phase system).

Let us make explicit the bias component in the phase quantities and let us denote the new quantities with the apex. It is

$$x_a = x'_a + x_0$$

$$x_b = x'_b + x_0$$

$$x_c = x'_c + x_0$$

where

$$x_0 = \frac{1}{3}(x_a + x_b + x_c)$$

Note that

$$x_0 = \frac{1}{3k_1 k_2} x_\gamma$$

Of course, the three-phase system  $x'_a, x'_b, x'_c$  has no bias component, i.e. it is

$$x'_a + x'_b + x'_c = 0$$

It can be easily shown that only the terms  $x'_a, x'_b, x'_c$  contribute to  $x_\alpha, x_\beta$ . Indeed, by substituting the above expression of  $x_a, x_b$  and  $x_c$  into the equations of expression of  $x_\alpha$  and  $x_\beta$ , it is

$$x_\alpha = k_I \left( x'_a - \frac{1}{2} x'_b - \frac{1}{2} x'_c \right)$$

$$x_\beta = k_I \left( \frac{\sqrt{3}}{2} x'_b - \frac{\sqrt{3}}{2} x'_c \right)$$

Therefore, if the three-phase quantities have a bias component, it does not appear in the transformed quantities. In particular,  $x_\alpha$  becomes equal to

$$x_\alpha = k_1 \frac{3}{2} x_a'$$

Vice versa, it can be affirmed that  $x_\alpha$  and  $x_\beta$  does not contain any bias component whatever are  $x_a$ ,  $x_b$  and  $x_c$ . This means that going back from  $x_\alpha$  and  $x_\beta$  to  $x_a$ ,  $x_b$  and  $x_c$  gives a correct result only if the three-phase quantities is free from any bias component. If not, the  $\gamma$  component of the Clarke transformation must be also taken into account.

## 2) Representation of the Clark transformation by vector

Let us consider a Cartesian plane  $\alpha, \beta$  and let the three axes a,b,c be placed as in the figure (i.e. axis a aligned along axis  $\alpha$ , axis b rotated of  $2\pi/3$  with respect to axis a, axis c rotated of  $4\pi/3$  with respect to axis a).

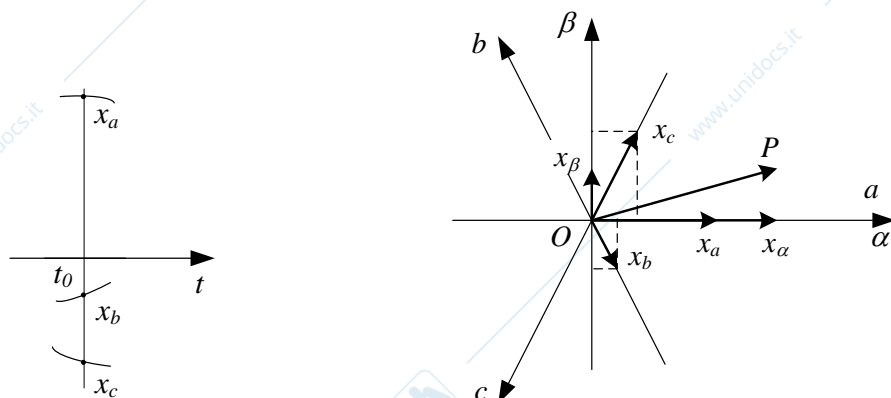
Then let us consider three vectors  $\mathbf{x}_a$ ,  $\mathbf{x}_b$ ,  $\mathbf{x}_c$  along the three axes a,b,c whose amplitudes are  $x_a$ ,  $x_b$ ,  $x_c$  respectively. Let us now calculate the projections of the three vectors on the axis  $\alpha$ . The algebraic sum of the three amplitudes multiplied by the coefficient  $k_1$  gives the component  $x_\alpha$  above defined.

$$x_\alpha = k_1 \left[ x_a + x_b \cos\left(\frac{2\pi}{3}\right) + x_c \cos\left(\frac{4\pi}{3}\right) \right]$$

$$x_\alpha = k_1 \left( x_a - \frac{1}{2}x_b - \frac{1}{2}x_c \right)$$

Multiplication of  $x_\alpha$  by the versor of the axis  $\alpha$  gives a vector that is representative of the projections of the three vectors  $\mathbf{x}_a$ ,  $\mathbf{x}_b$ ,  $\mathbf{x}_c$  along the axis  $\alpha$ :

$$\mathbf{x}_\alpha = x_\alpha \mathbf{vers}(\alpha)$$



(with  $K_1=1$ )

Let us do the same for the projections of the three vectors along the axis  $\beta$ . The algebraic sum of the three amplitudes multiplied by the coefficient  $k_1$  gives the component  $x_\beta$  above defined.

$$x_\beta = k_1 \left[ x_b \sin\left(\frac{2\pi}{3}\right) + x_c \sin\left(\frac{4\pi}{3}\right) \right]$$

$$x_{\beta} = k_1 \left( \frac{\sqrt{3}}{2} x_b - \frac{\sqrt{3}}{2} x_c \right)$$

Multiplication of  $x_{\beta}$  by the versor of the axis  $\beta$  gives a vector that is representative of the projections of the three vectors  $\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c$  along the axis  $\beta$

$$\mathbf{x}_{\beta} = x_{\beta} \mathbf{vers}(\beta)$$

If the three-phase system does not have the bias component, the vector representation of the three-phase system of vector  $\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c$  is defined as

$$\mathbf{x}_{\alpha\beta} = x_{\alpha} \mathbf{vers}(\alpha) + x_{\beta} \mathbf{vers}(\beta)$$

If  $x_a + x_b + x_c \neq 0$ , the plane  $\alpha, \beta$  is extended by including the axis  $\gamma$  orthogonal to the plane  $\alpha, \beta$  and directed as per the right-hand rule. A vector  $\mathbf{x}_{\gamma}$  is then defined along the axis  $\gamma$ ,

$$\mathbf{x}_{\gamma} = x_{\gamma} \mathbf{vers}(\gamma)$$

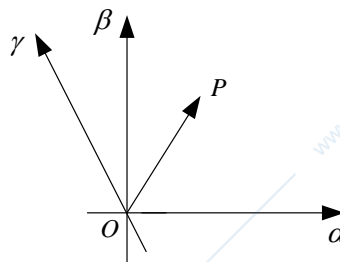
the amplitude of which is equal to the sum of  $x_a + x_b + x_c$  multiplied by the coefficient  $k_1 k_2$ , i.e. to

$$x_{\gamma} = k_1 k_2 (x_a + x_b + x_c)$$

The vector representation of the three-phase system is defined as

$$\mathbf{x}_{\alpha\beta\gamma} = x_{\alpha} \mathbf{vers}(\alpha) + x_{\beta} \mathbf{vers}(\beta) + x_{\gamma} \mathbf{vers}(\gamma)$$

and is given by the vector OP in the figure. It represents the Clarke transformation in vector form. For a three-phase system with no bias component the vector OP lies on the plane  $\alpha, \beta$ .



