CLASSICAL ELECTRODYNAMICS

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Note: Sections marked by an asterisk contain supplementary material, and can be skipped on a first reading.

Indice

Capitolo 1

Maxwell equations and their properties

We discuss the properties of the Maxwell equations.

1.1 Maxwell equations

Maxwell equations, formulated on 1861, hold both in relativity and in quantum mechanics, with the only difference that in quantum mechanics the electric field is an operator and not a simple vector. Maxwell completed the electromagnetism laws, formulated by Gauss, Ampère, Faraday and others, adding the displacement current to the Ampère law, in order to remove some inconsistency. Before Maxwell, the laws of the electric and magnetic phenomena were described by the following equations:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.2}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.4}$$

The laws described by Eqs.(1.1),(1.3) and (1.4) were obtained empirically in static conditions (i.e. $\partial/\partial t = 0$). However, Eq.(1.4) implies that $\nabla \cdot \mathbf{J} = 0$, in contrast with the continuity equation $\nabla \cdot \mathbf{J} + \partial\rho/\partial t = 0$. Using Eq.(1.3) this can be written as $\nabla \cdot \mathbf{J} + (\partial/\partial t)(\nabla \cdot \mathbf{D}) = 0$ and so $\nabla \cdot (\mathbf{J} + \partial \mathbf{D}/\partial t) = 0$. Hence, by the substitution

$\mathbf{J} \to \mathbf{J} + \partial \mathbf{D}/\partial t$, Eq.(1.4) becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.5}$$

which complete the Maxwell equations set. The term $\partial \mathbf{D}/\partial t$ is said 'displacement current', and it is responsible for the existence of the electromagnetic waves [1].

Eq.(1.1)-(1.5) are macroscopic equations, obtained by a spatial average over a volume much larger than the atomic dimensions, containing a large number of atoms. To them we must add the relations

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{1.6}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{1.7}$$

where $\mathbf{P}$ is the polarization (electric dipole moment for volume units) and $\mathbf{M}$ is the magnetization (magnetic dipole moment for volume units); ρ is the charge density and $\mathbf{J}$ is the current density, supposed assigned externally. These equations describe the general behavior of the e.m. field in the matter, provided the dependence of $\mathbf{P}$ and $\mathbf{M}$ on the fields is known.

1.2 Scalar and vector potentials

It is possible to write the Maxwell equations in terms of the scalar and vector potentials of the e.m. fields. In fact, Eq.(1.1) implies that

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{1.8}$$

where $\mathbf{A}$ is the vector potential. From the other homogeneous Maxwell equation (1.2),

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0.$$

A field whose curl is zero admits a potential, so we can write

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi. \tag{1.9}$$

[1] Maxwell reduced all of the current knowledge on electromagnetism into a linked set of 20 differential equations in 20 variables. The present version of the equations was formulated by Oliver Heaviside on 1873. He wrote: "*I remember my first look at the great treatise of Maxwell's when I was a young man. I saw that it was great, greater and greatest, with prodigious possibilities in its power..It took me several years before I could understand as much as I possibly could.*"

where for convenience we introduced a negative sign in front of ϕ, which is the scalar potential. Eqs.(1.8) and (1.9) are equivalent to the two homogeneous Maxwell equations, (1.1) and (1.2). Let's now consider the two non homogeneous equations, (1.3) and (1.5). From them it follows the continuity equation:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t} \nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}$$

so that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0, \tag{1.10}$$

which expresses the conservation of the total electric charge. Using Eq.(1.6) and by substituting Eq.(1.9) in Eq.(1.3), one finds

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho \\ \epsilon_0 \nabla \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) &= \rho - \nabla \cdot \mathbf{P} \equiv \rho_1 \end{aligned} \tag{1.11}$$

where $\rho_1 = \rho - \nabla \cdot \mathbf{P}$ is the effective charge density [in fact we can write Eq.(1.3) in the form $\nabla \cdot \mathbf{E} = \rho_1/\epsilon_0$]. Finally, we can write Eq.(1.11) as

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_1}{\epsilon_0}, \tag{1.12}$$

where $\nabla^2 = \nabla \cdot \nabla$. In the similar way, we can insert the relations (1.6) and (1.7) in Eq.(1.5):

$$\begin{aligned} \nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) &= \mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0 \mathbf{E} + \mathbf{P}) \\ \nabla \times (\nabla \times \mathbf{A}) &= \mu_0 \mathbf{J} + \mu_0 \nabla \times \mathbf{M} + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) + \mu_0 \frac{\partial \mathbf{P}}{\partial t} \end{aligned}$$

since $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ and $\epsilon_0 \mu_0 = 1/c^2$,

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{J} + \mu_0 \nabla \times \mathbf{M} - \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \phi + \mu_0 \frac{\partial \mathbf{P}}{\partial t}$$

Grouping the terms and introducing the D'Alemembert operator $\Box = (1/c^2)\partial^2/\partial t^2 - \nabla^2$,

$$\Box \mathbf{A} + \nabla \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = \mu_0 \mathbf{J}_1 \tag{1.13}$$

where $\mathbf{J}_1 = \mathbf{J} + \nabla \times \mathbf{M} + \partial \mathbf{P}/\partial t$ is the effective density current. Notice that it is easy to check that

$$\nabla \cdot \mathbf{J}_1 + \frac{\partial \rho_1}{\partial t} = 0, \tag{1.14}$$

so $\mathbf{J}_1$ and ρ_1 satisfy the continuity equation.

1.2.1 Gauge transformation

The potentials **A** and ϕ are not determined in a unitary way by the fields **E** and **B**. In fact, Eq.(1.8) implies that **A** is defined up a gradient of a scalar function, $\boldsymbol{\nabla}\chi$, since, if $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$ then $\boldsymbol{\nabla} \times \mathbf{A}' = \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\chi = \mathbf{B}$. Then, it follows that

$$\mathbf{E} = -\frac{\partial \mathbf{A}'}{\partial t} - \boldsymbol{\nabla}\phi' = -\frac{\partial}{\partial t}(\mathbf{A} + \boldsymbol{\nabla}\chi) - \boldsymbol{\nabla}\phi' = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\left(\phi' + \frac{\partial \chi}{\partial t}\right).$$

The transformation

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi \tag{1.15}$$

$$\phi \rightarrow \phi' = \phi - \frac{\partial \chi}{\partial t} \tag{1.16}$$

is said gauge transformation and the Maxwell equations are invariant under gauge transformation.

1.2.2 Lorenz gauge

They exist different gauge transformation choices, and the best known is the *Lorenz* gauge:

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{1.17}$$

With this choice Eqs.(1.12) and (1.13) become

$$\Box\,\phi = \frac{\rho_1}{\epsilon_0} \tag{1.18}$$

$$\Box\,\mathbf{A} = \mu_0 \mathbf{J}_1 \tag{1.19}$$

The Lorenz gauge choice makes the vector and the scalar potentials to evolve in a similar way, both propagating into the space. Notice that if **A** and ϕ do not satisfy the Lorenz condition, it is possible to perform a gauge transformation into $\mathbf{A}'$ and ϕ' such that these do it:

$$\boldsymbol{\nabla} \cdot (\mathbf{A} + \boldsymbol{\nabla}\chi) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi - \frac{\partial \chi}{\partial t}\right) = \boldsymbol{\nabla} \cdot \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0,$$

so that χ is the solution of the equation:

$$\Box\,\chi = \left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right).$$

Once the function χ has been determined, infinite other solutions exist satisfying the Lorenz gauge such that $\Box\,\chi = 0$.

1.2.3 Coulomb gauge

In the laser physics, a more popular gauge is the *Coulomb* gauge, defined by

$$\boldsymbol{\nabla}\cdot\mathbf{A}=0 \tag{1.20}$$

[as for the Lorenz Gauge, it always exists a scalar function χ transforming $\mathbf{A}$ such that (1.20) holds, solution of $\nabla^2\chi=-\boldsymbol{\nabla}\cdot\mathbf{A}$]. The Coulomb gauge is named also *transverse* or *radiation* gauge. In fact, any vector $\mathbf{G}(\mathbf{r})$ can be decomposed in two parts, one transverse, $\mathbf{G}_\perp(\mathbf{r})$, and one longitudinal, $\mathbf{G}_\parallel(\mathbf{r})$, such that $\mathbf{G}(\mathbf{r})=\mathbf{G}_\perp(\mathbf{r})+\mathbf{G}_\parallel(\mathbf{r})$ where $\boldsymbol{\nabla}\cdot\mathbf{G}_\perp=0$ and $\boldsymbol{\nabla}\times\mathbf{G}_\parallel=0$. For a plane wave $\mathbf{G}(\mathbf{r})\propto\exp[i\mathbf{k}\cdot\mathbf{r}]$ the interpretation of these definition is transparent (in fact $\mathbf{k}\cdot\mathbf{G}_\perp=0$ and $\mathbf{k}\times\mathbf{G}_\parallel=0$). In the Coulomb gauge, Eqs.(1.12) and (1.13) become:

$$\nabla^2\phi = -\frac{\rho_1}{\epsilon_0}, \tag{1.21}$$

$$\Box\,\mathbf{A} = \mu_0\mathbf{J}_\perp \tag{1.22}$$

where

$$\mathbf{J}_\perp=\mathbf{J}_1-\epsilon_0\frac{\partial\boldsymbol{\nabla}\phi}{\partial t}. \tag{1.23}$$

The Coulomb gauge allows to separate the propagating field $\mathbf{A}$ from the electrostatic field, ϕ. In fact, Eq.(1.21) has the solution

$$\phi(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int_V d^3r'\frac{\rho_1(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}$$

equal to the instantaneous Coulomb potential generated by the charge density $\rho_1(\mathbf{r},t)$ [N.B. this does not contradict the relativity principles, since only the fields $\mathbf{E}$ and $\mathbf{B}$ have a physical meaning, not the potentials. It is possible to show that in fact $\mathbf{E}$ and $\mathbf{B}$ propagate with the finite speed c].

The transverse current density $\mathbf{J}_\perp$ is a transverse vector: in fact

$$\boldsymbol{\nabla}\cdot\mathbf{J}_\perp=\boldsymbol{\nabla}\cdot\left(\mathbf{J}_1-\epsilon_0\frac{\partial\boldsymbol{\nabla}\phi}{\partial t}\right)=\boldsymbol{\nabla}\cdot\mathbf{J}_1-\epsilon_0\frac{\partial}{\partial t}\nabla^2\phi=\boldsymbol{\nabla}\cdot\mathbf{J}_1+\frac{\partial\rho_1}{\partial t}=0$$

where the last equality follows from the continuity equation (1.14). Hence, $\mathbf{J}_\parallel=\epsilon_0(\partial/\partial t)\boldsymbol{\nabla}\phi$ is the longitudinal component of the current density, since $\boldsymbol{\nabla}\times\mathbf{J}_\parallel=\epsilon_0(\partial/\partial t)\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi=0$. Furthermore,

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}-\boldsymbol{\nabla}\phi=\mathbf{E}_\perp+\mathbf{E}_\parallel, \tag{1.24}$$

with $\mathbf{E}_\perp = -\partial\mathbf{A}/\partial t$ and $\mathbf{E}_\parallel = -\boldsymbol{\nabla}\phi$. The equations for the longitudinal and transverse electric fields are

$$\boldsymbol{\nabla}\cdot\mathbf{E}_\parallel = \frac{\rho_1}{\epsilon_0}, \tag{1.25}$$

$$\Box\,\mathbf{E}_\perp = -\mu_0\frac{\partial\mathbf{J}_\perp}{\partial t}. \tag{1.26}$$

Is is now clear the name *Coulomb* (see Eq.(1.21)) or *radiation* (see Eq.(1.26)) given to this gauge. The Coulomb gauge is useful in QED, since it may require to quantize only the potential vector $\mathbf{A}$ and not the scalar potential ϕ.

A typical situation in laser physics is to consider the interaction of the radiation with atoms such that $\mathbf{M} = 0$ (no magnetization), $\mathbf{J} = 0$, $\rho = 0$ (no free current of charges) and $\boldsymbol{\nabla}\cdot\mathbf{P} = 0$. With these assumption, $\phi = 0$,

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} \tag{1.27}$$

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{1.28}$$

and $\mathbf{J}_\perp = \mathbf{J}_1 = \partial\mathbf{P}/\partial t$. The fundamental equation in the laser physics theory is

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2} \tag{1.29}$$

An oscillating polarization generates a transverse radiation field (with $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ and $\mathbf{B}\cdot\mathbf{E} = 0$). A different situation occurs in radiation emitted by accelerated charges, as for instance in the free electron laser. In these cases the source is the current density $\mathbf{J}$ and the equation for the field is

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial\mathbf{J}}{\partial t}. \tag{1.30}$$

1.3 Conservation of energy of the electromagnetic fields.

From the Maxwell equations it is possible to obtain the conservation laws of the energy and of the momentum. Let's consider the Maxwell equations in the presence of free charges and currents. For a single charge q the rate of doing work by external electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ is $q\mathbf{v}\cdot\mathbf{E}$, where $\mathbf{v}$ is the velocity of the charge [the magnetic field does not work, since the Lorentz force is perpendicular to the velocity]. For a continuous distribution of charge and current, the total rate of doing work by the fields in a finite volume V is

$$\int_V \mathbf{J} \cdot \mathbf{E} \, d\mathbf{r} \tag{1.31}$$

This power is the e.m. energy converted for unit time in mechanical or thermal energy. It must be balanced by a corresponding rate of decrease of energy in the electromagnetic field within the volume V. This can be obtained from the Maxwell equations. Using Eq.(1.5) to eliminate $\mathbf{J}$

$$\int_V \mathbf{J} \cdot \mathbf{E} d\mathbf{r} = \int_V \left[\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right] d\mathbf{r} \tag{1.32}$$

Using the vectorial identity

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H})$$

and Eq.(1.2):

$$\int_V \mathbf{J} \cdot \mathbf{E} d\mathbf{r} = -\int_V \left[\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) \right] d\mathbf{r} \tag{1.33}$$

To proceed further we assume the the macroscopic medium is *linear* in its electric and magnetic properties. We assume also that the medium is dispersionless (see next section). Then the first and second term represent the time derivative of the energy density

$$u = \frac{1}{2} [\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}] \tag{1.34}$$

whereas the third term is the flux of the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \tag{1.35}$$

Since the volume V is arbitrary, we obtain a differential equation for the energy conservation:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} \tag{1.36}$$

The time derivative of the electromagnetic energy in a volume V plus the flux of energy traversing the surface enclosing V in the unit time is equal to the negative of the work done per unit time by the field on the sources inside the volume V. The Poynting vector $\mathbf{S}$ is the energy flux and has the dimensions of Watt/m^2.