Note that even in the presence of a bias component, the Clarke transformation has an useful property: the three axes  $\alpha, \beta, \gamma$  are orthogonal each other. This means that the projection of each axis on the other two is zero.

### 3) Representation of the Clarke transformation by space phasor

Let us consider only the  $\alpha, \beta$  components of the vector OP. If the plane  $\alpha, \beta$  is interpreted as the complex plane ( $\alpha \rightarrow \text{Re}, \beta \rightarrow \text{Im}$ ), the vector OP is described by means of the complex quantity  $\bar{x}_{\alpha\beta}$ . This quantity is termed space phasor as it represents phase-related quantities. Differently from a (time) phasor, a space phasor

- represents a three-phase system of quantities,

- is defined for any time function of the quantities, whether periodic or not; if periodic, whether sinusoidal or not
- is not fixed on the plane but moves on it with the time
- all its the components  $x_\alpha$ ,  $x_\beta$ ,  $x_\gamma$  have a physical meaning as they are representative of a three quantities.

In rectangular form, it is

$$\bar{x}_{\alpha\beta} = x_\alpha + jx_\beta$$

$$x_\alpha = \operatorname{Re}(\bar{x}_{\alpha\beta})$$

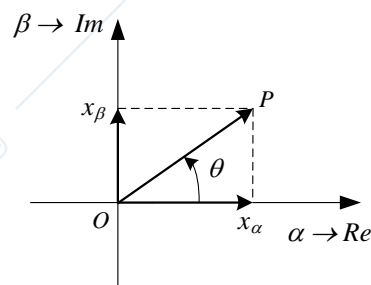
$$x_\beta = \operatorname{Im}(\bar{x}_{\alpha\beta})$$

where  $x_\alpha$  and  $x_\beta$  are the  $\alpha, \beta$  components of the space phasor.

In polar form, the expression of a space phasor is

$$\bar{x}_{\alpha\beta} = x_{\alpha\beta} e^{j\theta_{\alpha\beta}}$$

where  $x_{\alpha\beta}$  is the magnitude and  $\theta_{\alpha\beta}$  the argument of the space phasor.



The relationships between the parameters in the polar and rectangular forms of a space phasor are

$$x_{\alpha\beta} = \sqrt{x_\alpha^2 + x_\beta^2}$$

$$\theta_{\alpha\beta} = \operatorname{arctg} 2 \left( \frac{x_\beta}{x_\alpha} \right)$$

and

$$x_\alpha = x_{\alpha\beta} \cos \theta_{\alpha\beta}$$

$$x_\beta = x_{\alpha\beta} \sin \theta_{\alpha\beta}$$

If the three-phase system has a bias component, the space phasor  $\bar{x}_{\alpha\beta}$  does not fully represent the phase quantities but it must be supplemented with the quantity  $x_\gamma$ .

#### 4) Example

Let us consider a three-phase sinusoidal system with initial phase  $\theta_0$ . It is

$$x_a = \sqrt{2}X \cos(\Omega t + \theta_0)$$

$$x_b = \sqrt{2}X \cos(\Omega t - \frac{2\pi}{3} + \theta_0)$$

$$x_c = \sqrt{2}X \cos(\Omega t - \frac{4\pi}{3} + \theta_0)$$

The transformed quantities  $x_\alpha$  and  $x_\beta$  are

$$x_\alpha = \frac{3}{2}k_1\sqrt{2}X \cos(\Omega t + \theta_0)$$

$$x_\beta = \frac{3}{2}k_1\sqrt{2}X \sin(\Omega t + \theta_0)$$

and the space phasor in polar form is given by

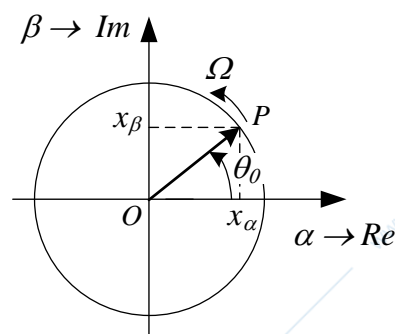
$$\bar{x}_{\alpha\beta} = \frac{3}{2}k_1\sqrt{2}X e^{j(\Omega t + \theta_0)}$$

The space phasor fully represents the three-phase sinusoidal system since it does not have any zero-sequence component.

As a function of time, the space phasor rotates on the complex plane at the angular speed  $\Omega$ , describing a circle of radius

$$R = \frac{3}{2}k_1\sqrt{2}X$$

which is equal to the modulus of the space phasor. The figure below draws the space phasor at  $t=0$ .



### 5) Power expressions

In terms of phase quantities

$$p_{abc} = v_a i_a + v_b i_b + v_c i_c = \mathbf{v}_{abc}^T \mathbf{i}_{abc}$$

In terms of transformed quantities

$$p_{\alpha\beta\gamma} = v_\alpha i_\alpha + v_\beta i_\beta + v_\gamma i_\gamma = \mathbf{v}_{\alpha\beta\gamma}^T \mathbf{i}_{\alpha\beta\gamma}$$

By substituting the Clarke transformation of voltages and currents in  $p_{abc}$ , its expression becomes

$$p_{abc} = \mathbf{v}_{abc}^T \mathbf{i}_{abc} = \mathbf{v}_{\alpha\beta\gamma}^T (\mathbf{C}^{-1})^T \mathbf{C}^{-1} \mathbf{i}_{\alpha\beta\gamma}$$

6) Selection of the coefficients  $k_1$  and  $k_2$ 

Coefficients  $k_1$  and  $k_2$  can be selected arbitrarily, provided that the Clarke matrix is invertible. It is useful to select them by convenience criteria.

Selection #1

Let  $k_1 = \frac{2}{3}$  and  $k_2 = \frac{1}{2}$ . The Clarke matrix becomes

$$\mathbf{C} = \frac{2}{3} \begin{vmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

Its inverse exists and is equal to

$$\mathbf{C}^{-1} = \begin{vmatrix} 1 & 0 & 1 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 1 \end{vmatrix}$$

The extended expressions of  $x_\alpha$ ,  $x_\beta$ ,  $x_\gamma$  become

$$x_\alpha = \frac{2}{3} \left( x_a - \frac{1}{2} x_b - \frac{1}{2} x_c \right)$$

$$x_\beta = \frac{2}{3} \left( \frac{\sqrt{3}}{2} x_b - \frac{\sqrt{3}}{2} x_c \right)$$

$$x_\gamma = \frac{1}{3} (x_a + x_b + x_c)$$

The power in terms of transformed quantities is

$$p_{abc} = \frac{3}{2} (v_\alpha i_\alpha + v_\beta i_\beta) + 3v_\gamma i_\gamma = \frac{3}{2} (v_\alpha i_\alpha + v_\beta i_\beta + 2v_\gamma i_\gamma)$$

Then, by selection #1, the transformation is not invariant to the power. For a three-phase system with no neutral connection, it is

$$p_{abc} = \frac{3}{2} (v_\alpha i_\alpha + v_\beta i_\beta)$$

Let us consider a sinusoidal three-phase system

$$x_a = \sqrt{2}X \cos(\Omega t)$$

$$x_b = \sqrt{2}X \cos\left(\Omega t - \frac{2\pi}{3}\right)$$

$$x_c = \sqrt{2}X \cos(\Omega t - \frac{4\pi}{3})$$

The transformed quantities are

$$x_\alpha = \sqrt{2}X \cos(\Omega t)$$

$$x_\beta = \sqrt{2}X \sin(\Omega t)$$

The magnitude of the transformed quantities is equal to the magnitude of the phase quantities. Then, by selection #1, the transformation is invariant to the magnitude of the sinusoidal quantities. Selection #1 is used when dealing with voltages and currents.

Note that if the coefficients  $k_1$  and  $k_2$  had selected equal to  $k_1 = \frac{2}{3}$  and  $k_2 = \frac{1}{\sqrt{2}}$ , the Clarke matrix, its inverse and the power relationship would become

$$C = \frac{2}{3} \begin{vmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$C^{-1} = \begin{vmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$p_{abc} = \frac{3}{2} (v_\alpha i_\alpha + v_\beta i_\beta + v_\gamma i_\gamma)$$

### Selection #2

Let  $k_1 = \sqrt{\frac{2}{3}}$  and  $k_2 = \frac{1}{\sqrt{2}}$ . The Clarke matrix becomes

$$C = \sqrt{\frac{2}{3}} \begin{vmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

Its inverse exists and is equal to

$$\mathbf{C}^{-1} = \sqrt{\frac{2}{3}} \begin{vmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

It can be easily proved that the matrix is orthogonal, i.e. that

$$\mathbf{C}^T = \mathbf{C}^{-1}$$

The extended expressions of  $x_\alpha$ ,  $x_\beta$ ,  $x_\gamma$  are

$$x_\alpha = \sqrt{\frac{2}{3}} \left( x_a - \frac{1}{2}x_b - \frac{1}{2}x_c \right)$$

$$x_\beta = \sqrt{\frac{2}{3}} \left( \frac{\sqrt{3}}{2}x_b - \frac{\sqrt{3}}{2}x_c \right)$$

$$x_\gamma = \frac{1}{\sqrt{3}} (x_a + x_b + x_c)$$

The power in terms of transformed quantities is

$$p_{abc} = \mathbf{v}_{\alpha\beta\gamma}^T \mathbf{I} \mathbf{i}_{\alpha\beta\gamma} = p_{\alpha\beta\gamma}$$

where  $\mathbf{I}$  is the unit (or identity) matrix. Then, by selection #2, the transformation is invariant to the power.

Let us consider a sinusoidal three-phase system

$$x_a = \sqrt{2}X \cos(\Omega t)$$

$$x_b = \sqrt{2}X \cos\left(\Omega t - \frac{2\pi}{3}\right)$$

$$x_c = \sqrt{2}X \cos\left(\Omega t - \frac{4\pi}{3}\right)$$

The transformed quantities are

$$x_\alpha = \sqrt{3}X \cos(\Omega t)$$

$$x_\beta = \sqrt{3}X \sin(\Omega t)$$

The magnitude of the transformed quantities differs from the magnitude of the phase quantities. Then, by selection #2, the transformation is not invariant to the magnitude of the sinusoidal quantities. Selection #2 is used when dealing with powers. Then it is commonly used hereafter.

#### 7) Space phasor in terms of phase quantities

By selection #2, the transformed quantities become

$$x_\alpha = \sqrt{\frac{2}{3}} \left( x_a - \frac{1}{2}x_b - \frac{1}{2}x_c \right)$$

$$x_\beta = \sqrt{\frac{2}{3}} \left( \frac{\sqrt{3}}{2} x_b - \frac{\sqrt{3}}{2} x_c \right) = \frac{1}{\sqrt{2}} (x_b - x_c)$$

$$x_\gamma = \sqrt{\frac{1}{3}} (x_a + x_b + x_c)$$

The space phasor  $\bar{x}_{\alpha\beta}$  can be expressed as

$$\bar{x}_{\alpha\beta} = x_\alpha + jx_\beta = \sqrt{\frac{2}{3}} \left( x_a + x_b e^{j\frac{2\pi}{3}} + x_c e^{j\frac{4\pi}{3}} \right)$$

being

$$e^{j\frac{2\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

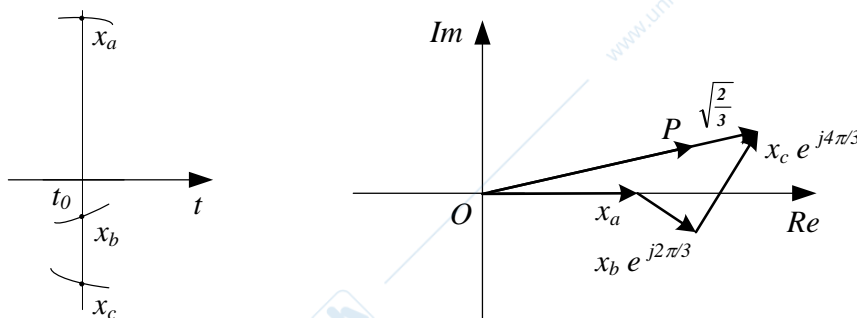
The latter quantity is a space phasor with unit magnitude and argument of  $2\pi/3$ . It is called the rotating operator. Indeed, when it multiplies another quantity, it rotates such a quantity of  $2\pi/3$  without changing its magnitude.