The emphasis so far has been on the energy of the electromagnetic fields only. The work done per unit time per unit volume by the field $\mathbf{J} \cdot \mathbf{E}$ is a conversion of electromagnetic energy into mechanical or heat energy. We can think this as a rate of increase of energy of the charged particles per unit volume, We can interpret the Poynting's theorem for the fields **E** and **B** as a statement of the conservation of energy of the combined system of particles and fields. If the total energy of the particles within V is E_{mech} and no particles move out of the volume, we have

$$\frac{dE_{mech}}{dt} = \int_V \mathbf{J} \cdot \mathbf{E} \, d\mathbf{r}.$$

The Poynting's theorem expresses the conservation of energy for the combined system as

$$\frac{d}{dt}(E_{mech} + E_{field}) = -\oint_S \mathbf{S} \cdot \mathbf{n} \, da \qquad (1.37)$$

where the field energy inside V (in vacuum, where $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$) is

$$E_{field} = \int_V u \, d\mathbf{r} = \frac{\epsilon_0}{2} \int_V (\mathbf{E}^2 + c^2 \mathbf{B}^2) \, d\mathbf{r}. \qquad (1.38)$$

1.4 The Poyntings's theorem for harmonic fields

Let consider a field varying sinusoidally in time

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}\left\{\mathbf{E}(\mathbf{r}) e^{-i\omega t}\right\} = \frac{1}{2}\left\{\mathbf{E}(\mathbf{r}) e^{-i\omega t} + \mathbf{E}^*(\mathbf{r}) e^{i\omega t}\right\} \qquad (1.39)$$

where **E** is complex. Then the product of two quantities, as $\mathbf{J} \cdot \mathbf{E}$ is

$$\begin{aligned} \mathbf{J}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) &= \frac{1}{4}\left\{\mathbf{J}(\mathbf{r}) e^{-i\omega t} + \mathbf{J}^*(\mathbf{r}) e^{i\omega t}\right\} \cdot \left\{\mathbf{E}(\mathbf{r}) e^{-i\omega t} + \mathbf{E}^*(\mathbf{r}) e^{i\omega t}\right\} \\ &= \frac{1}{2}\text{Re}\left\{\mathbf{J}^*(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) e^{-2i\omega t}\right\} \qquad (1.40) \end{aligned}$$

so that the average value (in time) is

$$\langle \mathbf{J}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \rangle = \frac{1}{2}\text{Re}\left\{\mathbf{J}^*(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})\right\} \qquad (1.41)$$

The Maxwell equations for the harmonic fields become

$$\nabla \cdot \mathbf{B} = 0 \quad , \qquad \nabla \cdot \mathbf{D} = \rho \qquad (1.42)$$

$$\nabla \times \mathbf{E} - i\omega \mathbf{B} = 0 \quad , \qquad \nabla \times \mathbf{H} + i\omega \mathbf{D} = \mathbf{J} \qquad (1.43)$$

The derivation of the Poynting's theorem follows the same lines as before:

$$\frac{1}{2}\int_V \mathbf{J}^* \cdot \mathbf{E} \, d\mathbf{r} = \frac{1}{2}\int_V \mathbf{E} \cdot [\nabla \times \mathbf{H}^* - i\omega \mathbf{D}^*] \, d\mathbf{r} \qquad (1.44)$$

Using the vectorial identity

$$\boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H}^* \cdot (\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}^*) = i\omega \mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H}^*)$$

we obtain

$$\frac{1}{2}\int_V \mathbf{J}^* \cdot \mathbf{E}\, d\mathbf{r} = \frac{1}{2}\int_V \left[i\omega(\mathbf{H}^* \cdot \mathbf{B} - \mathbf{E} \cdot \mathbf{D}^*) - \boldsymbol{\nabla} \cdot (\mathbf{E} \times \mathbf{H}^*)\right] d\mathbf{r}. \tag{1.45}$$

We define the Poynting vector

$$\mathbf{S} = \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*) \tag{1.46}$$

and the harmonic electric and magnetic energy densities

$$u_e = \frac{1}{4}(\mathbf{E} \cdot \mathbf{D}^*) \qquad , \qquad u_m = \frac{1}{4}(\mathbf{B} \cdot \mathbf{H}^*) \tag{1.47}$$

such that the Poynting's theorem takes the form:

$$\frac{1}{2}\int_V \mathbf{J}^* \cdot \mathbf{E}\, d\mathbf{r} + 2i\omega \int_V (u_e - u_m) d\mathbf{r} + \oint_S \mathbf{S} \cdot \mathbf{n}\, da = 0 \tag{1.48}$$

It is a complex equation whose real part gives the conservation of energy for the time-averaged quantities and whose imaginary part relates to the stored energy and its alternating flow. If the energy densities u_e and u_m have real volume integrals, as occurs for systems with lossless dielectric and perfect conductors, the real part of Eq.(1.48) is

$$\frac{1}{2}\int_V \mathrm{Re}(\mathbf{J}^* \cdot \mathbf{E}) d\mathbf{r} + \oint_S \mathrm{Re}(\mathbf{S} \cdot \mathbf{n})\, da = 0 \tag{1.49}$$

showing that the steady-state time-average rate of doing work on the source in V by the fields is equal to the average flow of power into the volume V through the boundaries surface S.

1.5 Momentum of the electromagnetic fields

The electromagnetic field transports not only energy by also momentum. A photon of energy $\hbar\omega$ has also a momentum $\hbar\mathbf{k}$, where $\omega = ck$ in vacuum. Classically, it is possible to introduce the momentum density of the electromagnetic field in vacuum. In a dielectric medium this concept is more complicated and ambiguous, as we will see in the next section. So for the moment we consider the electromagnetic field in vacuum only.

The force acting on a charged particle is the Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

For a continuous charge distribution, the second Newton law states that if $\mathbf{P}_{mech}$ is the sum of all the impulses of the particles inside a volume V, then

$$\frac{d\mathbf{P}_{mech}}{dt} = \int_V (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, d\mathbf{r}. \tag{1.50}$$

We use now the Maxwell equations to express ρ and $\mathbf{J}$ in terms of the fields $\mathbf{E}$ and $\mathbf{B}$. Using Eqs.(1.3) and (1.5) in vacuum (i.e. with $\mathbf{D} = \epsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$)

$$\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \epsilon_0 \left[\mathbf{E}\,(\boldsymbol{\nabla} \cdot \mathbf{E}) + \mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} - c^2\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})\right].$$

We can write

$$\mathbf{B} \times \frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}.$$

Then, adding $c^2\,\mathbf{B}\,(\nabla \cdot \mathbf{B}) = 0$ and using Eq.(1.2), we obtain

$$\begin{aligned}\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} &= \epsilon_0 \left[\mathbf{E}\,(\boldsymbol{\nabla} \cdot \mathbf{E}) + c^2\mathbf{B}\,(\nabla \cdot \mathbf{B})\right. \\ &- \left.\mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) - c^2\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B}) - \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B})\right].\end{aligned} \tag{1.51}$$

In this way the time derivative of the impulse is

$$\begin{aligned}&\frac{d}{dt}\left[\mathbf{P}_{mech} + \int_V \epsilon_0(\mathbf{E} \times \mathbf{B})d\mathbf{r}\right] = \\ &\epsilon_0 \int_V \left[\mathbf{E}\,(\boldsymbol{\nabla} \cdot \mathbf{E}) - \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{E}) + c^2\mathbf{B}\,(\boldsymbol{\nabla} \cdot \mathbf{B}) - c^2\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})\right] d\mathbf{r}\end{aligned} \tag{1.52}$$

The term on the left-hand side of the equation is the time derivative of the total momentum, equal to the sum of the mechanical momentum $\mathbf{P}_{mech}$ and the electromagnetic momentum

$$\mathbf{P}_{field} = \epsilon_0 \int_V (\mathbf{E} \times \mathbf{B})d\mathbf{r} = \frac{1}{c^2} \int_V (\mathbf{E} \times \mathbf{H})d\mathbf{r} \tag{1.53}$$

The momentum density of the electromagnetic field in vacuum is

$$\mathbf{g} = \epsilon_0\mathbf{E} \times \mathbf{B} = \mathbf{D} \times \mathbf{B} = \frac{1}{c^2}\mathbf{E} \times \mathbf{H} = \frac{1}{c^2}\mathbf{S}. \tag{1.54}$$

In order to see that the above equation expresses the conservation of the total momentum, we must prove that the right-hand side of the equation is a surface integral of the flux of momentum, i.e. that the function in the volume integral is the divergence of a tensor. Let's consider the x-component of the electric part of it:

$$\begin{aligned}
[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})]_x &= E_x(\partial_x E_x + \partial_y E_y + \partial_z E_z) \\
&- E_y(\partial_x E_y - \partial_y E_x) + E_z(\partial_z E_x - \partial_x E_z) \\
&= \partial_x(E_x^2) + \partial_y(E_x E_y) + \partial_z(E_x E_z) \\
&- \frac{1}{2}\partial_x(E_x^2 + E_y^2 + E_z^2)
\end{aligned}$$

Indicating the cartesian coordinates as x_α with $\alpha = 1, 2, 3$, this means that we can write the α-component as

$$[\mathbf{E}(\nabla\cdot\mathbf{E}) - \mathbf{E}\times(\nabla\times\mathbf{E})]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta}\left(E_\alpha E_\beta - \frac{1}{2}\mathbf{E}\cdot\mathbf{E}\,\delta_{\alpha\beta}\right)$$

so that the vector can be written as the divergence of a tensor of rank two. Adding the magnetic part, we define the *strain Maxwell'a tensor* $T_{\alpha\beta}$ as:

$$T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B}\right)\delta_{\alpha\beta}\right] \tag{1.55}$$

The balance equation of the momentum can be written as

$$\frac{d}{dt}\left(\mathbf{P}_{mech} + \mathbf{P}_{field}\right)_\alpha = \sum_\beta \int_V d\mathbf{r}\frac{\partial}{\partial x_\beta}T_{\alpha\beta} \tag{1.56}$$

and applying the theorem of the divergence to the volume integral,

$$\frac{d}{dt}\left(\mathbf{P}_{mech} + \mathbf{P}_{field}\right)_\alpha = \oint_S da\sum_\beta T_{\alpha\beta}n_\beta \tag{1.57}$$

where $\mathbf{n}$ is a versor orthogonal to the closed surface S and pointing outside. Hence, the equation represents a conservation law for the momentum, where $\sum_\beta T_{\alpha\beta}n_\beta$ is the flux of the force per unit of surface through the surface S. This means that it is the force acting on the particles and the fields inside the volume V. It can be used to calculate the forces acting on material bodies immersed in electromagnetic fields.

1.6 Momentum of the electromagnetic fields in a dielectric medium*

Let's now extend the previous result in the material. Using the macroscopic Maxwell equations,

$$\rho\mathbf{E} + \mathbf{J}\times\mathbf{B} = \left[\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) + \mathbf{B}\times\frac{\partial\mathbf{D}}{\partial t} - \mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right)\right]$$

We can write

$$\mathbf{B}\times\frac{\partial\mathbf{D}}{\partial t} = -\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}$$

Then, adding $\mathbf{B}\left(\nabla\cdot\mathbf{H}\right) = 0$, we obtain

$$\begin{aligned}\rho\mathbf{E} + \mathbf{J}\times\mathbf{B} &= \left[\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) + \mathbf{B}\left(\nabla\cdot\mathbf{H}\right)\right.\\ &- \left.\mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right) - \mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right) - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})\right]\end{aligned} \tag{1.58}$$

In this way the time derivative of the mechanical impulse is

$$\begin{aligned}&\frac{d}{dt}\left[\mathbf{P}_m + \int_V(\mathbf{D}\times\mathbf{B})d\mathbf{r}\right] = \\ &\int_V\left[\mathbf{E}\left(\boldsymbol{\nabla}\cdot\mathbf{D}\right) - \mathbf{D}\times\left(\boldsymbol{\nabla}\times\mathbf{E}\right) + \mathbf{B}\left(\boldsymbol{\nabla}\cdot\mathbf{H}\right) - \mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{H}\right)\right]d\mathbf{r}\end{aligned} \tag{1.59}$$

Assuming a linear relation between $\mathbf{D}$ and $\mathbf{E}$ and between $\mathbf{B}$ and $\mathbf{H}$, we can write the α-component of the force density as

$$\frac{d}{dt}\left[\mathbf{p}_m + \mathbf{D}\times\mathbf{B}\right]_\alpha = \sum_\beta\frac{\partial}{\partial x_\beta}T^M_{\alpha\beta} \tag{1.60}$$

where $\mathbf{P}_m = \int_V \mathbf{p}_m d\mathbf{r}$ and

$$T^M_{\alpha\beta} = E_\alpha D_\beta + B_\alpha H_\beta - \frac{1}{2}\left(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}\right)\delta_{\alpha\beta} \tag{1.61}$$

is the Minkowski form of the Maxwell's tensor. These equations suggest that the total momentum density is the mechanical momentum density $\mathbf{p}_m$ plus

$$\mathbf{g}_M = \mathbf{D}\times\mathbf{B} \tag{1.62}$$

in terms of which

$$\left[\rho\mathbf{E} + \mathbf{J}\times\mathbf{B} + \frac{\partial\mathbf{g}_M}{\partial t}\right]_\alpha = \sum_\beta\frac{\partial}{\partial x_\beta}T^M_{\alpha\beta} \tag{1.63}$$

Let's us recall the Poynting's theorem, Eq.(1.36),

$$\nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E} - \frac{\partial u}{\partial t}$$

The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ gives the flux of electromagnetic energy. The right-hand side is the rate of change of the total energy, that of the field plus that of the material medium, but only energy attributable to the field actually propagates out of any given volume element of the medium. In other words, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ gives the energy flux of the field in the medium as well as in free space. From this assumption, and the relation $\mathbf{g} = \mathbf{S}/c^2$ of special relativity theory for the momentum density associated with any process, electromagnetic or otherwise, by which energy is transported with a flux $\mathbf{S}$ [2], we are led to assign to the field a momentum density

$$\mathbf{g}_A = \frac{\mathbf{E} \times \mathbf{H}}{c^2} \tag{1.64}$$

This defines the Abraham momentum density. Using $\mathbf{D} \times \mathbf{B} = (n^2/c^2)\mathbf{E} \times \mathbf{H} = (1/c^2)(1 + n^2 - 1)\mathbf{E} \times \mathbf{H}$ for an effectively non dispersive and isotropic linear medium with refractive index n, we can write

$$\left[\rho \mathbf{E} + \mathbf{J} \times \mathbf{B} + \mathbf{f}^A + \frac{\partial \mathbf{g}_A}{\partial t}\right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} T^M_{\alpha\beta} \tag{1.65}$$

where

$$\mathbf{f}^A = \frac{1}{c^2}(n^2 - 1)\frac{\partial}{\partial t}\mathbf{E} \times \mathbf{H} \tag{1.66}$$

is the so-called Abraham force density. Equation (1.63) suggests that the force density acting on the material medium is

$$(\mathbf{f}_M)_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} T^M_{\alpha\beta} - \left(\frac{\partial \mathbf{g}_M}{\partial t}\right)_\alpha \tag{1.67}$$

whereas according to Eq. (1.65) the force density acting on the medium is

$$(\mathbf{f}_A)_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta} T^M_{\alpha\beta} - \left(\frac{\partial \mathbf{g}_A}{\partial t}\right)_\alpha \tag{1.68}$$

i.e.

$$\mathbf{f}_A = \mathbf{f}_M + \frac{\partial \mathbf{g}_M}{\partial t} - \frac{\partial \mathbf{g}_A}{\partial t} = \mathbf{f}_M + \mathbf{f}^A \tag{1.69}$$

[2] For instance, for a plane wave $|\mathbf{S}| = c(\mathcal{E}/V)$ where $\mathcal{E}$ is the energy in the volume V. For the theory of relativity, a massless particle carries a momentum $p = \mathcal{E}/c$, so that the momentum density is $\mathbf{p}/V = \mathbf{S}/c^2$.

Thus in the Minkowski formulation the force acting on the particles of the dielectric medium is obtained by subtracting $\partial \mathbf{g}_M/\partial t$ from $\sum_\beta \partial T^M_{\alpha\beta}/\partial x_\beta$, suggesting that the field momentum density is $\mathbf{g}_M$. In the Abraham interpretation as just described, however, the force on the medium is obtained by subtracting $\partial \mathbf{g}_A/\partial t$ from $\sum_\beta \partial T^M_{\alpha\beta}/\partial x_\beta$ under the assumption that the momentum density of the field is $\mathbf{g}_A$. In this interpretation there appears the force density $\mathbf{f}^A$ that, together with the Lorentz force density $\rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$, gives the total force on the material medium, as is clear from Eq.(1.65).

Of course both the Minkowski and Abraham momentum densities, $\mathbf{g}_M$ and $\mathbf{g}_A$, are defined in terms of measurable quantities and are themselves therefore measurable in principle. Either momentum density will comport with conservation of linear momentum, the only difference being in how we choose to apportion the total momentum between the field and the material medium. This is of course obvious from the equation

$$\rho\mathbf{E} + \mathbf{J}\times\mathbf{B} + \frac{\partial \mathbf{g}_M}{\partial t} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} + \mathbf{f}^A + \frac{\partial \mathbf{g}_A}{\partial t}$$

On the left-hand side we could interpret $\mathbf{g}_M$ as field momentum density; on the right-hand side we could interpret $\mathbf{g}_A$ as field momentum density, but then, compared with the left side, we have an additional (Abraham) force (and momentum) density associated with the medium. The generally accepted view is that $\mathbf{g}_A$ is the momentum density of the field.

Discussion*

For a monochromatic plane wave along $\hat{z}$ in vacuum, with $\omega = ck$,

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_0 \cos(kz - \omega t)$$
$$\mathbf{B}(z,t) = \hat{\mathbf{y}} \frac{E_0}{c} \cos(kz - \omega t)$$

the energy density, the Poynting vector and the momentum density are

$$u = \frac{1}{2}\epsilon_0 E_0^2, \qquad \mathbf{S} = \hat{\mathbf{z}}\frac{1}{2}c\epsilon_0 E_0^2 = \hat{\mathbf{z}}cu, \qquad \mathbf{g} = \hat{\mathbf{z}}\frac{\epsilon_0}{2c}E_0^2 = \hat{\mathbf{z}}\frac{u}{c} \tag{1.70}$$

where we replace $\cos^2(kz - \omega t)$ by its average, $1/2$. We can express these quantities in terms of photons by writing $u = q\hbar\omega/V$, where q is the average number of photons in a volume V. This implies that $\mathbf{g} = q(\hbar\omega/cV)\hat{\mathbf{z}}$, consistent with a single photon in vacuum having a linear momentum $\mathbf{p} = \hbar\mathbf{k}$ with $\mathbf{k} = (\omega/c)\hat{\mathbf{z}}$ and with the requirement of special relativity that the energy E and linear momentum $\mathbf{p}$ of a particle with zero rest mass satisfy $E = |\mathbf{p}|c = pc$.