#### 8) Construction of the space phasor

The following equation

$$\bar{x}_{\alpha\beta} = \sqrt{\frac{2}{3}} \left( x_a + x_b e^{j\frac{2\pi}{3}} + x_c e^{j\frac{4\pi}{3}} \right)$$

suggests a direct way of constructing the space phasor OP. Let  $x_a$ ,  $x_b$  and  $x_c$  be the amplitudes of the three phase quantities at the time  $t_0$ . The space phasor OP at the time  $t_0$  is constructed as shown (Note that  $x_b$  and  $x_c$  have negative values)



#### 9) Clark anti-transformation

By means of the matrix  $\mathbf{C}^{-1}$ , the phase quantities can be expressed in terms of the transformed quantities as follows:

$$x_a = \sqrt{\frac{2}{3}} \left( x_\alpha + \frac{1}{\sqrt{2}} x_\gamma \right)$$

$$x_b = \sqrt{\frac{2}{3}} \left( -\frac{1}{2}x_\alpha + \frac{\sqrt{3}}{2}x_\beta + \frac{1}{\sqrt{2}}x_\gamma \right)$$

$$x_c = \sqrt{\frac{2}{3}} \left( -\frac{1}{2}x_\alpha - \frac{\sqrt{3}}{2}x_\beta + \frac{1}{\sqrt{2}}x_\gamma \right)$$

The bias component is clearly given by  $\frac{1}{\sqrt{3}}x_\gamma$ . Then

$$x_a = x'_a + \frac{1}{\sqrt{3}}x_\gamma$$

$$x_b = x'_b + \frac{1}{\sqrt{3}}x_\gamma$$

$$x_c = x'_c + \frac{1}{\sqrt{3}}x_\gamma$$

where

$$x'_a = \sqrt{\frac{2}{3}}x_\alpha$$

$$x'_b = \sqrt{\frac{2}{3}} \left( -\frac{1}{2}x_\alpha + \frac{\sqrt{3}}{2}x_\beta \right)$$

$$x'_c = \sqrt{\frac{2}{3}} \left( -\frac{1}{2}x_\alpha - \frac{\sqrt{3}}{2}x_\beta \right)$$

and  $x'_a$  is obtained as

$$x'_a = \sqrt{\frac{2}{3}}x_\alpha = \sqrt{\frac{2}{3}} \operatorname{Re}(\bar{x}_{\alpha\beta})$$

If the space phasor  $\bar{x}_{\alpha\beta}$  is multiplied by  $e^{-j\frac{2\pi}{3}}$ , one obtains

$$\bar{x}_{\alpha\beta} e^{-j\frac{2\pi}{3}} = \sqrt{\frac{2}{3}} \left( x_a e^{-j\frac{2\pi}{3}} + x_b + x_c e^{j\frac{2\pi}{3}} \right)$$

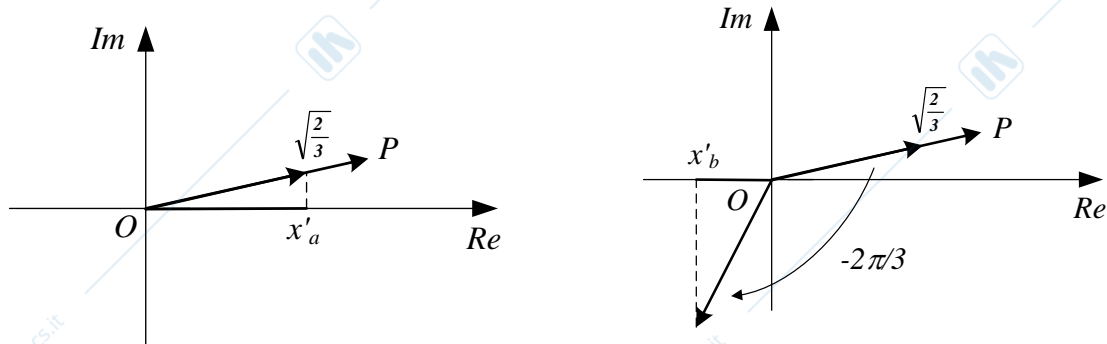
By looking at the procedure used to obtain  $x'_a$ , one can readily realize that

$$x'_b = \sqrt{\frac{2}{3}} \operatorname{Re} \left( \bar{x}_{\alpha\beta} e^{-j\frac{2\pi}{3}} \right)$$

In a similar way, it is

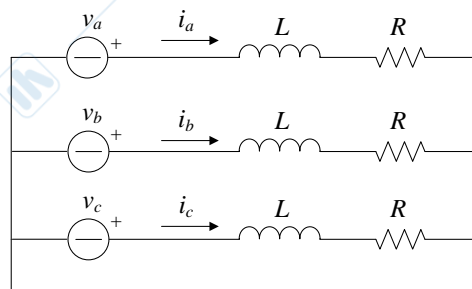
$$x'_c = \sqrt{\frac{2}{3}} \operatorname{Re} \left( \bar{x}_{\alpha\beta} e^{-j\frac{4\pi}{3}} \right)$$

From the analytical process used to execute the anti-transformation, it can be easily derived the geometrical process. It is illustrated in the figure.



#### 10) Analysis of a three-phase circuit by Clarke transformation

Let us consider a three-phase system with neutral connection and let the load (f.i. of RL type) be balanced. The three-phase source voltages are denoted with  $v_a, v_b, v_c$  and are arbitrary functions of time; the currents flowing into the load are denoted with  $i_a, i_b, i_c$ .