We can extend these considerations heuristically to a plane wave with $\omega = ck/n$ in the case of a dielectric medium with refractive index $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ at the frequency ω. For utmost simplicity we assume for now that both absorption and dispersion are negligible at frequency ω, so that n may be taken to be real and independent on ω. We will also assume that $\mu \approx \mu_0$, generally an excellent approximation at optical frequencies. Then the cycle-averaged energy density is $u = (1/2)\epsilon E_0^2$. However, the form of the momentum density depends on which form of Eq.1.54 we use. If we use the $(1/c^2)\mathbf{E} \times \mathbf{H}$ form, we obtain the (cycle-averaged) momentum density

$$\mathbf{g}_A = \hat{\mathbf{z}}\frac{n\epsilon_0}{2c}E_0^2 = \hat{\mathbf{z}}\frac{u}{cn} \tag{1.71}$$

In terms of photons, if we write $u = q\hbar\omega/V$, then $\mathbf{g}_A = \hat{\mathbf{z}}(q\hbar\omega/cnV)$, implying that the momentum of a photon in a dielectric medium is

$$p_A = \frac{1}{n}\frac{\hbar\omega}{c}. \tag{1.72}$$

It is often asserted, however, that the momentum of a photon in a dielectric medium is $\hbar$ times the wave vector:

$$p_M = \hbar|\mathbf{k}| = n\frac{\hbar\omega}{c}. \tag{1.73}$$

In fact, this form follows directly from the $\mathbf{B} \times \mathbf{D}$ form of Eq.(1.62):

$$\mathbf{g}_M = \hat{\mathbf{z}}\frac{n\epsilon}{2c}E_0^2 = \hat{\mathbf{z}}\frac{nu}{c} \tag{1.74}$$

Here p_A is the Abraham expression for the photon momentum in a dielectric medium, whereas p_M is the Minkowski expression. There is a long-standing question as to which expression is correct, not yet fully resolved.

We present two basic examples of the transfer of momentum between light and matter. These examples are used to support the interpretation of the Abraham momentum as the momentum of the field, whereas the Minkowski momentum is the momentum that the field imparts to atoms in a dielectric medium. For simplicity we assume in each of these examples that dispersion is negligible at the frequency ω considered.

The Balazs Block*

We consider a block of mass M, refractive index n, and thickness a. The block is initially at rest on a frictionless surface. A single-photon pulse of frequency ω is incident on the block, which is assumed to be non absorbing at frequency ω and to have anti-reflection coatings on its front and back surfaces. If the photon momentum is p_{in} inside the block and p_{out} in the vacuum outside it, the block will pick up a momentum $MV = p_{out} - p_{in}$ when the pulse enters. Outside the block the photon momentum is $p_{out} = mc$, where $m = E/c^2 = \hbar\omega/c^2$ is the mass associated with the photon. Similarly $p_{in} = mv$, where v is the velocity of light in the block. If there is no dispersion, $v = v_p = c/n$, and the momentum of the photon in the block is evidently given by $p_{in} = mc/n = \hbar\omega/nc = p_A$, the Abraham photon momentum. The crucial assumption in this argument, originally made in essentially this way by Balazs, is that the velocity of light in the dispersionless medium is the phase velocity v_p. Together with momentum conservation, this assumption leads to the conclusion that the momentum of the field has the Abraham form. This can in principle be tested experimentally. Conservation of momentum requires that $MV = m(c - v)$. When the pulse exits the block, the block recoils and comes to rest and is left with a net displacement

$$\Delta x = V\Delta t = \frac{m}{M}(c - v)\frac{a}{v} = \frac{\hbar\omega}{Mc^2}(n - 1)a$$

as a result of the light having passed through it. If the photon momentum inside the block were assumed to have the Minkowski form $n\hbar\omega/c$, however, the displacement of the block would be in the opposite direction, contradicting the Newton's law.

The Doppler effect*

In the absence of an experimental test the example just considered does not prove that the momentum of a photon in a dielectric medium is $p_A = \hbar\omega/nc$, but only makes it plausible. We next consider an example where the answer - the Doppler shift in a medium of refractive index n - is known, and see what it says about p_A versus p_M. This example is based on an argument of Fermi's that the Doppler effect is a consequence of recoil. Consider an atom of mass M inside a dielectric medium with refractive index $n(\omega)$. The atom is assumed to have a sharply defined transition frequency ω_0 and to be moving initially with a velocity v away from a source of light of frequency ω. Because the light in the atom's reference frame has a Doppler-shifted frequency $\omega(1 - nv/c)$ determined by the phase velocity c/n of light in the medium, the atom can absorb a only photon if $\omega(1 - nv/c) = \omega_0$, or

$$\omega \sim \omega_0(1 + nv/c)$$

We associate with a photon in the medium a momentum p, and consider the implications of (nonrelativistic) energy and momentum conservation. The initial energy is $E_i = \hbar\omega + (1/2)Mv^2$, and the final energy, after the atom has absorbed a photon, is $\hbar\omega_0 + (1/2)Mv'^2$, where v' is the velocity of the atom after absorption. The initial momentum is $p + Mv$, and the final momentum is just Mv'. Therefore

$$\frac{1}{2}M(v'^2 - v^2) \sim Mv(v' - v) = Mv(p/M) = \hbar(\omega - \omega_0)$$

or

$$\omega \sim \omega_0 + \frac{pv}{\hbar} \sim \omega_0(1 + nv/c)$$

so that $p = \hbar\omega_0(n/c) = p_M$.

The first example suggests that the momentum of the photon is p_A, while the second example seems at first thought to suggest that it is p_M. There is no doubt about the (first-order) Doppler shift in a dielectric medium being $nv\omega/c$, as we have assumed, but does this imply that the momentum of a photon in a dielectric is $n\hbar\omega/c$? It is possible to see that the total force exerted by a single photon plane monochromatic wave on the particles of a dielectric, including the Abraham force, suggests a momentum density of magnitude

$$\frac{p_{med}}{V} = \left(n - \frac{1}{n}\right)\frac{\hbar\omega}{cV}$$

if dispersion is negligible. Now from our conclusion from energy and momentum conservation that the known Doppler shift implies that an absorber (or emitter) inside a dielectric recoils with momentum $n\hbar\omega/c$, all we can logically deduce is that a momentum $n\hbar\omega/c$ is taken from (or given to) the combined system of the field and the dielectric medium. Given that the medium acquires the momentum density above, from the force exerted on it by the propagating field, and that the atom recoils with momentum $n\hbar\omega/c$, we can attribute to the field, by conservation of momentum, a momentum

$$n\frac{\hbar\omega}{c} - p_{med} = \frac{1}{n}\frac{\hbar\omega}{c} = p_A.$$

In other words, the kinetic momentum of the field is the Abraham momentum, consistent with our discussion of the example of the Balazs block. The momentum $n\hbar\omega/c$ evidently gives the momentum not of the field as such but of the combined system of field plus dielectric; it is the momentum density equal to the total energy density $u = \hbar\omega/V$ for a monochromatic field divided by the phase velocity c/n of the propagating wave. Experiments on the recoil of objects immersed in dielectric media indicate that the recoil momentum is $n\hbar\omega/c$ per unit of energy $\hbar\omega$ of the field, just as in the Doppler effect. A consistent interpretation of these examples, therefore, is that the field carries a momentum p_A per photon, but that there is also a momentum p_{med} imparted by the field to the medium. An atom that absorbs or emits a photon of frequency ω in the medium therefore recoils with the momentum $p_A + p_{med} = p_M$, as *if* the photon momentum were the Minkowski photon momentum p_M.

1.7 Wave equation and Green functions

We consider the equation for the electric field $\mathbf{E}$ in the presence of a dielectric with polarization $\mathbf{P}$:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{1.75}$$

It has the structure of a wave equation, where each component is the solution of an equation of the form

$$\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\mathbf{r}, t) \tag{1.76}$$

where $f(\mathbf{r}, t)$ is a known distribution of sources. It is useful to express the solution of this equation using the Green function. Let's assume a case without boundary conditions and eliminate the temporal dependence by a Fourier transform:

$$\Psi(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\mathbf{r}, \omega) e^{-i\omega t} d\omega \tag{1.77}$$

$$f(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\mathbf{r}, \omega) e^{-i\omega t} d\omega \tag{1.78}$$

so that Eq.(1.76) becomes

$$(\nabla^2 + k^2)\Psi(\mathbf{r}, \omega) = -4\pi f(\mathbf{r}, \omega) \tag{1.79}$$

where $k = \omega/c$. The associated Green function satisfies the equation

$$(\nabla^2 + k^2) G_k(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \tag{1.80}$$

such that the $\Psi(\mathbf{r}, \omega)$ can be written as

$$\Psi(\mathbf{r}, \omega) = \int G_k(\mathbf{r}, \mathbf{r}') f(\mathbf{r}', \omega)\, d\mathbf{r}'. \tag{1.81}$$

If there are not boundary surfaces, the Green function may depend only on $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and must have a spherical symmetry, i.e. must depends only on $R = |\mathbf{R}|$. From the expression of the laplacian in spherical coordinates,

$$\frac{1}{R}\frac{d^2}{dR^2}(RG_k) + k^2 G_k = -4\pi\delta(R).$$

Here RG_k satisfies in all the space except $R = 0$ the homogeneous equation

$$\frac{d^2}{dR^2}(RG_k) + k^2(RG_k) = 0$$

with solution $RG_k(R) = Ae^{ikR} + Be^{-ikR}$. The delta function is relevant only for $R \to 0$, so the equation reduces to the Poisson equation, since $kR \ll 1$. Then

$$\lim_{kR\to 0} G_k(R) = \frac{1}{R}$$

and the general solution is

$$G_k(R) = AG_k^{(+)}(R) + BG_k^{(-)}(R) \tag{1.82}$$

where

$$G_k^{(\pm)}(R) = \frac{e^{\pm ikR}}{R} \tag{1.83}$$

where $A + B = 1$. The two solutions correspond to a divergent and convergent spherical wave, respectively. The time-dependent Green functions are

$$G^{(\pm)}(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm ikR}}{R}e^{-i\omega\tau}d\omega = \frac{1}{R}\delta\left(\tau \mp \frac{R}{c}\right) \tag{1.84}$$

where the last expression hold for a non dispersive medium, with $k = \omega/c$. The function $G^{(+)}$ is called *retarded Green function*, and $G^{(-)}$ is called *advanced Green function*. The particular solutions of the wave equation are

$$\Psi^{(\pm)}(\mathbf{r},t) = \int\int G^{(\pm)}(\mathbf{r}-\mathbf{r}',t-t')f(\mathbf{r}',t')d\mathbf{r}'dt'. \tag{1.85}$$

At this solution sometimes is necessary to add a solution of the homogeneous equation, depending on the initial conditions. For instance, if for $t \to -\infty$ there is an incoming wave Ψ_{in} solution of the homogeneous equation, then, if the source f is localized in space and time, then

$$\Psi(\mathbf{r},t) = \Psi_{in}(\mathbf{r},t) + \int\int G^{(+)}(\mathbf{r}-\mathbf{r}',t-t')f(\mathbf{r}',t')d\mathbf{r}'dt'. \tag{1.86}$$

It means that in the past there was only the wave Ψ_{in}. On the contrary, if for $t \to \infty$ there is only a wave Ψ_{out}, then the solution is

$$\Psi(\mathbf{r},t) = \Psi_{out}(\mathbf{r},t) + \int\int G^{(-)}(\mathbf{r}-\mathbf{r}',t-t')f(\mathbf{r}',t')d\mathbf{r}'dt' \tag{1.87}$$

It means that once switched off the source f, there is only Ψ_{out}.

From it, we can write the expression of the retarded potential $\mathbf{A}$ and ϕ, solutions of the wave equations in the Lorentz gauge

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho_1}{\epsilon_0} \tag{1.88}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0 \mathbf{J}_1 \tag{1.89}$$

Using the retarded Green function, they are

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int d\mathbf{r}' \left[\frac{\rho_1(\mathbf{r}', t')}{R}\right]_{\text{rit}} \tag{1.90}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d\mathbf{r}' \left[\frac{\mathbf{J}_1(\mathbf{r}', t')}{R}\right]_{\text{rit}} \tag{1.91}$$

where the squared parenthesis means that the time t' must be evaluated at the time $t' = t - |\mathbf{r} - \mathbf{r}'|/c$.

Capitolo 2

Plane electromagnetic waves and linear dispersive media

We describe the propagation of plane waves in linear dielectric media.

2.1 Electromagnetic plane waves in a dielectric medium

A fundamental fact of the Maxwell's equation is the existence of solutions in the form of propagating waves, carrying energy from one point of the space to an other. The simplest example of these waves is that of a transverse plane wave. We consider a medium with uniform electric and magnetic constant, ϵ and μ, and infinitely extended. In absence of other sources, the Maxwell equations for each monochromatic component are

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \quad , \qquad \boldsymbol{\nabla} \cdot \mathbf{D} = 0 \tag{2.1}$$

$$\boldsymbol{\nabla} \times \mathbf{E} - i\omega \mathbf{B} = 0 \quad , \qquad \boldsymbol{\nabla} \times \mathbf{H} + i\omega \mathbf{D} = 0 \tag{2.2}$$

where for a linear, isotropic and uniform media, $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. We assume ϵ and μ real (so we neglect losses). Then, the equations for $\mathbf{E}$ and $\mathbf{B}$ are

$$\boldsymbol{\nabla} \times \mathbf{E} - i\omega \mathbf{B} = 0 \quad , \qquad \boldsymbol{\nabla} \times \mathbf{B} + i\omega\mu\epsilon \mathbf{E} = 0 \tag{2.3}$$

The other two equations with the divergence follow from these (since $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{V} = 0$). Combining the two equations we obtain the Helmoltz equation:

$$(\nabla^2 + \mu\epsilon\omega^2) \left\{ \begin{array}{c} \mathbf{E} \\ \mathbf{B} \end{array} \right\} = 0 \tag{2.4}$$

Considering a plane wave propagating along x, proportional to $e^{ikx-i\omega t}$, then we obtain

$$k = \sqrt{\mu\epsilon}\omega = \frac{n\omega}{c} \tag{2.5}$$

where $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0} = c\sqrt{\mu\epsilon}$ is the refraction index. The general solution is

$$u(x,t) = ae^{ikx-i\omega t} + be^{-ikx-i\omega t} = ae^{ik(x-vt)} + be^{-ik(x+vt)} \tag{2.6}$$

where $v = \omega/k = c/n$ is the phase velocity. If the medium is without dispersion, such that μ and ϵ are independent on ω, then by linear superposition of the elementary solutions the general solution is

$$u(x,t) = f(x-vt) + g(x+vt)$$

where f and g are arbitrary functions.

Let's now consider an electromagnetic plane wave with frequency ω and wave vector $\mathbf{k} = k\mathbf{n}$, where $\mathbf{n}\cdot\mathbf{n} = 1$, satisfying the Maxwell equations. Taking the electric and magnetic fields as the real parts of the complex quantities

$$\mathbf{E} = \mathbf{E}_0 e^{ik\mathbf{n}\cdot\mathbf{r}-i\omega t} \quad , \quad \mathbf{B} = \mathbf{B}_0 e^{ik\mathbf{n}\cdot\mathbf{r}-i\omega t}, \tag{2.7}$$

they satisfy the Maxwell equations if $k^2 = \mu\epsilon\omega^2$ and

$$\mathbf{n}\cdot\mathbf{E}_0 = 0 \quad , \quad \mathbf{n}\cdot\mathbf{B}_0 = 0 \tag{2.8}$$

i.e. **E** and **B** are both perpendicular to the direction of propagation. The curl equations say that

$$\mathbf{B} = \frac{n}{c}\mathbf{n}\times\mathbf{E} \tag{2.9}$$

If $\mathbf{n}$ is real, they have the same phase. We introduce a set of three unit vectors mutual orthogonal, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ such that $\mathbf{E}_0 = \mathbf{e}_1 E_0$, $\mathbf{B}_0 = \mathbf{e}_2(n/c)E_0$. The wave here described is a transverse wave propagating along the direction $\mathbf{n}$, carrying an average flux of energy given by the real part of the Poynting vector

$$\mathbf{S} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* = \frac{n}{2c\mu}|E_0|^2\mathbf{n}$$

The average energy density is

$$u = \frac{1}{4}(\mathbf{E}\cdot\mathbf{D}^* + \mathbf{B}\cdot\mathbf{H}^*) = \frac{\epsilon}{2}|E_0|^2$$

so that $S = (c/n)u = vu$, where the phase velocity $v = c/n$ is also the energy transport velocity.