Voltage equations of the circuit are

$$v_a = Ri_a + L \frac{di_a}{dt}$$

$$v_b = Ri_b + L \frac{di_b}{dt}$$

$$v_c = Ri_c + L \frac{di_c}{dt}$$

In space phasor form, the voltage equations can be written as

$$\mathbf{v}_{abc} = R\mathbf{i}_{abc} + L \frac{d\mathbf{i}_{abc}}{dt}$$

Clarke transformation of the voltage equations is obtained by multiplying the voltage space phasor equation by the Clarke matrix  $\mathbf{C}$

$$\mathbf{C}\mathbf{v}_{abc} = R\mathbf{C}\mathbf{i}_{abc} + L \frac{d\mathbf{C}\mathbf{i}_{abc}}{dt}$$

obtaining

$$\mathbf{v}_{\alpha\beta\gamma} = R\mathbf{i}_{\alpha\beta\gamma} + L\frac{d\mathbf{i}_{\alpha\beta\gamma}}{dt}$$

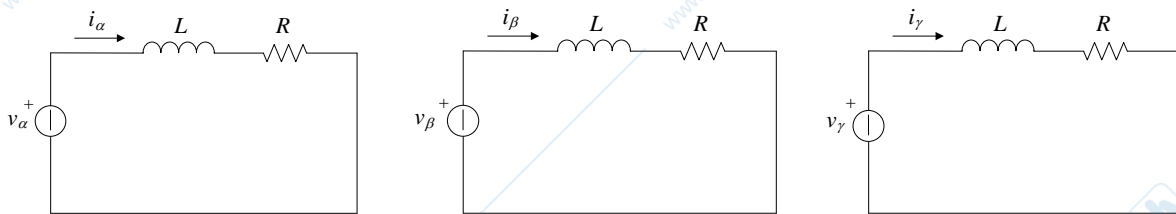
The extended expression of the resulting voltage space phasor equation in terms of  $\alpha, \beta, \gamma$ , it is

$$v_{\alpha} = Ri_{\alpha} + L\frac{di_{\alpha}}{dt}$$

$$v_{\beta} = Ri_{\beta} + L\frac{di_{\beta}}{dt}$$

$$v_{\gamma} = Ri_{\gamma} + L\frac{di_{\gamma}}{dt}$$

The transformed voltage equations of the circuit are the voltage equations of three independent circuits in the transformed phases  $\alpha, \beta, \gamma$ , where the voltage applied to each phase determines the current in the same phase. Note that this occurs irrespectively of the time functions of the applied voltages.



Let us now assume that there is no neutral connection. Above equations and circuits reduce to the ones of the transformed phases  $\alpha$  and  $\beta$ . By using the space phasor representation for voltages and currents, it is

$$\bar{v} = v_{\alpha} + jv_{\beta}$$

$$\bar{i} = i_{\alpha} + ji_{\beta}$$

The relevant voltage space phasor equation is defined as

$$\bar{v} = \underbrace{R\bar{i}}_{\bar{v}_R} + L\underbrace{\frac{d\bar{i}}{dt}}_{\bar{v}_L}$$

The space phasor of the currents can be written as

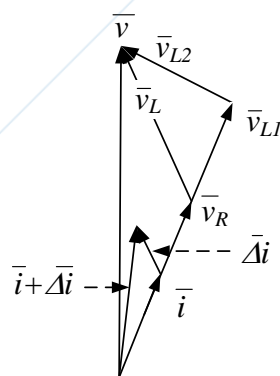
$$\bar{i} = ie^{j\theta_i}$$

Then  $\bar{v}_L$  is given by

$$\bar{v}_L = L\frac{d\bar{i}}{dt} = L\frac{di}{dt}e^{j\theta_i} + Lij\frac{d\theta_i}{dt}e^{j\theta_i} = \underbrace{L\frac{di}{dt}e^{j\theta_i}}_{\bar{v}_{L1}} + \underbrace{j\frac{d\theta_i}{dt}Li\bar{i}}_{\bar{v}_{L2}}$$

i.e. it consists of two terms: one proportional to the change in the current space phasor magnitude and the other one to the change in the current space phasor argument. The first term is aligned along the space phasor of the current whilst the other term is orthogonal to it. The voltage  $\bar{v}_L$ , in turn, is aligned with  $d\bar{i}$ . Note that the finite expression of the time rate is  $\frac{di}{dt} = \frac{\Delta i}{\Delta t}$ , where  $\Delta i$  is the magnitude of  $\Delta\bar{i}$ .

The space phasor equation of the  $\alpha, \beta$  circuits has the graphical representation shown in the figure.



For a sinusoidal three-phase system of applied voltages, the three-phase system of currents is sinusoidal, and the voltage and current expressions are

$$v_a = \sqrt{2}V \cos(\Omega t)$$

$$v_b = \sqrt{2}V \cos\left(\Omega t - \frac{2\pi}{3}\right)$$

$$v_c = \sqrt{2}V \cos\left(\Omega t - \frac{4\pi}{3}\right)$$

$$i_a = \sqrt{2}I \cos(\Omega t - \varphi)$$

$$i_b = \sqrt{2}I \cos\left(\Omega t - \frac{2}{3}\pi - \varphi\right)$$

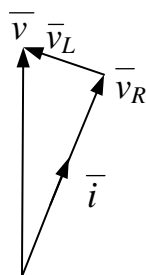
$$i_c = \sqrt{2}I \cos\left(\Omega t - \frac{4}{3}\pi - \varphi\right)$$

The voltage equations, the space phasor of the voltage equations and the graphical representation of the voltage space phasor equation become

$$v_\alpha = Ri_\alpha + j\Omega Li_\alpha$$

$$v_\beta = Ri_\beta + j\Omega Li_\beta$$

$$\bar{v} = \underbrace{R\bar{i}}_{\bar{v}_R} + \underbrace{j\Omega L\bar{i}}_{\bar{v}_L} = \underbrace{R\bar{i}}_{\bar{v}_R} + L \underbrace{\frac{d\bar{i}}{dt}}_{\bar{v}_L}$$



### 11) dq0 (Park) transformation

The Park transformation transforms a three-phase system of quantities in another three-phase system. Let us start from the three-phase system made of  $x_\alpha, x_\beta, x_\gamma$ . The Park transformation in matrix form is defined as

$$\mathbf{x}_{dq0} = \mathbf{P} \mathbf{x}_{\alpha\beta\gamma}$$

where

$$\mathbf{x}_{dq0} = \begin{bmatrix} x_d \\ x_q \\ x_0 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \cos\theta_p & \sin\theta_p & 0 \\ -\sin\theta_p & \cos\theta_p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where  $\theta_p$  is an angle that is a (arbitrary) function of the time, i.e.  $\theta_p = f(t)$ . Therefore the coefficients of  $\mathbf{P}$  are a function of time. This is the additional feature of the Park transformation compared to the Clarke transformation. In extended form, the expression of the Park-transformed quantities is

$$x_d = x_\alpha \cos\theta_p + x_\beta \sin\theta_p$$

$$x_q = -x_\alpha \sin\theta_p + x_\beta \cos\theta_p$$

$$x_0 = x_\gamma$$

Note that

i) the matrix  $\mathbf{P}$  is orthogonal, i.e. its inverse coincides with the transpose

$$\mathbf{P}^{-1} = \mathbf{P}^T = \begin{bmatrix} \cos\theta_p & -\sin\theta_p & 0 \\ \sin\theta_p & \cos\theta_p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ii) the Park transformation transforms only the  $\alpha, \beta$  components of the three-phase system whilst it does not make any transformation on the  $\gamma$  component

### 12) Representation of the Park transformation by vector

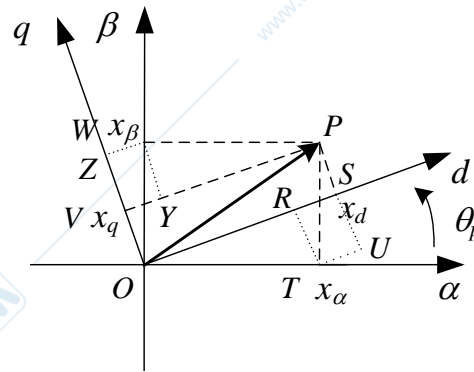
Let us consider the Cartesian plane  $\alpha, \beta$  and a vector  $OP$

$$\mathbf{x}_{\alpha\beta} = x_\alpha \mathbf{vers}(\alpha) + x_\beta \mathbf{vers}(\beta)$$

On the plane  $\alpha, \beta$  another reference frame is drawn, rotating of  $\theta_p$  in anticlockwise way with respect to the reference frame  $\alpha, \beta$ . The axes of the new reference frame are termed  $d, q$  as well as the relevant plane. The Park transformation represents the vector  $OP$  in the  $d, q$  frame

$$\mathbf{x}_{dq} = x_d \mathbf{vers}(d) + x_q \mathbf{vers}(q)$$

Demonstration:



$$x_d = OS = OR + RS = OR + TU = x_\alpha \cos \theta_p + x_\beta \sin \theta_p$$

$$x_q = OV = OZ - ZV = OZ - WY = x_\beta \cos \theta_p - x_\alpha \sin \theta_p$$

### 13) Representation of the Park transformation by space phasor

If the plane d,q is interpreted as a complex plane ( $d \rightarrow \text{Re}$ ,  $q \rightarrow \text{Im}$ ), the vector OP is described by means of the complex quantity  $\bar{x}_{dq}$  that, as for the Clarke transformation, is called space phasor. It may be written in polar and rectangular forms as follows

$$\bar{x}_{dq} = x_{dq} e^{j\theta_{dq}}$$

$$\bar{x}_{dq} = x_d + jx_q$$

By substituting the expressions of  $x_d$  and  $x_q$  as a function of  $x_\alpha$  and  $x_\beta$  in the rectangular form, it is

$$\begin{aligned} \bar{x}_{dq} = x_d + jx_q &= x_\alpha \cos \theta_p + x_\beta \sin \theta_p + j x_\beta \cos \theta_p - j x_\alpha \sin \theta_p = x_\alpha (\cos \theta_p - j \sin \theta_p) + j x_\beta (\cos \theta_p - j \sin \theta_p) = \\ &= (x_\alpha + jx_\beta)(\cos \theta_p - j \sin \theta_p) = (x_\alpha + jx_\beta) e^{-j\theta_p} = \bar{x}_{\alpha\beta} e^{-j\theta_p} = x_{\alpha\beta} e^{j(\theta_{\alpha\beta} - \theta_p)} \end{aligned}$$

The last equation, here rewritten

$$\bar{x}_{dq} = x_{\alpha\beta} e^{j(\theta_{\alpha\beta} - \theta_p)}$$

shows that the Park transformation introduces a clockwise rotation of  $-\theta_p(t)$  in the space phasor in the  $\alpha, \beta$  plane whilst it leaves unchanged the magnitude of the space phasor. Then

$$x_{dq} = x_{\alpha\beta}$$

$$\theta_{dq} = \theta_{\alpha\beta} - \theta_p$$

#### Example

The power-invariant Clarke transformation of a three-phase sinusoidal system is expressed as

$$x_\alpha = \sqrt{3}X \cos(\Omega t + \theta_0)$$

$$x_\beta = \sqrt{3}X \sin(\Omega t + \theta_0)$$

and its space phasor representation as

$$\bar{x}_{\alpha\beta} = \sqrt{3}X e^{j(\Omega t + \theta_0)}$$

Let

$$\theta_p = \Omega t$$

The Park transformation of the space phasor in the  $\alpha, \beta$  plane becomes

$$\bar{x}_{dq} = \sqrt{3}X e^{j\theta_0}$$

This equation shows that the space phasor in the d,q plane is stationary; it is due to the fact that the selected d,q plane is made rotating at the same speed of  $\bar{x}_{\alpha\beta}$ .

#### 14) Park transformation in terms of phase quantities

The Park transformation can be directly applied to a three-phase system  $x_a, x_b$  and  $x_c$  by

$$\mathbf{x}_{dq0} = \mathbf{PC}\mathbf{x}_{abc}$$

$$\mathbf{T} = \mathbf{PC}$$

The matrix  $\mathbf{T}$  is invertible since they are invertible both  $\mathbf{C}$  and  $\mathbf{P}$ . By selecting for  $\mathbf{C}$  the power-invariant version,  $\mathbf{T}$  is orthogonal too. Its expression is

$$\mathbf{T} = \sqrt{\frac{2}{3}} \begin{vmatrix} \cos \theta_p & \cos\left(\theta_p - \frac{2}{3}\pi\right) & \cos\left(\theta_p - \frac{4}{3}\pi\right) \\ -\sin \theta_p & -\sin\left(\theta_p - \frac{2}{3}\pi\right) & -\sin\left(\theta_p - \frac{4}{3}\pi\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

and its inverse is

$$\mathbf{T}^{-1} = \sqrt{\frac{2}{3}} \begin{vmatrix} \cos \theta_p & -\sin \theta_p & \frac{1}{\sqrt{2}} \\ \cos\left(\theta_p - \frac{2\pi}{3}\right) & -\sin\left(\theta_p - \frac{2\pi}{3}\right) & \frac{1}{\sqrt{2}} \\ \cos\left(\theta_p - \frac{4\pi}{3}\right) & -\sin\left(\theta_p - \frac{4\pi}{3}\right) & \frac{1}{\sqrt{2}} \end{vmatrix}$$