2.2 Linear and circular polarization

The plane wave (2.7) is a wave where the electric field has a constant direction $\mathbf{e}_1$, and it is said to be *linearly polarized* with polarization vector $\mathbf{e}_1$. We can form a more general wave by superposition of two linearly polarized wave along $\mathbf{e}_1$ ad $\mathbf{e}_2$:

$$E(\mathbf{r},t) = [\mathbf{e}_1 E_1 + \mathbf{e}_2 E_2] e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tag{2.10}$$

where E_1 and E_2 are in general complex. If they have the same phase, (2.10) is still a linearly polarized wave, whose polarization vector forms an angle $\theta = \arctan(E_2/E_1)$ with respect to $\mathbf{e}_1$ and whose amplitude is $E = \sqrt{E_1^2 + E_2^2}$. If E_1 and E_2 have different phases, the wave (2.10) is *elliptically polarized*. To discuss a simpler case, we consider the *circular polarization*, where E_1 and E_2 have the same modulus and a phase difference of 90°. The wave (2.10) becomes

$$\mathbf{E}(\mathbf{r},t) = E_0[\mathbf{e}_1 \pm i\mathbf{e}_2] e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tag{2.11}$$

We choose the axis such that the propagation is along the positive z-axis and $\mathbf{e}_1$ and $\mathbf{e}_2$ are along the x and y axis, respectively. Then $E_x(\mathbf{r},t) = E_0 \cos(kz - \omega t)$ and $E_y(\mathbf{r},t) = \mp E_0 \sin(kz - \omega t)$. In a given point of the space, this wave describes a vector with constant module rotating with angular velocity ω counter-clockwise (when the sign is minus) or clockwise (when the sign is plus). The two waves have positive and negative helicity. In the first case the wave has a positive angular momentum (for $\mathbf{e}_1 + i\mathbf{e}_2$) and negative angular momentum (for $\mathbf{e}_1 - i\mathbf{e}_2$). We introduce the unit vectors

$$\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2) \tag{2.12}$$

satisfying the relations: $\mathbf{e}_\pm^* \cdot \mathbf{e}_\mp = 0$, $\mathbf{e}_\pm^* \cdot \mathbf{n} = 0$ and $\mathbf{e}_\pm^* \cdot \mathbf{e}_\pm = 1$. Then we can write

$$\mathbf{E}(\mathbf{r},t) = [E_+\mathbf{e}_+ + E_-\mathbf{e}_-] e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tag{2.13}$$

where E_+ and E_- are complex amplitudes. If E_+ and E_- have different modules but the same phase, it represents a wave elliptically polarized with the principal axis along $\mathbf{e}_1$ and $\mathbf{e}_2$. The ratio of the major and minor semi-axis is $(1+r)/(1-r)$ with $r = E_-/E_+$. If the amplitudes have a phase difference, $E_-/E_+ = re^{i\alpha}$, the ellipse is rotated by $\alpha/2$. If $r = \pm 1$, we obtain the case of a linear polarization. The polarization of the wave can be quantified by the so-called Stokes parameters, which can be determined by measuring the intensity in the presence of linear and circular polarizers. We refer to the textbook for a complete discussion of the Stokes parameters (see Jackson, Sec. 7.2).

2.3 Reflection and refraction at a plane surface between two dielectric media

We consider the reflection and the refraction of a monochromatic plane wave at a plane interface of separation between two media with different indexes. We divide their features into kinematic and dynamical properties. The firsts concern the direction of the reflected and refracted beams, and follow from the wave character of the phenomenon and from the boundary conditions at $z = 0$. The seconds follow from the physical properties of the electric and magnetic fields and from their boundary conditions.

We consider two media below and above the plane $z = 0$ (see figure 2.1), with dielectric constant and magnetic permeability ϵ, μ for $z < 0$ and ϵ', μ' for $z > 0$. The refraction indexes are $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ and $n' = \sqrt{\epsilon'\mu'/\epsilon_0\mu_0}$, respectively. A plane wave with frequency ω and wave vector $\mathbf{k}$ is incident on the plane $z = 0$ from below with an angle θ with respect to the normal to the plane $z = 0$. The refracted and reflected waves have the *same frequency* and wave-vectors $\mathbf{k}'$ and $\mathbf{k}''$, making the angles θ' and θ'' with respect to the unit vector $\mathbf{n}$, pointing upward. The incident, refracted and reflected waves are

$$\mathbf{E} = \mathbf{E}_0\, e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \qquad \mathbf{B} = \frac{n}{ck}\mathbf{k}\times\mathbf{E} \qquad \text{INCIDENT} \tag{2.14}$$

$$\mathbf{E}' = \mathbf{E}'_0 e^{i\mathbf{k}'\cdot\mathbf{r}-i\omega t}, \qquad \mathbf{B}' = \frac{n'}{ck'}\mathbf{k}'\times\mathbf{E}' \qquad \text{REFRACTED} \tag{2.15}$$

$$\mathbf{E}'' = \mathbf{E}''_0 e^{i\mathbf{k}''\cdot\mathbf{r}-i\omega t}, \qquad \mathbf{B}'' = \frac{n}{ck}\mathbf{k}''\times\mathbf{E}'' \qquad \text{REFLECTED} \tag{2.16}$$

The module of the wave vectors are $|\mathbf{k}| = |\mathbf{k}''| = k = n\omega/c$ and $|\mathbf{k}'| = k' = n'\omega/c$. The spatial and temporal variations of all the three waves must be the same at $z = 0$, so that their phase factors must be equal at $z = 0$,

$$(\mathbf{k}\cdot\mathbf{r})_{z=0} = (\mathbf{k}'\cdot\mathbf{r})_{z=0} = (\mathbf{k}''\cdot\mathbf{r})_{z=0} \tag{2.17}$$

These equations contain the kinematic aspects of refraction and reflection:

1. The wave vectors of the incident, refracted and reflected waves must lie in the same plane;

2. $k\sin\theta = k'\sin\theta' = k''\sin\theta''$. Since $k'' = k$, then

$$\theta'' = \theta \qquad \text{(reflection law)} \tag{2.18}$$

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and since $k'/k = n'/n$ then

$$n \sin\theta = n' \sin\theta' \qquad \text{(Snell's or Cartesio's law of refraction)} \tag{2.19}$$

The dynamical properties are contained in the boundary conditions for the electric and magnetic fields at the interface $z = 0$: the orthogonal components of $\mathbf{D}$ and $\mathbf{B}$ and the longitudinal components of $\mathbf{E}$ and $\mathbf{H}$ must be continue. Since $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, the four conditions are:

$$[\epsilon(\mathbf{E}_0 + \mathbf{E}_0'') - \epsilon'\mathbf{E}_0']_\perp = 0 \tag{2.20}$$

$$[\mathbf{k} \times \mathbf{E}_0 + \mathbf{k}'' \times \mathbf{E}_0'' - \mathbf{k}' \times \mathbf{E}_0']_\perp = 0 \tag{2.21}$$

$$[\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0']_\parallel = 0 \tag{2.22}$$

$$\left[\frac{1}{\mu}(\mathbf{k} \times \mathbf{E}_0 + \mathbf{k}'' \times \mathbf{E}_0'') - \frac{1}{\mu'}\mathbf{k}' \times \mathbf{E}_0'\right]_\parallel = 0 \tag{2.23}$$

We consider two cases: (a) the incident wave has its electric field perpendicular to the incidence plane (i.e. the plane containing $\mathbf{k}$ and $\mathbf{n}$) and (b) the incident wave has its electric field parallel to the incidence plane.

$\mathbf{E}_0$ perpendicular to the incident plane.

In this case the electric field points inward the page of fig.2.1. The magnetic field is such the energy flux is in the direction of the wave vectors. From Eqs.(2.22) and (2.23):

$$E_0 + E_0'' - E_0' = 0 \tag{2.24}$$

$$\frac{k}{\mu}(E_0 - E_0'')\cos\theta - \frac{k'}{\mu'}E_0'\cos\theta' = 0 \tag{2.25}$$

Assuming $\mu \sim \mu'$ and $k = n(\omega/c)$, $k' = n'(\omega/c)$, we obtain

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta}{n\cos\theta + n'\cos\theta'} = \frac{2n\cos\theta}{n\cos\theta + \sqrt{n'^2 - n^2\sin^2\theta}} \tag{2.26}$$

$$\frac{E_0''}{E_0} = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'} = \frac{n\cos\theta - \sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \sqrt{n'^2 - n^2\sin^2\theta}} \tag{2.27}$$

where we used the Snell's law $n\sin\theta = n'\sin\theta'$ in order to express $\cos\theta'$ as a function of the incidence angle θ.

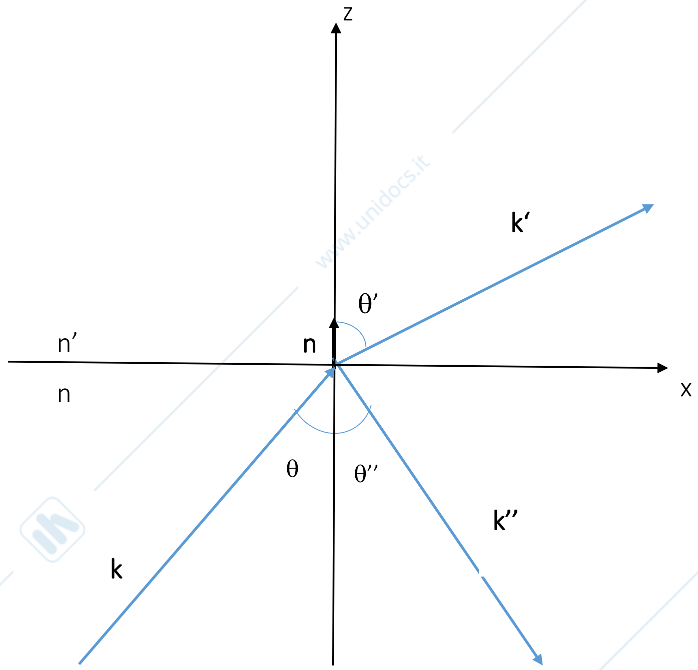

Figura 2.1: Scheme of reflection and refraction from a plane interface between two media with indexes n an n'. A plane wave with wave vector $\mathbf{k}$ is incident on the plane at $z = 0$ from below; the refracted and reflected waves have wave vectors $\mathbf{k}'$ and $\mathbf{k}''$.

$\mathbf{E}_0$ parallel to the incident plane.

In this case the magnetic field is perpendicular to the page of fig.2.1, pointing upward. We use Eqs.(2.22) and (2.23), assuming again $\mu \sim \mu'$:

$$(E_0 - E_0'')\cos\theta - E_0'\cos\theta' = 0 \tag{2.28}$$
$$n(E_0 + E_0'') - n'E_0' = 0 \tag{2.29}$$

which give

$$\frac{E_0'}{E_0} = \frac{2n\cos\theta}{n'\cos\theta + n\cos\theta'} = \frac{2nn'\cos\theta}{n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \tag{2.30}$$
$$\frac{E_0''}{E_0} = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'} = \frac{n'^2\cos\theta - n\sqrt{n'^2 - n^2\sin^2\theta}}{n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}} \tag{2.31}$$

Notice that for normal incidence, $\theta = 0$, the electric field is always parallel to the interface. Then, for the conditions (2.22) and (2.23) we obtain

$$\frac{E_0'}{E_0} = \frac{2n}{n+n'}, \qquad \frac{E_0''}{E_0} = \frac{n-n'}{n+n'} \tag{2.32}$$

The reflected wave in the case of normal incidence suffers a phase inversion if $n' > n$. In general, E_0 and E_0' have always the same sign, so that the incident and refracted waves are always *in phase* at the surface of separation. For the reflected wave we must discuss the different cases:

a) When **E** is perpendicular to the incident plane, we can write

$$\frac{E_0''}{E_0} = \frac{\sin(\theta' - \theta)}{\sin(\theta + \theta')} \tag{2.33}$$

Since $\theta' < \theta$ if $n < n'$, then the reflected wave and the incident wave have opposite phases if $n < n'$ and are in phase if $n > n'$;

b) When **E** is parallel to the incident plane, we can write

$$\frac{E_0''}{E_0} = \frac{\tan(\theta - \theta')}{\tan(\theta + \theta')} \tag{2.34}$$

If $n < n'$ then the reflected wave and the incident are in phase if $\theta + \theta' < \pi/2$ and with opposite phase if $\theta + \theta' > \pi/2$. The situation is reversed if $n > n'$.

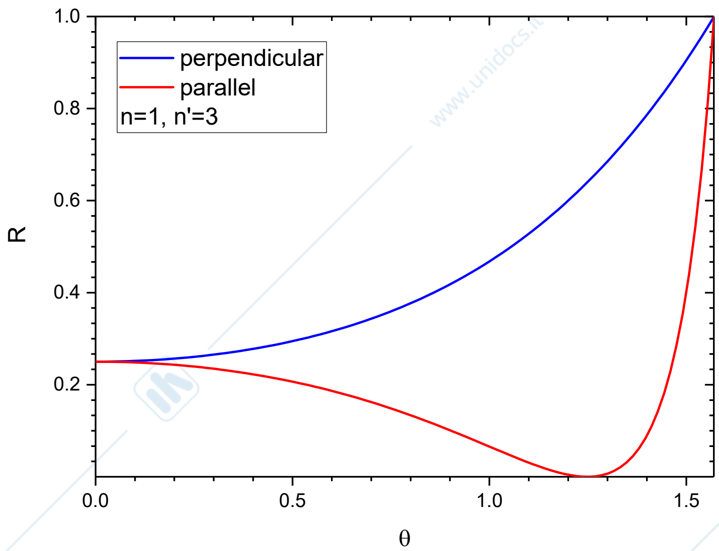

Figura 2.2: Reflection coefficient $R = |E_0''/E_0|^2$ as a function of the incident angle θ for $n = 1$ and $n' = 3$, for the incident field perpendicular and parallel to the incidence plane.

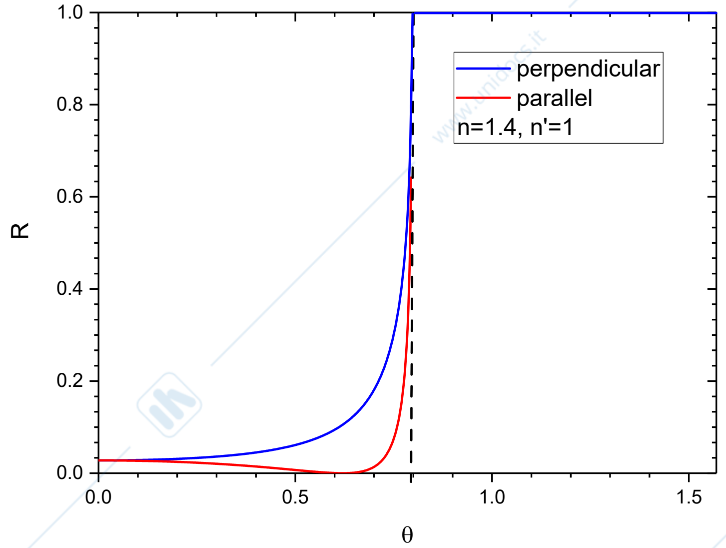

Figura 2.3: Reflection coefficient $R = |E_0''/E_0|^2$ as a function of the incident angle θ for $n = 1.4$ and $n' = 1$, for the incident field perpendicular and parallel to the incidence plane. The dashed line corresponds to the limit angle $\theta_L = \arcsin(n/n') = 0.795 = 45.6°$

Notice that for the case of electric field parallel to the incident plane, the reflection wave may vanish when $n' \cos\theta = n \cos\theta'$ i.e., using the Snell's law, when $\sin(2\theta') = \sin(2\theta)$ or $\theta' + \theta = \pi/2$. In this case the refracted beam would form a right angle with respect to the reflected beam, and the electric field of the reflected wave would point along the direction of the refracted beam. The incidence angle which meets this condition is called Brewster's angle, and it is equal to $\theta_B = \arctan(n'/n)$. Hence, when a non-polarized wave is incident on a refractive surface at the Brewster's angle, the reflected wave is totally linear polarized perpendicularly to the incidence plane, and so parallel to the surface. This effect can be used to eliminate the reflected light, for instance using polarized glasses with vertical polarization.

2.4 Total internal reflection

An other important effect is the total internal reflection, for light coming from a medium with $n > n'$. In this case if $n > n'$ then $\theta > \theta'$ and the maximum value $\theta' = \pi/2$ occurs when $\theta = \arcsin(n'/n) = \theta_L$. When $\theta = \theta_L$ the refractive wave propagates along the interface and there is not energy transport in the second medium. The light is totally reflected in the incoming medium. When $\theta > \theta_L$, $\sin\theta' > 1$, and this mean that θ' is a complex number with a cosine purely imaginary:

$$\cos\theta' = \sqrt{1 - \frac{\sin^2\theta}{\sin^2\theta_L}} = i\sqrt{\frac{\sin^2\theta}{\sin^2\theta_L} - 1} = i\xi$$

This implies that the refractive wave has a propagation factor

$$e^{i\mathbf{k}'\cdot\mathbf{r}} = e^{ik'(\cos\theta' z + \sin\theta' x)} = e^{-k'\xi z} e^{ik'\sin\theta' x}$$

It means that the refracted wave propagates along the interface with an amplitude exponentially decreasing as the distance from the interface increases. Also if the wave is non zero in the second medium, there is no energy transport of it, as can be verified calculating the average Poynting vector orthogonal to the surface:

$$\mathbf{S}\cdot\mathbf{n} = \frac{1}{2}\mathrm{Re}[\mathbf{n}\cdot(\mathbf{E}' \times \mathbf{H}'^*)] = \frac{1}{2\omega\mu'}\mathrm{Re}[(\mathbf{n}\cdot\mathbf{k}')|E_0'|^2)]$$

But $\mathbf{n}\cdot\mathbf{k}' = k'\cos\theta'$ which is purely imaginary, so that $\mathbf{S}\cdot\mathbf{n} = 0$. The reflected wave in the case of total internal reflection is $E_0''/E_0 = (1 - ih)/(1 + ih)$ where $h = \sin\theta_L \xi / \cos\theta$ for $\mathbf{E}$ perpendicular to the incident plane and $h = \xi/(\sin\theta_L \cos\theta)$ for $\mathbf{E}$ parallel to the incident plane. The module is unity, but the reflected light suffers a phase change, different for the two kinds of incidence and depending on the incidence angle. The phase shift is $\tan(\psi/2) = \sqrt{\sin^2\theta - \sin^2\theta_L}/\cos\theta$ for incidence perpendicular to the incidence plane and $\tan(\psi/2) = \sqrt{\sin^2\theta - \sin^2\theta_L}/(\sin^2\theta_L\cos\theta)$ for incidence parallel to the incidence plane. Hence, in general in the case of an incident light linearly polarized in a plane which is neither parallel or perpendicular to the incident plane, the total reflection changes a linear polarization into an elliptical polarization.

The total internal reflection is the origin of the *Goos-Hänchen effect*, where a beam with a finite transverse extension is displaced after internal total reflection.

Total internal reflection has as an important application in the light propagation in optical fibers.

2.5 Linear dielectric media

We now reconsider the initial assumption of a linear, isotropic dielectric medium described by a macroscopic polarization vector $\mathbf{P}(\mathbf{r}, t)$. The Maxwell equation is

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2} \tag{2.35}$$

We express the electric field and the polarization in Fourier plane waves, in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega \tilde{\mathbf{E}}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{2.36}$$

$$\mathbf{P}(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega \tilde{\mathbf{P}}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{2.37}$$

This has the advantage that the differential equation transforms into an algebraic equation for the Fourier amplitudes:

$$\left\{-k^2 + \frac{\omega^2}{c^2}\right\}\tilde{\mathbf{E}}(\mathbf{k}, \omega) = -\mu_0\omega^2\tilde{\mathbf{P}}(\mathbf{k}, \omega). \tag{2.38}$$

Let's now introduce the *linearity* hypotesis:

$$\tilde{\mathbf{P}}(\mathbf{k}, \omega) = \epsilon_0\chi(\omega)\tilde{\mathbf{E}}(\mathbf{k}, \omega) \tag{2.39}$$

where $\chi(\omega)$ is the electric susceptivity (we assume the medium is isotropic, such that $\mathbf{P}$ and $\mathbf{E}$ are parallel (otherwise χ will be a tensor) and homogeneous, such that $\chi(\omega)$ is independent on the wave vector $\mathbf{k}$. By substitution, we obtain

$$\left\{-k^2 + \frac{\omega^2}{c^2} + \epsilon_0\mu_0\omega^2\chi(\omega)\right\}\tilde{\mathbf{E}}(\mathbf{k}, \omega) = 0 \tag{2.40}$$

and

$$k^2(\omega) = \frac{\omega^2}{c^2}\{1+\chi(\omega)\} \tag{2.41}$$

Suppose $\omega > 0$ and $\chi(\omega) = \chi_1(\omega) + i\chi_2(\omega)$ a complex function of ω. Then, if $\chi(\omega) \ll 1$, $k(\omega) = \beta + i\alpha/2$ where $\beta = (\omega/c)(1+\chi_1(\omega)/2)$ and $\alpha = (\omega/c)\chi_2(\omega)$. A plane wave of the form $\exp[i(kx-\omega t)]$ propagating inside the medium has the form $e^{-\alpha x/2}e^{i(\beta x-\omega t)}$. Hence, the imaginary part $\chi_2(\omega)$ is responsible of an exponential growth or decay, whereas the real part $\chi_1(\omega)$ is responsible of a phase variation (dispersion), such that the wave has a phase velocity

$$v_p = \frac{\omega}{\beta} = \frac{c}{1+\chi_1(\omega)/2}.$$

Hence, each monochromatic field component travels with a different phase velocity. Since a wave packet is obtained overlapping many monochromatic plane wave components, in a dispersive medium the wave packet will be deformed.

2.5.1 Group velocity

Let consider a component of the electric field, $E(x,t)$, propagating along the x-axis as a wave packet[1]:

$$E(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}(k)e^{ikx-i\omega(k)t}dk \tag{2.42}$$

We assume k and $\omega(k)$ reals, and in this way we exclude dissipation effects. The Fourier transform $\tilde{E}(k)$ is given by the Fourier transform of the spatial amplitude $E(x,t)$ evaluated at $t=0$:

$$E(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}(k)e^{ikx}dk \to \tilde{E}(k) = \int_{-\infty}^{\infty}E(x,0)e^{-ikx}dx. \tag{2.43}$$

If $E(x,0)$ represents an harmonic wave e^{ik_0x} for all x, then $\tilde{E}(k) = 2\pi\delta(k-k_0)$, representing a monochromatic traveling wave $e^{ik_0x-i\omega(k_0)t}$, as required. If at $t=0$ $E(x,0)$ represents a finite wave train with a length of order Δx, then the amplitude $\tilde{E}(k)$ is not a delta function. Rather, it is a peaked function with a width of the order of Δk, centered around k_0 which is the dominant wave number in the modulated wave $E(x,0)$. If Δx and Δk are defined as the rms deviation from the average

[1] We express $\omega(k)$ as a function of k instead of $k(\omega)$, supposing that it is possible to invert the dispersion relation (2.41). This is convenient when one considers an initial condition problem (Cauchy problem).

of x and k, then $\Delta x \, \Delta k \sim 1$. This means that short wave trains with only a few wavelengths present have a very wide distribution of wave numbers of monochromatic waves, and conversely that long sinusoidal wave trains are almost monochromatic.

If the distribution $\tilde{E}(k)$ is fairly sharply peaked around k_0, the integral is appreciable only around k_0. If $\omega(k)$ depends weakly on k then it can be expanded around k_0:

$$\omega(k) = \omega_0 + (k - k_0)\omega'(k_0) + \dots$$

(where $\omega_0 = \omega(k_0)$) and the integral performed. Thus

$$\begin{aligned} E(x,t) &= \frac{1}{2\pi} e^{-i[\omega_0 - k_0\omega'(k_0)]t} \int_{-\infty}^{\infty} \tilde{E}(k) e^{ik[x-\omega'(k_0)t]} dk \\ &= E(x - v_g t, 0) e^{i[k_0\omega'(k_0) - \omega_0]t} \end{aligned}$$

This show that, apart from an overall phase factor, the pulse travels along undistorted in shape with a velocity, called *group velocity*, $v_g = \omega'(k_0)$. Hence, far from the regions where $\omega(k)$ varies abruptly (for instance near a resonance), it is possible to define a group velocity which describes to energy transport of the wave packet.

An important parameter is the (relative) dielectric constant, $\epsilon_r(\omega) = \epsilon(\omega)/\epsilon_0 = 1 + \chi(\omega)$, such that $k(\omega) = (\omega/c)n(\omega)$, where $n(\omega) = \sqrt{\epsilon_r(\omega)}$ is the refraction index. With these definitions, the phase velocity is $v_p = c/n(\omega)$ and the group velocity $v_g^{-1} = dk/d\omega = (n + \omega dn/d\omega)/c$, so that

$$v_g = \frac{c}{n(\omega) + \omega(dn/d\omega)}$$

Hence,

a) for $dn/d\omega = 0$ there is no dispersion and $v_g = v_p = c/n$;

b) for $dn/d\omega > 0$ and $n > 1$ (normal dispersion) $v_g < v_p < c$;

c) for $dn/d\omega$ large and negative (anomalous dispersion) v_g differs greatly from the phase velocity, often becoming larger than c.

In general the last case is associated with large absorption and an abrupt change of $n(\omega)$, so that the approximations made are no longer valid and the behavior of the pulse is much more complicated.

2.5.2 Causality and analyticity

We now discuss the relationship between the principle of causality and the mathematical properties of the dielectric susceptivity. In the hypothesis of the linear medium, $\tilde{P}(\omega) = \epsilon_0\chi(\omega)\tilde{E}(\omega)$, the electric displacement is

$$\tilde{D}(\omega) = \epsilon_0\tilde{E}(\omega) + \tilde{P}(\omega) = \epsilon_0(1+\chi(\omega))\tilde{E}(\omega) = \epsilon_0\epsilon_r(\omega)\tilde{E}(\omega)$$

The dependence on ω in $\chi(\omega)$ means that the response of the medium to the electromagnetic field is non local in time. Taking the Fourier transform,

$$\begin{aligned} P(t) &= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{+\infty} d\omega\chi(\omega)\tilde{E}(\omega)e^{-i\omega t} \\ &= \frac{\epsilon_0}{2\pi}\int_{-\infty}^{+\infty} d\omega\chi(\omega)\left\{\int_{-\infty}^{+\infty} dt' E(t')e^{i\omega t'}\right\}e^{-i\omega t} \\ &= \epsilon_0\int_{-\infty}^{+\infty} dt' E(t')\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\chi(\omega)e^{-i\omega(t-t')} \\ &= \int_{-\infty}^{+\infty} d\tau E(t-\tau)G(\tau) \end{aligned} \tag{2.44}$$

where

$$G(\tau) = \frac{\epsilon_0}{2\pi}\int_{-\infty}^{+\infty} d\omega\chi(\omega)e^{-i\omega\tau}. \tag{2.45}$$

$P(t)$ is a convolution product between E and G. In general, τ can be positive or negative. However, if $\tau < 0$ the polarization $P(t)$ should depend from the electric field at times earlier than t, in violation of the causality principle (it can not be polarization before the arrival of the electric field!). So, we must impose that $G(\tau) = 0$ for $\tau < 0$, so that:

$$\mathbf{P}(\mathbf{r},t) = \int_0^{+\infty} d\tau\mathbf{E}(\mathbf{r},t-\tau)G(\tau). \tag{2.46}$$

If $\chi(\omega) = \chi_0$ does not depend on ω, $G(\tau) = \epsilon_0\chi_0\delta(\tau)$ and $\mathbf{P}(\mathbf{r},t) = \epsilon_0\chi_0\mathbf{E}(\mathbf{r},t)$: the medium's response to the electric field is instantaneous. The relation (2.46) is the most general spatially local, linear and causal relation that can be written between $\mathbf{P}$ and $\mathbf{E}$ in a uniform isotropic medium. Its validity transcends any specific model of $\epsilon(\omega)$. This relation has several interesting consequences. If $\chi(\omega)$ is viewed as a function in the complex ω plane, with $\omega = \omega_1 + i\omega_2$, then the integral of Eq.(2.45) can be evaluated by the residues integral, closing the path on the real axis with a semi-circle of radius R in the upper or lower half-plane; due to the factor $\exp(-i\omega\tau) \propto \exp(\omega_2\tau)$, the path must be closed in the upper half-plane when $\tau < 0$, or in the lower half-plane when $\tau > 0$. Since $G(\tau) = 0$ for $\tau < 0$, then $\chi(\omega)$ must be an analytic function in the upper half-plane. Hence, $\chi(\omega)$ does not have singularities in the upper half-plane.

2.5.3 Kramers-Kronig relations

Let's consider the integral:

$$I(\omega) = \oint_C \frac{\chi(\omega')}{\omega' - \omega} d\omega' \tag{2.47}$$

on a path C composed by the real axis excluding the pole $\omega' = \omega$ by a small half circle clockwise of radius ϵ and a large half circle counter-clockwise with radius R. Since the path C is closed and $\chi(\omega)$ is analytic in the upper half plane, then $I(\omega) = 0$. In the limit of large R and small ϵ the integral is equal to:

$$I(\omega) = P \int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega} d\omega' - i\pi\chi(\omega) \tag{2.48}$$

where P means principal part. Therefore

$$\chi(\omega) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega} d\omega'. \tag{2.49}$$

By separating the real and imaginary part, $\chi = \chi_1 + \chi_2$, we find

$$\chi_1(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi_2(\omega')}{\omega' - \omega} d\omega' \tag{2.50}$$

$$\chi_2(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi_1(\omega')}{\omega' - \omega} d\omega' \tag{2.51}$$

These are the *Kramers-Kronig* (or dispersion) relations. The relations express the connection between absorption and dispersion. They are of very general validity, following from the assumption of the causality connection between the polarization and the electric field. Empirical knowledge of $\chi_2(\omega)$ from absorption studies allows the calculation of $\chi_1(\omega)$. It is also possible to show that, since P and E are real, then $G(\tau) = G^*(\tau)$ whereas $\chi_1(\omega) = \chi_1(-\omega)$ (even function) and $\chi_2(\omega) = -\chi_2(-\omega)$ (odd function).

2.5.4 The envelope equations*

Let's return to the Maxwell equation, assuming a propagation of a plane wave along the z-axis,

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2} \tag{2.52}$$

and let's assume that the field is a quasi-monochromatic wave, with a distribution peaked around a value $\omega = ck$. Then the electric field $E(z,t)$ and the polarization $P(z,t)$ have the form:

$$E(z,t) = \frac{1}{2}\left\{E_0(z,t)e^{i(kz-\omega t)} + \text{c.c.}\right\} \tag{2.53}$$

$$P(z,t) = \frac{1}{2}\left\{P_0(z,t)e^{i(kz-\omega t)} + \text{c.c.}\right\} \tag{2.54}$$

where c.c. means complex conjugate. If the field is quasi-monochromatic, the complex amplitudes $E_0(z,t)$ and $P_0(z,t)$ are slowly varying in z and t within a spatial and temporal period $2\pi/k$ and $2\pi/\omega$:

$$\frac{\partial E_0}{\partial z} \ll kE_0 \qquad \frac{\partial E_0}{\partial t} \ll \omega E_0 \tag{2.55}$$

$$\frac{\partial P_0}{\partial z} \ll kP_0 \qquad \frac{\partial P_0}{\partial t} \ll \omega P_0 \tag{2.56}$$

Then, we can write Eq.(2.52) as

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(E_0 e^{i\psi} + \text{c.c.}\right) = \mu_0 \frac{\partial^2}{\partial t^2}\left(P_0 e^{i\psi} + \text{c.c.}\right) \tag{2.57}$$

where $\psi = k(z - ct)$. Since

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) E_0 e^{i\psi} = e^{i\psi}\left(\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t}\right)$$

$$\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right) E_0 e^{i\psi} = e^{i\psi}\left(\frac{\partial E_0}{\partial z} - \frac{1}{c}\frac{\partial E_0}{\partial t}\right) + 2ikE_0 e^{i\psi}$$

Then

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E_0 e^{i\psi} = 2ike^{i\psi}\left(\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t}\right) + e^{i\psi}\left(\frac{\partial^2 E_0}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_0}{\partial t^2}\right) \tag{2.58}$$

and

$$\frac{\partial^2}{\partial t^2} P_0 e^{i\psi} = e^{i\psi}\left\{\frac{\partial^2 P_0}{\partial t^2} - 2i\omega\frac{\partial P_0}{\partial t} - \omega^2 P_0\right\} \tag{2.59}$$

Neglecting the second derivatives of E_0 and the second and first derivatives of P_0, we find

$$2ik\left(\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t}\right) = -\mu_0\omega^2 P_0$$

or, since $\omega = ck$ and $c^2 = 1/\epsilon_0\mu_0$,

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = \frac{ik}{2\epsilon_0} P_0. \tag{2.60}$$

The approximation (2.55) and (2.56) is named *'Slowly Varying Envelope approximation' (SVEA)* and is very useful in laser and atomic physics. It allows to reduce the second-order Maxwell equation to a first-order equation for the slowly varying complex envelope propagating in the forward direction.