The expression of the space phasor  $\bar{x}_{dq}$  in terms of the phase quantities  $x_a, x_b$  and  $x_c$  can be readily obtained from the expression of  $\bar{x}_{\alpha\beta}$ . It is

$$\bar{x}_{dq} = \sqrt{\frac{2}{3}} \underbrace{\left( x_a + x_b e^{j\frac{2\pi}{3}} + x_c e^{j\frac{4\pi}{3}} \right)}_{\bar{x}_{\alpha\beta}} e^{-j\theta_p}$$

#### 15) Correspondence between vector and space phasor operations

Let us consider two vectors on the plan  $\alpha, \beta$

$$\mathbf{x}_1 = (x_{\alpha 1}, x_{\beta 1}), \quad \mathbf{x}_2 = (x_{\alpha 2}, x_{\beta 2})$$

Their internal (or dot/scalar) product is a scalar quantity given by the following expressions

$$\begin{aligned}\mathbf{x}_1 \cdot \mathbf{x}_2 &= X_1 X_2 \cos(\varphi_2 - \varphi_1) \\ \mathbf{x}_1 \cdot \mathbf{x}_2 &= x_{\alpha 1} x_{\alpha 2} + x_{\beta 1} x_{\beta 2}\end{aligned}$$

Note that

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2 \cdot \mathbf{x}_1$$

The properties of the internal product are

- If the two vectors are orthogonal, their internal product is zero (and vice versa).
- The internal product is commutative (i.e. the result does not depend on the order of the vectors).

Their cross (or vector/external) product is a vector given by the following expressions

$$\begin{aligned}\mathbf{x}_1 \wedge \mathbf{x}_2 &= X_1 X_2 \sin(\varphi_2 - \varphi_1) \mathbf{vers}(\boldsymbol{\gamma}) \\ \mathbf{x}_1 \wedge \mathbf{x}_2 &= (x_{\alpha 1} x_{\beta 2} - x_{\alpha 2} x_{\beta 1}) \mathbf{vers}(\boldsymbol{\gamma})\end{aligned}$$

where the direction of  $\boldsymbol{\gamma}$  is orthogonal to  $\mathbf{x}_1$ ,  $\mathbf{x}_2$  and is obtained by the right-hand rule. The orientation of the resulting vector depends on the sign of its amplitude. The amplitude  $Am$  of the vector is given by

$$Am(\mathbf{x}_1 \wedge \mathbf{x}_2) = X_1 X_2 \sin(\varphi_2 - \varphi_1) = (x_{\alpha 1} x_{\beta 2} - x_{\alpha 2} x_{\beta 1})$$

Note that

$$Am(\mathbf{x}_1 \wedge \mathbf{x}_2) = -Am(\mathbf{x}_2 \wedge \mathbf{x}_1)$$

The properties of the cross product are

- If the two vectors are parallel, the cross product is zero (and vice versa).
- The cross product is non-commutative (i.e. the result depends on the order of the vectors)

### Space phasors

Let us consider two space phasors  $\bar{x}_1$  and  $\bar{x}_2$  on the  $\alpha, \beta$  plane

$$\bar{x}_1 = X_1 e^{j\varphi_1} = x_{\alpha 1} + jx_{\beta 1}, \quad \bar{x}_2 = X_2 e^{j\varphi_2} = x_{\alpha 2} + jx_{\beta 2}$$

### Internal product

Let us define the internal product of two space phasors as

$$\bar{x}_1 \cdot \bar{x}_2 = \text{Re}(\bar{x}_1^* \bar{x}_2)$$

It can be expressed in two ways, depending on whether it is used the polar or the rectangular form

$$\begin{aligned}\text{Re}(\bar{x}_1^* \bar{x}_2) &= \text{Re}[X_1 X_2 e^{j(\varphi_2 - \varphi_1)}] = X_1 X_2 \cos(\varphi_2 - \varphi_1) \equiv \mathbf{x}_1 \cdot \mathbf{x}_2 \\ \text{Re}(\bar{x}_1^* \bar{x}_2) &= x_{\alpha 1} x_{\alpha 2} + x_{\beta 1} x_{\beta 2} \equiv \mathbf{x}_1 \cdot \mathbf{x}_2\end{aligned}$$

These results shows that the internal product of the two space vectors is equal to the internal product of the two vectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$  represented by the space phasors. Then it has the same properties of the internal product of the two vectors; f.i. it is commutative

$$Re(\bar{x}_1^* \bar{x}_2) = Re(\bar{x}_2 \bar{x}_1^*)$$

### Example

Voltage and current space vectors in a system with no bias component are given by

$$\begin{aligned}\bar{v}_{\alpha\beta} &= v_\alpha + jv_\beta \\ \bar{i}_{\alpha\beta} &= i_\alpha + ji_\beta\end{aligned}$$

and the instantaneous power by

$$p_{\alpha\beta} = v_\alpha i_\alpha + v_\beta i_\beta$$

It can be written as the internal product of the voltage space phasor by the current space phasor

$$p_{\alpha\beta} = \bar{v}_{\alpha\beta} \cdot \bar{i}_{\alpha\beta}$$

### Cross product

Let us define the cross product of two space phasors as

$$\bar{x}_1 \wedge \bar{x}_2 \equiv Im(\bar{x}_1^* \bar{x}_2)$$

It can be expressed in two ways, depending on whether the polar or the rectangular form is used

$$\begin{aligned}Im(\bar{x}_1^* \bar{x}_2) &= Im[X_1 X_2 e^{j(\varphi_2 - \varphi_1)}] = X_1 X_2 \sin(\varphi_2 - \varphi_1) \equiv Am(\mathbf{x}_1 \wedge \mathbf{x}_2) \\ Im(\bar{x}_1^* \bar{x}_2) &= (x_{\alpha 1} x_{\beta 2} - x_{\alpha 2} x_{\beta 1}) \equiv Am(\mathbf{x}_1 \wedge \mathbf{x}_2)\end{aligned}$$

These results shows that the cross product of the two space vectors is equal to the amplitude of the cross product of the two vectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$  represented by the two space vectors. If convenient, it can be thought that the cross product is directed along the axis  $\gamma$ , orthogonal to the plane  $\alpha, \beta$  and obtained by the right-hand rule. The orientation along the axis  $\gamma$  depends on the sign of  $Am$ .