Setting $P_0 = \epsilon_0 \chi E_0$ with $\chi = \chi_1 + i\chi_2$ and writing $E_0 = |E_0| \exp(i\phi)$, Eq.(2.60) splits into:

$$\frac{\partial |E_0|}{\partial z} + \frac{1}{c}\frac{\partial |E_0|}{\partial t} = -\frac{k}{2}\chi_2 |E_0| \tag{2.61}$$

$$\frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial \phi}{\partial t} = \frac{k}{2}\chi_1 \tag{2.62}$$

Since the intensity is $I = c\epsilon_0 |E_0|^2/2$, Eq.(2.61) is

$$\frac{\partial I}{\partial z} + \frac{1}{c}\frac{\partial I}{\partial t} = -k\chi_2 I \tag{2.63}$$

showing that the imaginary part (named also 'in quadrature' part) of χ is responsible of absorption or amplification of the wave propagating through the material, whereas the real part (or 'in phase' part) of χ is responsible of its dispersion.

In the stationary case, $I(z) = I(0)\exp[-\alpha z]$, where $\alpha = k\chi_2$ is the linear absorption coefficient, and $\phi(z) = (k/2)\chi_1 z$ such that the refractive index is $n = 1 + \chi_1/2$.

2.6 Electromagnetic energy in dispersive material with losses

The electromagnetic energy has been defined neglecting dispersion, i.e assuming ϵ and μ real and independent on the frequency. In general, the materials are dispersive, so we want to extend the Poynting's theorem to the case where ϵ and μ are complex and depending on the frequency. We introduce the Fourier transforms in time:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi}\int d\omega \tilde{\mathbf{E}}(\mathbf{r}, \omega) e^{-i\omega t} \tag{2.64}$$

$$\mathbf{D}(\mathbf{r}, t) = \frac{1}{2\pi}\int d\omega \tilde{\mathbf{D}}(\mathbf{r}, \omega) e^{-i\omega t} \tag{2.65}$$

and as before we make the hypothesis of linearity for each Fourier component:

$$\tilde{\mathbf{D}}(\mathbf{r}, \omega) = \epsilon(\omega)\tilde{\mathbf{E}}(\mathbf{r}, \omega) \tag{2.66}$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \mu(\omega)\tilde{\mathbf{H}}(\mathbf{r}, \omega) \tag{2.67}$$

Since the fields are real, then $\tilde{\mathbf{E}}(\mathbf{r},-\omega)=\tilde{\mathbf{E}}^*(\mathbf{r},\omega)$ and $\tilde{\mathbf{D}}(\mathbf{r},-\omega)=\tilde{\mathbf{D}}^*(\mathbf{r},\omega)$, so that $\epsilon(-\omega)=\epsilon^*(\omega)$ (the same is true for the magnetic field). As a consequence, the term $\mathbf{E}\cdot(\partial\mathbf{D}/\partial t)$ is not simply equal to $(1/2)(\partial/\partial t)(\mathbf{E}\cdot\mathbf{D})$. Using the Fourier transform, we write

$$\begin{aligned}
\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} &= \frac{1}{(2\pi)^2}\int d\omega'\int d\omega\,\tilde{\mathbf{E}}(\omega')[-i\omega\epsilon(\omega)]\cdot\tilde{\mathbf{E}}(\omega)e^{-i(\omega+\omega')t}\\
&= \frac{1}{4\pi^2}\int d\omega''\int d\omega\,\tilde{\mathbf{E}}^*(\omega'')[-i\omega\epsilon(\omega)]\cdot\tilde{\mathbf{E}}(\omega)e^{-i(\omega-\omega'')t}\\
&= \frac{1}{8\pi^2}\int d\omega\int d\omega'\tilde{\mathbf{E}}^*(\omega')[-i\omega\epsilon(\omega)+i\omega'\epsilon^*(\omega')]\cdot\tilde{\mathbf{E}}(\omega)e^{-i(\omega-\omega')t}
\end{aligned}$$

where in the second line we changed $\omega'\to-\omega''$, and in the second term of the third line we changed $\omega\to-\omega'$ and $\omega''\to-\omega$. Supposing that the fields have their frequency components in a small interval $\omega-\omega'$ where $\epsilon(\omega)$ varies slowly, we can expand the factor $\omega'\epsilon^*(\omega')$ around $\omega'=\omega$:

$$[-i\omega\epsilon(\omega)+i\omega'\epsilon^*(\omega')]\approx 2\omega\,\mathrm{Im}\,\epsilon(\omega)+i(\omega'-\omega)\frac{d}{d\omega}(\omega\epsilon^*(\omega))+\ldots$$

By substitution we obtain

$$\begin{aligned}
\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} &= \frac{1}{4\pi^2}\int d\omega'\int d\omega\,[\omega\,\mathrm{Im}\,\epsilon(\omega)\,]\tilde{\mathbf{E}}^*(\omega')\cdot\tilde{\mathbf{E}}(\omega)e^{-i(\omega-\omega')t}\\
&+\frac{1}{8\pi^2}\frac{\partial}{\partial t}\int d\omega'\int d\omega\frac{d}{d\omega}[\omega\epsilon^*(\omega)]\,\tilde{\mathbf{E}}^*(\omega')\cdot\tilde{\mathbf{E}}(\omega)e^{-i(\omega-\omega')t}. \qquad (2.68)
\end{aligned}$$

A similar expression holds for $\mathbf{H}\cdot(\partial\mathbf{B}/\partial t)$ with $\mu(\omega)$ in place of $\epsilon(\omega)$. We observe that if ϵ and μ are real and independent on the frequency, since

$$\begin{aligned}
\frac{1}{4\pi^2}\int d\omega'\int d\omega\,\tilde{\mathbf{E}}^*(\omega')\cdot\tilde{\mathbf{E}}(\omega)e^{-i(\omega-\omega')t} &= |\mathbf{E}(t)|^2\\
\frac{1}{4\pi^2}\int d\omega'\int d\omega\,\tilde{\mathbf{H}}^*(\omega')\cdot\tilde{\mathbf{H}}(\omega)e^{-i(\omega-\omega')t} &= |\mathbf{H}(t)|^2
\end{aligned}$$

we obtain the usual relation for the energy density:

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\left[\epsilon|\mathbf{E}(t)|^2+\mu|\mathbf{H}(t)|^2\right]=\frac{\partial u}{\partial t}.$$

The first term of (2.68) describes the dissipation of the energy (proportional to the imaginary part of the dielectric constant) whereas the second term in the derivative describes the effective energy density:

$$u_{\text{eff}}=\frac{1}{8\pi^2}\int d\omega'\int d\omega\left\{\frac{d(\omega\epsilon^*)}{d\omega}\tilde{\mathbf{E}}^*(\omega')\cdot\tilde{\mathbf{E}}(\omega)+\frac{d(\omega\mu^*)}{d\omega}\,\tilde{\mathbf{H}}^*(\omega')\cdot\tilde{\mathbf{H}}(\omega)\right\}e^{-i(\omega-\omega')t}. \qquad (2.69)$$

For slowly varying fields $\mathbf{E}(t) = \mathbf{E}_0(t)\cos(\omega_0 t + \phi)$ and $\mathbf{H}(t) = \mathbf{H}_0(t)\cos(\omega_0 t + \phi)$, with a slowly varying envelope (with $\partial \mathbf{E}_0/\partial t \ll \omega_0 \mathbf{E}_0$), its Fourier transform is

$$\tilde{\mathbf{E}}(\omega) \approx \pi \mathbf{E}_0[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)].$$

Neglecting the fast oscillating terms as $e^{2i\omega_0 t}$, the effective energy density reduces to

$$u_{\text{eff}} = \frac{1}{4}\left\{ \text{Re}\left[\frac{d(\omega\epsilon)}{d\omega}\right]_{\omega=\omega_0} |\mathbf{E}_0|^2 + \text{Re}\left[\frac{d}{d\omega}(\omega\mu)\right]_{\omega=\omega_0} |\mathbf{H}_0|^2 \right\}. \tag{2.70}$$

Neglecting the imaginary parts of ϵ and μ, since the energy density must be positive the following inequality must be satisfied:

$$\frac{d}{d\omega}(\omega\epsilon) > 0 \tag{2.71}$$

$$\frac{d}{d\omega}(\omega\mu) > 0 \tag{2.72}$$

2.7 Metamaterials and negative refraction*

S. Veselago, in a paper published in 1968, discussed the consequences for e.m. wave propagating in a hypotetical material for which both the electric permittivity ϵ and the magnetic permeability μ were simultaneously negative. Because no naturally occurring material has ever been demonstrated with negative ϵ and μ, Veselago investigated the reasons of such apparent asymmetry. He concluded that such materials not only should be possible, but if ever found, they would exhibit remarkable properties, as negative index of refraction.

Originally, Veselago referred to these materials as *'left handed'*, because the wave vector is antiparallel to the usual right-handed cross product of the electric and magnetic fields. From the Maxwell equation, a plane wave propagating in a dielectric medium with relative dielectric and magnetic constants ϵ and μ with a wave vector $\mathbf{k}$ results in the relation $k^2 = \epsilon\mu\omega^2 = n^2\omega^2/c^2$, with $n^2 = \epsilon\mu$. Neglecting losses, we usually take the positive value of the square root $n = \sqrt{\epsilon\mu}$, but nothing preclude to take also its negative value. In this expression ϵ and μ are present with their product. But what happens if they are both negative? In order to understand the effect of changing the sign of ϵ and μ simultaneously, let consider the Maxwell equation where ϵ and μ appear separately:

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} \quad , \qquad \nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} \tag{2.73}$$

where we assumed $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ for a given frequency ω. For a plane wave propagating with wave vector $\mathbf{k}$,

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H} \quad , \qquad \mathbf{k} \times \mathbf{H} = -\omega\epsilon\mathbf{E} \tag{2.74}$$

If ϵ and μ are both positive, then the vector $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ form a right-handed triple (as $(\mathbf{x}, \mathbf{y}, \mathbf{z})$), whereas if ϵ and μ are both negative, they form a left-handed triple. Let's consider now the Poynting vector

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{2.75}$$

which always is directed outward with respect the source, independently on the sign of ϵ and μ. According to (2.75), $\mathbf{S}$ forms with $\mathbf{E}$ and $\mathbf{H}$ a right-handed triple. Then is clear that in the materials where ϵ and μ are both positive, $\mathbf{S}$ and $\mathbf{k}$ have the same direction, whereas in the materials where ϵ and μ are both negative, $\mathbf{S}$ and $\mathbf{k}$ have opposite direction. Since the phase velocity is $\mathbf{v}_p = (\omega/k)\hat{\mathbf{k}}$ (where $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$), we conclude that the e.m. waves propagate in the materials with ϵ and μ both negative with a *negative* phase velocity, opposite to the propagation direction of the energy. Since we can also write $\mathbf{k} = (\omega/c)n\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector in the direction of $\mathbf{S}$, we can say that in the materials with ϵ and μ negative the index of refraction is negative.

We observe that ϵ and μ can not be both negative without dispersion, otherwise the energy density of the electromagnetic field should be negative, which is impossible. These materials must be necessarily dispersive, with (2.71) and (2.72) not violated. In general, it is possible to have negative ϵ and μ if $\omega(d\epsilon/d\omega)$ and $\omega(d\mu/d\omega)$ are sufficiently positive. A negative refractive index implies many important consequences. The more striking is the reversal of wave refraction, as shown in fig.2.7. A negative-index material will refract light through a negative angle. The figure shows a simulation, but experiments in the microwave range has confirmed these predictions. In (a), a negative-index prism wedge with $\epsilon = -1$ and $\mu = -1$ deflects an electromagnetic beam by a negative angle relative to the normal to the surface: the beam emerges on the same side of the surface normal as the incident beam. In (b), a positive-index wedge, in contrast, will positively refract the same beam. Figure (c) and (d) show the deflection angle as a function of the frequency (vertical axis) for a beam traversing a negative-index wedge (c) and a positive-index wedge (d). In the negative-index wedge there is a strong dispersion with frequency: the condition $\epsilon = -1$ and $\mu = -1$ is realized only over a narrow bandwidth around 12GHz.

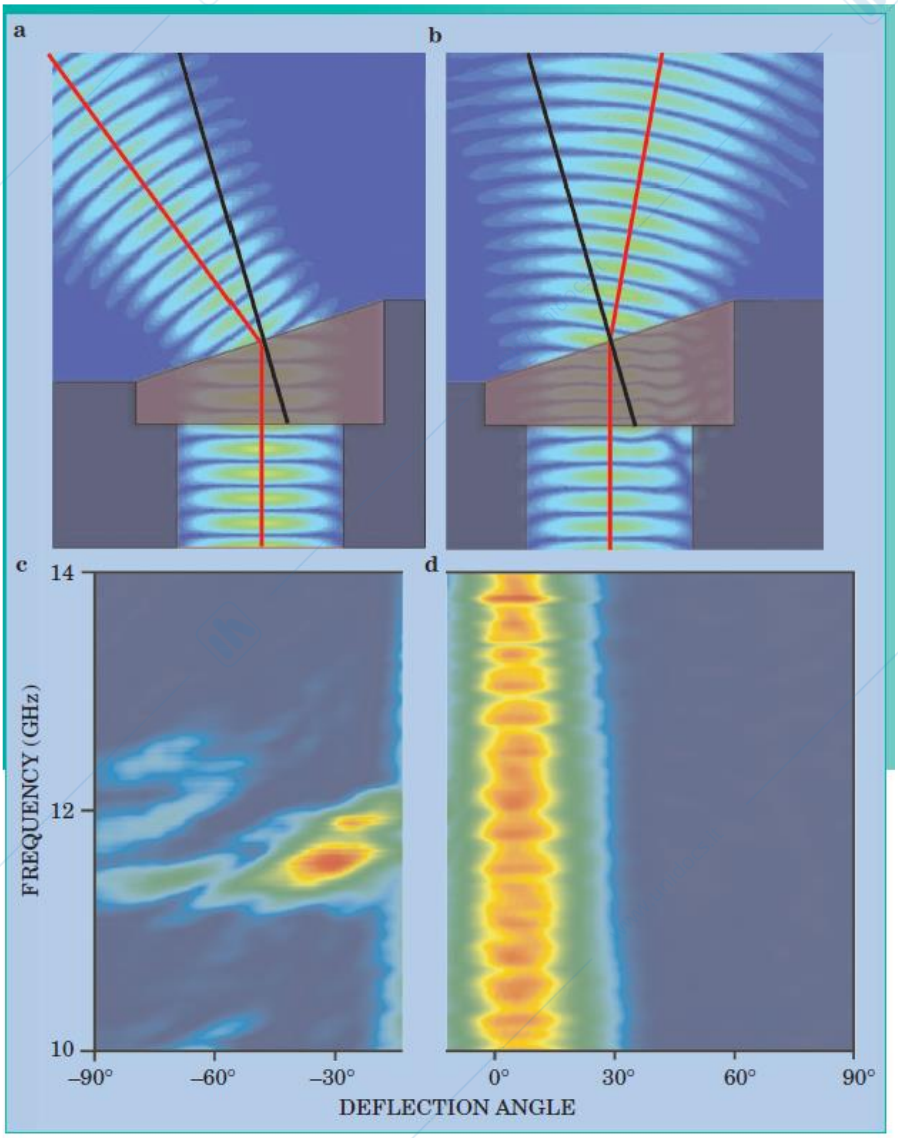

Figura 2.4: From J. Pendry and D. Smith, '*Reversing Light With Negative Refraction*', Physics Today 57, 37 (2004).

2.8 The Lorentz model

We develop a simple model of dispersion, considering the atomic medium as a collection of harmonically bound charges. We will neglect the difference between the applied field and the local field. The model is therefore appropriate only for substances of relatively low density. We describe the electron bound to the nucleus and oscillating around its equilibrium position with frequency ω_0 and a damping constant γ. For small oscillations around the equilibrium position, the force can be assumed linear, since the potential is $V(r) = V_0 + (\mathbf{r}-\mathbf{r}_0)\cdot\nabla V(r_0) + (1/2)|\mathbf{r}-\mathbf{r}_0|^2 V''(r_0)\cdots \sim V_0 + (1/2)m\omega_0^2 r^2$ (assuming $\mathbf{r}_0 = 0$). The equation of motion for an electron of charge $-e$ bound by a harmonic force and acted on by an electric field $\mathbf{E}(\mathbf{r},t)$ is

$$m[\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} + \omega_0^2\mathbf{r}] = -e\mathbf{E}(\mathbf{r},t).$$

We neglect the effect of the magnetic field. We make the additional assumption that the amplitude of the oscillations are small enough that the electric field can be evaluated at the average position of the electron. Since the spatial extension of the oscillation is of the order of the Bohr radius $a_0 \sim 10^{-10}\,m$ and the electric field oscillates at the optical wavelength $\lambda \sim 10^{-6}\,m$, the electric field can be assumed uniform over the atom if $\lambda \gg a_0$ (electric dipole approximation). If the field varies harmonically in time as $e^{-i\omega t}$, the dipole moment contributed by one electron is

$$\mathbf{p} = -e\mathbf{r} = \frac{e^2}{m}(\omega_0^2 - \omega^2 - i\omega\gamma)^{-1}\mathbf{E}$$

Since the polarization is $\mathbf{P} = n_a\mathbf{p} = \epsilon_0\chi(\omega)\mathbf{E}$, where n_a is the atomic density, then

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} \tag{2.76}$$

where $\omega_p = \sqrt{e^2 n_a/m\epsilon_0}$ is the plasma frequency ($\omega_p = 5.6\times 10^4\sqrt{n_a}\,s^{-1}$ where n_a is in atoms/cm^3). In spite of its simplicity, (2.76) is an accurate description of the atomic contribution to the dielectric constant.

The damping constant is generally small compared to the resonant frequency ω_0. This means that $\epsilon(\omega) = 1 + \chi(\omega)$ is approximately real for most frequencies. The factor $(\omega_0^2 - \omega^2)^{-1}$ is positive for $\omega < \omega_0$ and negative for $\omega > \omega_0$. Thus, at low frequencies $\epsilon(\omega)$ is greater than unity whereas for frequency larger than ω_0 $\epsilon(\omega)$ is less than one. In the neighborhood of ω_0 there is a rather violent behavior: the real part of the denominator in (2.76) vanishes and the term is large and purely imaginary.

The real and imaginary parts of the dielectric constant are

$$\mathrm{Re}\,\epsilon_r(\omega) = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$\mathrm{Im}\,\epsilon_r(\omega) = \frac{\gamma\omega\omega_p^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

The general features of the real and imaginary parts of $\epsilon(\omega)$ around the resonant frequency are shown in fig.2.5. Normal dispersion is associated with an increase of Re $\epsilon(\omega)$ with ω, anomalous dispersion with the reverse. Normal dispersion is seen to occur everywhere except in the neighborhood of the resonance frequency. And only where there is anomalous dispersion is the imaginary part of $\epsilon(\omega)$ appreciable. Since a positive imaginary part to $\epsilon(\omega)$ represents dissipation of energy from the electromagnetic wave into the medium, the regions where Im $\epsilon(\omega)$ is large are called regions of resonant absorption.

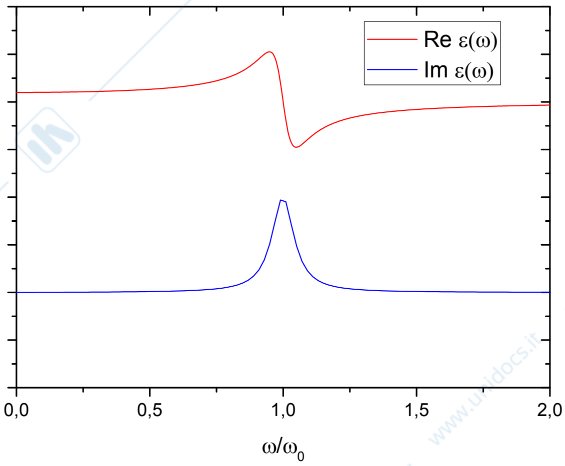

Figura 2.5: Real and imaginary parts of $\epsilon(\omega)$.

The model explains also the electric conductivity: if some electrons are 'free' in the sense of having $\omega_0 = 0$, the dielectric constant is singular at $\omega = 0$,

$$\epsilon_r(\omega) = \epsilon_d + i\frac{\omega_p^2}{\omega(\gamma - i\omega)}$$

where ϵ_d is the contribution of all the other dipoles. The singular behavior can be understood if we examine the Maxwell-Ampère equation,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

and assume that the medium obeys the Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ and, as before, $\mathbf{D} = \epsilon_0\epsilon_d\mathbf{E}$. With harmonic time dependence the equation becomes

$$\nabla \times \mathbf{H} = -i\omega\epsilon_0 \left[\epsilon_r(\omega) + i\frac{\sigma}{\epsilon_0\omega}\right]\mathbf{E}$$

If, on the other hand, we did not insert the Ohm's law explicitly but attributed instead all the properties of the medium to the dielectric constant, we would identify the quantity in the bracket with $\epsilon(\omega)$. A comparison with the expression of $\epsilon(\omega)$ yields an expression for the conductivity:

$$\sigma = \frac{\epsilon_0\omega_p^2}{\gamma - i\omega}.$$

Returning to dielectrics, at frequencies far above the resonant frequency the dielectric constant takes the simple form

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

and the wave number is given in this limit by

$$ck = \sqrt{\omega^2 - \omega_p^2}$$

In dielectric media this expression only applies for $\omega \gg \omega_0$. In certain situations, as in an electronic plasma or in the ionosphere, the electrons are free and the damping is negligible. Then, this expression holds over a wide range of frequencies, including $\omega < \omega_p$. For frequencies lower than the plasma frequency the wave number is purely imaginary. Such waves incident on a plasma are reflected and the field inside falls off exponentially with distance from the surface.

Near $\omega \sim \omega_0$ the real and imaginary parts of $\epsilon(\omega)$ are approximated by:

$$\mathrm{Re}\,\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{2\omega_0}\frac{\omega - \omega_0}{(\omega - \omega_0)^2 + (\gamma/2)^2} \tag{2.77}$$

$$\mathrm{Im}\,\epsilon_r(\omega) = \frac{\omega_p^2}{2\omega_0}\frac{\gamma/2}{(\omega - \omega_0)^2 + (\gamma/2)^2} \tag{2.78}$$

At resonance ($\omega = \omega_0$) $\mathrm{Re}\,\epsilon(\omega_0) = 1$ and $\mathrm{Im}\,\epsilon(\omega_0) = \chi_2(\omega_0) = \omega_p^2/\omega_0\gamma$, so that the absorption coefficient is $\alpha = k\chi_2 = \omega_p^2/c\gamma$.

2.8.1 $G(\tau)$ and $P(t)$ for the Lorentz model*

Let's now apply the previous results of sec.2.5.2 to the model (2.76). Viewed as a function in the complex ω plane, $\chi(\omega)$ is analytic in the upper half plane. In fact, it has two poles in the lower half plane, $\omega_{1,2} = -i\gamma/2 \pm \sqrt{\omega_0^2 - \gamma^2/4}$ (assuming $\gamma < \omega_0$). Therefore the kernel is

$$G(\tau) = \frac{\epsilon_0}{2\pi}\int_{-\infty}^{+\infty}\chi(\omega)e^{-i\omega\tau}d\omega = \epsilon_0\omega_p^2 e^{-\gamma\tau/2}\frac{\sin\sqrt{\omega_0^2-\gamma^2/4}\tau}{\sqrt{\omega_0^2-\gamma^2/4}} \quad \tau > 0. \tag{2.79}$$

whereas $G(\tau) = 0$ for $\tau < 0$. The kernel $G(\tau)$ is oscillatory with the characteristic frequency ω_0 of the medium and damped in time with the damping constant γ of the electronic oscillators. The nonlocality in time of the connection between **D** and **E** is thus confined to times of the order of γ^{-1}.

Notice that near $\omega \sim \omega_0$

$$\chi(\omega) \approx -\frac{\omega_p^2}{2\omega_0}\frac{1}{\omega - \omega_0 + i\gamma/2}$$

which has a single pole $\omega = \omega_0 - i\gamma/2$. Hence, the kernel is

$$G(\tau) = i\epsilon_0\frac{\omega_p^2}{2\omega_0}e^{-\gamma\tau/2 - i\omega_0\tau} \quad \tau > 0.$$

$G(\tau)$ is complex and has a phase lag of $\pi/2$ with respect the field at resonance. From it we can build the polarization as

$$P(t) = i\epsilon_0\frac{\omega_p^2}{2\omega_0}\int_0^{\infty}E(t-\tau)e^{-\gamma\tau/2 - i\omega_0\tau}d\tau \tag{2.80}$$

For the slowly varying envelopes of P and E (see sec.2.5.4):

$$P_0(t) = i\epsilon_0\frac{\omega_p^2}{2\omega_0}\int_{-\infty}^{t}E_0(t')e^{-(\gamma/2)(t-t')+i(\omega-\omega_0)(t-t')}dt' \tag{2.81}$$

It is easy to check that P_0 satisfies the following differential equation

$$\frac{\partial P_0}{\partial t} = i(\omega - \omega_0)P_0 - \frac{\gamma}{2}P_0 + i\epsilon_0\frac{\omega_p^2}{2\omega_0}E_0. \tag{2.82}$$

We observe that the electric field acts on the 'in quadrature' part of the polarization.

2.9 Surface plasmons*

Surface plasmons are e.m. waves which propagate at the interface between a metal and a dielectric, confined to the interface between the two media. They are due to the superficial charge induced by the electric field of the incident wave, polarized in the incident plane. These modes exist for frequencies ω such that $\text{Re}[\epsilon_M(\omega)] < -\epsilon_A$, where ϵ_A and ϵ_M are the relative dielectric constant of the dielectric medium and of the metal, respectively.

Let consider two media A and M with ϵ_A and ϵ_M, and a wave propagating along the interface $z = 0$, decaying exponentially away from the interface:

$$\mathbf{E}^{(A)}(x,z,t) = (A,0,B)e^{i(kx-\omega t)}e^{-\alpha_A z} \quad \text{for } z > 0 \tag{2.83}$$

$$\mathbf{E}^{(M)}(x,z,t) = (C,0,D)e^{i(kx-\omega t)}e^{+\alpha_M z} \quad \text{for } z < 0 \tag{2.84}$$

From the Maxwell equation (3.5),

$$\left(\nabla^2 + \frac{\epsilon\omega^2}{c^2}\right)\mathbf{E} = 0$$

(where we assumed $\mu = \mu_0$) we obtain:

$$\alpha_A = \sqrt{k^2 - \epsilon_A\omega^2/c^2}, \qquad \alpha_M = \sqrt{k^2 - \epsilon_M\omega^2/c^2} \tag{2.85}$$

The dispersion relation between ω, k and $\epsilon_{A,M}$ is obtained from the boundary condition at $z = 0$. From the continuity of $\mathbf{E}_\parallel$ and $\mathbf{D}_\perp$, we have $A = C$ and $\epsilon_A B = \epsilon_M D$. Since $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ in A and B, we have $ikA - \alpha_A B = 0$ and $ikC + \alpha_M D = 0$. From them we obtain

$$\frac{\epsilon_A}{\alpha_A} + \frac{\epsilon_M}{\alpha_M} = 0 \tag{2.86}$$

which combined with Eq.(2.85) gives

$$k(\omega) = \frac{\omega}{c}\sqrt{\frac{\epsilon_A\epsilon_M}{\epsilon_A + \epsilon_M}} \tag{2.87}$$

(notice the symmetry between ϵ_A and ϵ_M). At this point we assume ϵ_A real and positive for the dielectric medium A and $\epsilon_M = 1 - \omega_p^2/\omega^2$ for the metal M, so that the dispersion relation becomes

$$k(\omega) = \frac{\omega}{c}\sqrt{\frac{\epsilon_A(\omega^2 - \omega_p^2)}{(1+\epsilon_A)\omega^2 - \omega_p^2}} \tag{2.88}$$

The function $k(\omega)$ has two limits,

$$\lim_{\omega\to\infty} \frac{ck(\omega)}{\omega} = \sqrt{\frac{\epsilon_A}{1+\epsilon_A}}, \qquad \lim_{\omega\to 0} \frac{ck(\omega)}{\omega} = \sqrt{\epsilon_A} \tag{2.89}$$

and a vertical asymptote at $\omega = \omega_p/\sqrt{1+\epsilon_A}$. We can obtain from the study of the function $k(\omega)$ the following results:

1. For $\omega_p/\sqrt{1+\epsilon_A} < \omega < \omega_p$ there are no solutions.

2. For $\omega > \omega_p$, we have $\epsilon_M > 0$ and $k < (\omega/c)\sqrt{\epsilon_A}$. In this case the mode propagates in the media A and M. In fact, since $k^2 - \epsilon_A\omega^2/c^2 < 0$ and $k^2 - \epsilon_M\omega^2/c^2 = -(\omega^2/c^2)\epsilon_M^2/(\epsilon_A+\epsilon_M) < 0$ we have $\alpha_A = ik_z^{(A)}$ and $\alpha_M = ik_z^{(M)}$. These kind of plasmons are said '*radiative*' or '*Brewster's modes*'. In fact, for a given incident angle it is possible to satisfy the Brewster condition such that the reflected wave vanishes: assuming $k = (\omega/c)\sqrt{\epsilon_A}\sin\theta_B$ in Eq.(2.87), we obtain the Brewster relation $\tan\theta_B = \sqrt{\epsilon_M/\epsilon_A}$.

3. For $\omega < \omega_p/\sqrt{1+\epsilon_A}$ we obtain the *Fano modes*, which are confined to the interface: in fact for them $k > (\omega/c)\sqrt{\epsilon_A}$ so that $\alpha_A > 0$ and, since $\epsilon_A+\epsilon_M < 0$, also $k^2 - \epsilon_M\omega^2/c^2 = -(\omega^2/c^2)\epsilon_M^2/(\epsilon_A+\epsilon_M) > 0$ and $\alpha_M > 0$. These modes can not be excited from the medium A, since $k = (\omega/c)\sqrt{\epsilon_A}\sin\theta < (\omega/c)\sqrt{\epsilon_A}$, and the incident k is not sufficiently large to match the dispersion curve.

The modes described in the point (3) are the so-called *surface plasmons*. They can be excited using the total internal reflection technique. In fact, adding to the metal and dielectric layers an other dielectric P with refractive index $n_p > n_A = \sqrt{\epsilon_A}$ it is possible to increase the wave vector k such that it can match the dispersion curve. In fact, since $k = (\omega/c)n_p\sin\theta$, if $n_p\sin\theta > \sqrt{\epsilon_A}$ we can intersect the plasmonic dispersion curve. From the Snell's law, $n_p\sin\theta = \sqrt{\epsilon_A}\sin\theta'$, in order to have $n_p\sin\theta$ greater than $\sqrt{\epsilon_A}$ it must be $\sin\theta' > 0$ i.e. we must be in the total internal reflection condition. The light is totally reflected in the medium P and penetrates in the dielectric A with an exponential law $\exp[-\alpha_P(d-z)]$ (where d is the thickness of the layer A) with $\alpha_P = (\omega/c)\sqrt{n_P^2\sin^2\theta - \epsilon_A}$.