The cross product of two space phasors has the same properties of the cross product of two vectors; f.i. it is not commutative

$$Im(\bar{x}_1^* \bar{x}_2) \neq Im(\bar{x}_2^* \bar{x}_1)$$

### Useful relationships

It can be proved that

$$\begin{aligned}Am(\bar{x}_1 \wedge \bar{x}_2) &= \bar{x}_1 \cdot j\bar{x}_2 \\ \bar{x}_1 \cdot \bar{x}_2 &= Am(\bar{x}_1 \wedge j\bar{x}_2)\end{aligned}$$

The formulas show that

- the cross product of two space phasors can be calculated as the internal product of the two space vectors, where the second one is multiplied by  $j$  (i.e. rotated of  $\pi/2$ ),
- the internal produce of two space phasors can be calculated as the cross product of the space phasor, where the second one is multiplied by  $j$  (i.e. is rotated of  $\pi/2$ ).

## 16) Analysis of a circuit with time-variant load

The space phasor approach is a powerful tool for analyzing the behavior of very complex circuits. Let us consider a three-phase sinusoidal system of source voltages that supply an RL load with the resistance that is time-varying (f.i. a load made of an arc furnace). The space phasor of the three-phase system of source voltages obtained with the magnitude-invariant Clarke transformation is

$$\bar{v}_s = \sqrt{2}V e^{j\Omega t}$$

Let the varying resistance be expressed as

$$r = R_0[1 - m \cos(\omega_m t)]$$

where  $m$  is the magnitude and  $\omega_m$  is the angular frequency of the varying fraction of the resistance. Let  $m \ll 1$  and  $\omega_m < \Omega$ . For the load inductance  $L$  equal to zero, it is for the generic phase  $k$ , with  $k=a, b, c$

$$i_k = \frac{v_k}{r} \cong \frac{v_k}{R_0} [1 + m \cos(\omega_m t)]$$

By accounting for  $L$ , the current equations become quite involved. Let us assume that they can be expressed as

$$i_a = \sqrt{2}i \cos(\Omega t - \varphi)$$

$$i_b = \sqrt{2}i \cos\left(\Omega t - \frac{2\pi}{3} - \varphi\right)$$

$$i_c = \sqrt{2}i \cos\left(\Omega t - \frac{4\pi}{3} - \varphi\right)$$

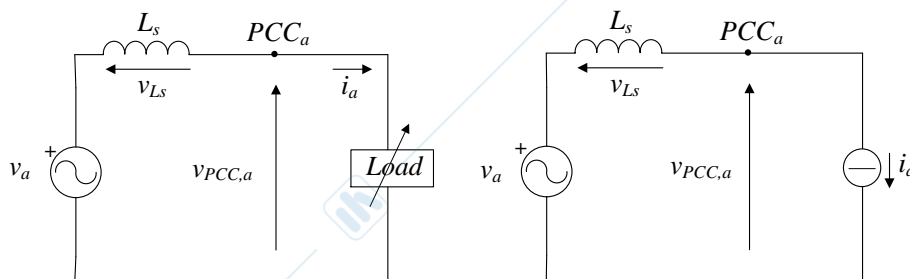
where

$$i = I[1 + m \cos(\omega_m t)]$$

$$I = \frac{V}{R_0}$$

Note that this assumption is equivalent to consider a load constituted by a three-phase current user.

The one-line scheme of the circuit, f.i. of the phase  $a$ , is shown in the figure, where  $v_a$  is the source voltage and  $L_s$  is the line inductance.



#### Constant load resistance operation

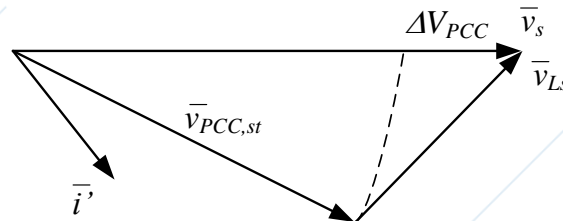
For constant load resistance,  $m=0$  and the space phasor of the currents is

$$\bar{i}' = \sqrt{2}I e^{j(\Omega t - \varphi)}$$

The space phasor of the PCC voltages is

$$\bar{v}_{PCC,st} = \bar{v}_s - \underbrace{L_s \frac{d\bar{i}'}{dt}}_{\bar{v}_{L_s}} = \bar{v}_s - j\Omega L_s \bar{i}'$$

where  $L_s$  is the source inductance and the subscript  $st$  stays for stationary. The voltage drop at PCC is shown in the figure. Note that this situation is the same as the one exposed in chapter A1.



### Varying load resistance operation

Under load resistance variation, the space phasor of the currents becomes

$$\bar{i} = [1 + m \cos(\omega_m t)] \bar{i}'$$

and the space phasor of the PCC voltages, calculated by expressing the  $\cos$  function according to the Euler formula, i.e.

$$\bar{i} = \left[ 1 + m \left( \frac{e^{j\omega_m t} + e^{-j\omega_m t}}{2} \right) \right] \bar{i}'$$

is

$$\bar{v}_{PCC} = \underbrace{\bar{v}_s - j\Omega L_s \bar{i}'}_{\bar{v}_{PCC,st}} - \underbrace{j \frac{m}{2} \Omega L_s \bar{i}' \left[ \left( 1 + \frac{\omega_m}{\Omega} \right) e^{j\omega_m t} + \left( 1 - \frac{\omega_m}{\Omega} \right) e^{-j\omega_m t} \right]}_{\bar{v}_{PCC,r}}$$

where the subscript  $r$  stays for rotating. This expression shows that the space phasor of the PCC voltages consists of two terms: one term ( $\bar{v}_{PCC,st}$ ) has constant magnitude and rotates synchronously with  $\bar{v}_s$ ; the other term ( $\bar{v}_{PCC,r}$ ) is composed by two space phasors of different magnitude that rotate in opposite directions at the angular speed  $\omega_m$  with respect to  $\bar{v}_{PCC}$ . Then the trajectory of the tip of  $\bar{v}_{PCC,r}$  describes an ellipse. The same occurs for the resulting space phasor  $\bar{v}_{PCC}$ . As a result, the amplitude of the space phasor of the PCC voltages fluctuates; this is underlined by the fluctuations of the peak amplitude of the PCC voltages.