A possible configuration is to take the vacuum as medium A, between a prism with index n_P as medium P and the metal M. A simpler configuration can be obtained noting the the plasmon dispersion relation (2.87) is symmetrical with respect to the interchange between ϵ_A and ϵ_M. Hence an equivalent configuration is that where the metal M is set between the prism and the vacuum. With a suitable thickness of

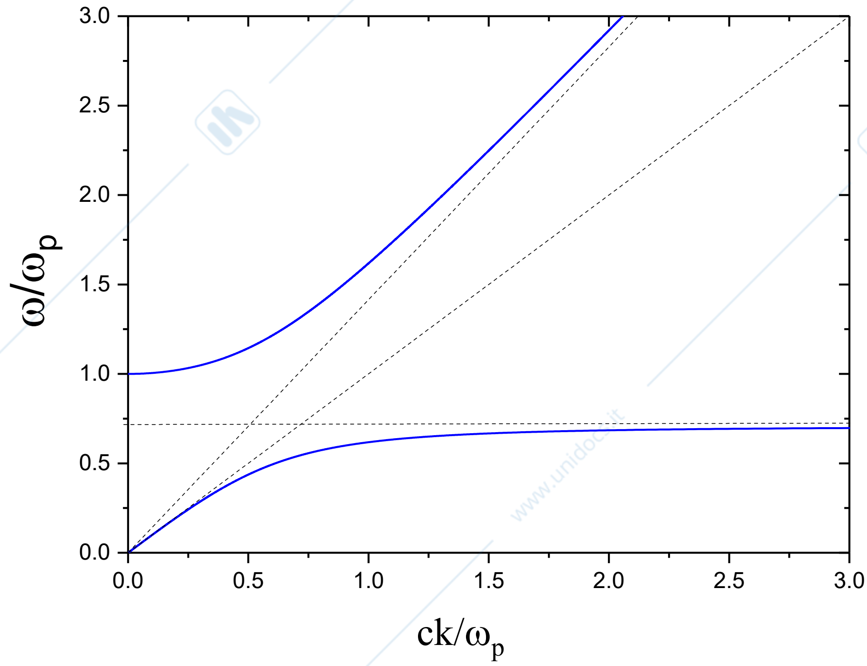

Figura 2.6: Plot of Eq.(2.90), with ω/ω_p as a function of ck/ω_p, with $\epsilon_A = 1$. The surface plasmon occurs for $0 < \omega < \omega_p/\sqrt{2}$.

the metallic film deposited on the prism face it is easy to excite the surface plasmon in the total internal reflection condition.

Assuming $\epsilon_A = 1$, we can invert Eq.(2.88) and obtain

$$\omega^2 = \frac{\omega_p^2}{2} + c^2k^2 \pm \sqrt{c^4k^4 + \frac{\omega_p^4}{4}} \tag{2.90}$$

The surface plasmon corresponds to the sign minus of this expression, whereas the sign plus gives the radiative plasmon.

Capitolo 3

Propagation in wave guides

We discuss the propagation of electromagnetic waves in guided structures, as metallic cylindrical wave guides, resonant cavities and dielectric wave guides.

An electromagnetic wave in free space tends to diffract from the main direction of propagation. A method of transport e.m. waves in a given direction is to propagate them in guided structures. We consider first the propagation in a cylindrical metallic wave guide. Then, we discuss the propagation in dielectric slabs and make some consideration on multi-mode and single-mode propagation in dielectric wave guides.

3.1 Cylindrical wave guides

A practical situation of great importance in the microwave range is the propagation of electromagnetic waves in hollow metallic cylinders. We assume a metallic wave guide with uniform transverse section and infinite length. The wave guide is filled by a dielectric with real and uniform ϵ and μ, with metallic boundary surfaces with infinite conductivity. For a given frequency ω the Maxwell's equations are

$$\boldsymbol{\nabla}\cdot\mathbf{E}=0 \quad , \qquad \boldsymbol{\nabla}\cdot\mathbf{B}=0 \tag{3.1}$$

$$\boldsymbol{\nabla}\times\mathbf{E}=i\omega\mathbf{B} \quad , \qquad \boldsymbol{\nabla}\times\mathbf{B}=-i\omega\mu\epsilon\mathbf{E} \tag{3.2}$$

from which it follows the Helmoltz equation

$$\left\{\nabla^2+\frac{n^2\omega^2}{c^2}\right\}\left\{\begin{array}{c}\mathbf{E}\\ \mathbf{B}\end{array}\right\}=0$$

where $n = c\sqrt{\epsilon\mu}$. If z is the wave guide axis, we look for solutions of the form

$$\mathbf{E}(x, y, z, t) = \mathbf{E}(x, y)e^{ikz-i\omega t} \tag{3.3}$$

$$\mathbf{B}(x, y, z, t) = \mathbf{B}(x, y)e^{ikz-i\omega t} \tag{3.4}$$

where k is a parameter which must be determined and can be, in general, real or complex. With this assumption, the wave equation is

$$\left\{\nabla_t^2 + \left(\frac{n^2\omega^2}{c^2} - k^2\right)\right\}\left\{\begin{array}{c}\mathbf{E}\\ \mathbf{B}\end{array}\right\} = 0 \tag{3.5}$$

where $\nabla_t^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Separating the fields into transverse and longitudinal to the z axis components,

$$\mathbf{E} = \mathbf{E}_t + E_z\hat{\mathbf{z}} \tag{3.6}$$

$$\mathbf{B} = \mathbf{B}_t + B_z\hat{\mathbf{z}} \tag{3.7}$$

the two curl Maxwell's equations become:

$$\frac{\partial \mathbf{E}_t}{\partial z} + i\omega(\hat{\mathbf{z}} \times \mathbf{B}_t) = \boldsymbol{\nabla}_t E_z \quad , \qquad \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_t \times \mathbf{E}_t) = i\omega B_z \tag{3.8}$$

$$\frac{\partial \mathbf{B}_t}{\partial z} - i\mu\epsilon\omega(\hat{\mathbf{z}} \times \mathbf{E}_t) = \boldsymbol{\nabla}_t B_z \quad , \qquad \hat{\mathbf{z}} \cdot (\boldsymbol{\nabla}_t \times \mathbf{B}_t) = -i\mu\epsilon\omega E_z \tag{3.9}$$

It is evident that if E_z and B_z are known the transverse components of $\mathbf{E}$ and $\mathbf{B}$ are determined. Assuming the z dependence in Eqs.(3.8) and (3.9), the transverse components are:

$$\mathbf{E}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\{k\boldsymbol{\nabla}_t E_z - \omega\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t B_z\} \tag{3.10}$$

$$\mathbf{B}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\{k\boldsymbol{\nabla}_t B_z + \mu\epsilon\omega\hat{\mathbf{z}} \times \boldsymbol{\nabla}_t E_z\} \tag{3.11}$$

Before considering the kind of fields that can exist inside a hollow cylinder, we discuss the degenerate solution called the transverse electromagnetic (TEM) wave, with $E_z = 0$ and $B_z = 0$. From the second equation in (3.8) and the divergence of $\mathbf{E}$ equal to zero,

$$\boldsymbol{\nabla}_t \times \mathbf{E}_{TEM} = 0 \quad , \quad \boldsymbol{\nabla}_t \cdot \mathbf{E}_{TEM} = 0 \tag{3.12}$$

with the boundary condition $\mathbf{E}_{TEM} = 0$ on S. We see that the dependence of $\mathbf{E}_{TEM}$ on x and y is given by the solution of an electrostatic problem in two dimensions: $\mathbf{E}_{TEM} = -\boldsymbol{\nabla}_t\phi$, where ϕ satisfies the Laplace equation $\nabla_t^2\phi = 0$ with boundary condition ϕ = constant on S. In a simply connected domain this boundary condition leads to $\phi = \text{constan}t$, and so $\mathbf{E}_{TEM} = 0$ inside the domain. So, these kinds of wave can not propagate inside a hollow cylinder. With a domain multiply connected, the boundary conditions can be different on the different surfaces, and so the Laplace equation has non trivial solutions. The electric field distribution on the transverse section of the wave guide corresponds to the electrostatic field between the armatures of a capacitor under a given potential difference. Hence, it can exist in coaxial cables, which have two or more cylindrical surfaces. Notice that for TEM modes, $k = n\omega/c$ and $\mathbf{B}_{TEM} = (n/c)(\hat{\mathbf{z}} \times \mathbf{E}_{TEM})$, i.e. the same as for plane waves in an infinite medium. We discuss an example of TEM modes in a coaxial cable at the end of this section.

The possible solutions in a hollow metallic wave guide can be divided in two kinds: in one the magnetic field is purely transverse, $B_z = 0$ (TM modes), and in the other the electric field is purely transverse, $E_z = 0$ (TE modes).

TE waves

From Eqs.(3.10) and (3.11) with $E_z = 0$,

$$E_x = i\frac{\omega}{\gamma^2}\frac{\partial B_z}{\partial y} \quad , \qquad E_y = -i\frac{\omega}{\gamma^2}\frac{\partial B_z}{\partial x} \tag{3.13}$$

$$B_x = i\frac{k}{\gamma^2}\frac{\partial B_z}{\partial x} \quad , \qquad B_y = i\frac{k}{\gamma^2}\frac{\partial B_z}{\partial y} \tag{3.14}$$

where

$$\gamma^2 = \mu\epsilon\omega^2 - k^2 = \frac{n^2}{c^2}\omega^2 - k^2 \tag{3.15}$$

and where B_z is the solution of

$$(\nabla_t^2 + \gamma^2)B_z = 0 \tag{3.16}$$

The boundary conditions require that $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$ on the surface, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface. Then, from $\mathbf{B}_t = (ik/\gamma^2)\boldsymbol{\nabla}_t B_z$, it follows that $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla}_t B_z = \partial B_z/\partial n = 0$.

TM waves

From Eqs.(3.10) and (3.11) with $B_z = 0$,

$$E_x = i\frac{k}{\gamma^2}\frac{\partial E_z}{\partial x} \quad , \qquad E_y = i\frac{k}{\gamma^2}\frac{\partial E_z}{\partial y} \tag{3.17}$$

$$B_x = -i\frac{\mu\epsilon\omega}{\gamma^2}\frac{\partial E_z}{\partial y} \quad , \qquad B_y = i\frac{\mu\epsilon\omega}{\gamma^2}\frac{\partial E_z}{\partial x} \tag{3.18}$$

where E_z is the solution of

$$(\nabla_t^2 + \gamma^2)E_z = 0 \tag{3.19}$$

where the boundary conditions of this equation come from the requirement that the tangential component of $\mathbf{E}$ on the surface must be zero. At this aim it is sufficient to require that $E_z = 0$ on the surface S.

Both the equations (3.16) and (3.19) specify an eigenvalue problem, defined by the boundary conditions. We see that $\gamma^2 > 0$ since the solution must be oscillating in order to satisfy the boundary conditions on the surfaces at both sides of the axis. There exists a discrete set of the eigenvalues γ_λ^2, with $\lambda = 1, 2\ldots$, called the *modes* of the wave guide. For a given frequency, the wave number is

$$k_\lambda = \sqrt{n^2\omega^2/c^2 - \gamma_\lambda^2} = (n/c)\sqrt{\omega^2 - \omega_\lambda^2} \tag{3.20}$$

where $\omega_\lambda = (c/n)\gamma_\lambda$ is the cut-off frequency of the mode λ: For $\omega > \omega_\lambda$, k_λ is real and the wave can propagate in the wave guide; for $\omega < \omega_\lambda$, k_λ is imaginary and the wave can not propagate (evanescent wave). For a given frequency, there may exist more than one mode propagating in the wave guide. Inverting Eq.(3.20) we obtain

$$\omega^2 = (c/n)^2(k^2 + \gamma^2) \tag{3.21}$$

The group velocity of the wave along the wave guide is

$$v_g = \frac{d\omega}{dk} = \frac{ck}{n\sqrt{k^2 + \gamma^2}} = \frac{c^2 k}{n^2 \omega} \tag{3.22}$$

Since $\omega > ck/n$, then $v_g < c/n$.

The average (in time) flux of energy is given by the longitudinal component of the Poynting vector

$$\overline{\mathbf{S}}_z = \frac{1}{2}\mathrm{Re}[\mathbf{E} \times \mathbf{H}^*]_z = \frac{\omega k}{2\gamma^4}\begin{cases} \epsilon|\boldsymbol{\nabla}_t E_z|^2 & \text{(TM)} \\ \mu|\boldsymbol{\nabla}_t H_z|^2 & \text{(TE)} \end{cases} \tag{3.23}$$

The flux of the total energy (i.e. the power) can be obtained integrating over the transverse section of the wave guide. Using the first Green's identity in the bidimensional case[1]

$$\int_S (\phi\nabla^2\psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi)da = \oint_C \phi\frac{\partial\psi}{\partial n}dl$$

[1] see Jackson, pag. 34.

we can write

$$\int_S |\nabla_t E_z|^2 da = \oint_C E_z \frac{\partial E_z^*}{\partial n} dl - \int_S E_z \nabla_t^2 E_z^* da = \gamma^2 \int_S |E_z|^2 da \tag{3.24}$$

where the integral on the contour C is zero for the boundary condition on E_z. A similar result is obtained for B_z. Then the average power is

$$P = \frac{\omega k}{2\gamma^2} \begin{cases} \epsilon \int_S |E_z|^2 da & \text{(TM)} \\ \mu \int_S |H_z|^2 da & \text{(TE)} \end{cases} \tag{3.25}$$

Rectangular wave guide

As a simple example, let consider a wave guide with rectangular cross section, with dimension a along x and b along y. Then solving the eigenvalue problem with the boundary conditions we obtain

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad (TM) \tag{3.26}$$

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \qquad (TE) \tag{3.27}$$

with

$$\gamma_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{3.28}$$

and cut-off frequencies

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu\epsilon}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \tag{3.29}$$

Notice that the lowest cut-off frequency for the TM solution is ω_{11} whereas for the TE solution, for $a > b$, is ω_{10}.

TEM mode in a coaxial cable

As an example of TEM mode, we calculate the TEM mode propagating in a transmission line consisting of two circular cylinders of metal with inner and outer radius a and b, with infinite conductivity and filled with a uniform lossless dielectric (ϵ, μ). A TEM mode is essential an electrostatic problem in two dimensions. We start finding the electric field between the two cylinder. By the Gauss theorem in cylindrical coordinates,

$$\mathbf{E}_t = \frac{A}{\rho} \hat{\rho}$$

where A is a constant to be determined. For propagation along $+z$, the magnetic field is

$$\mathbf{H}_t = \sqrt{\frac{\epsilon}{\mu}}\hat{\mathbf{z}} \times \mathbf{E}_t = \sqrt{\frac{\epsilon}{\mu}}\frac{A}{\rho}\hat{\phi}$$

At the surface of the inner cylinder ($\rho = a$), the magnitude of magnetic field is $H(a) = \sqrt{\epsilon/\mu}(A/a) = H_0$, so that

$$\mathbf{E}_t = \sqrt{\frac{\mu}{\epsilon}}H_0\frac{a}{\rho}\hat{\rho}, \quad \mathbf{H}_t = H_0\frac{a}{\rho}\hat{\phi}$$

The Poynting vector is

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2\frac{a^2}{\rho^2}\hat{\mathbf{z}}$$

so the power flow is

$$P = \int_A \mathbf{S} \cdot \hat{\mathbf{z}} da = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0|^2 \int_a^b \frac{a^2}{\rho^2}2\pi\rho d\rho = \sqrt{\frac{\mu}{\epsilon}}\pi a^2|H_0|^2 \ln\left(\frac{b}{a}\right). \qquad (3.30)$$

3.2 Resonant cavities

We consider now a closed metallic cavity. For simplicity, we consider the previous hollow cylinder with the end terminals closed by two surface perpendicular to the z axis, at $z = 0$ and $z = d$. Then, due to the reflection at the terminal surface, the field has a form

$$A\sin(kz) + B\cos(kz)$$

TE fields

For TE fields, the vanishing of the perpendicular component of the magnetic field imposes $H_z = 0$ at $z = 0$ and $z = d$, so that

$$H_z = \psi(x, y)\sin(p\pi z/d), \qquad (p = 1, 2, 3\dots) \qquad (3.31)$$

From Eqs.(3.13) and (3.14) with $E_z = 0$,

$$E_x = i\frac{\omega\mu}{\gamma^2}\sin kz\frac{\partial\psi}{\partial y}, \qquad E_y = -i\frac{\omega\mu}{\gamma^2}\sin kz\frac{\partial\psi}{\partial x} \qquad (3.32)$$

$$H_x = \frac{k}{\gamma^2}\sin kz\frac{\partial\psi}{\partial x}, \qquad H_y = \frac{k}{\gamma^2}\sin kz\frac{\partial\psi}{\partial y} \qquad (3.33)$$

where $k = p\pi/d$ and

$$\gamma^2 = \mu\epsilon\omega^2 - (p\pi/d)^2 \qquad (3.34)$$

TM fields

For the TM fields, we must impose the vanishing of $\mathbf{E}_t$ at $z = 0$ and $z = d$, so that it must have a dependence as $\sin(p\pi z/d)$: from the equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ this implies that E_z must have a dependence as

$$E_z = \psi(x, y) \cos(p\pi z/d), \qquad (p = 0, 1, 2 \dots) \tag{3.35}$$

From Eqs.(3.17) and (3.18) with $B_z = 0$,

$$E_x = -\frac{k}{\gamma^2} \cos kz \frac{\partial \psi}{\partial x} \quad , \qquad E_y = -\frac{k}{\gamma^2} \cos kz \frac{\partial \psi}{\partial y} \tag{3.36}$$

$$H_x = -i\frac{\omega\epsilon}{\gamma^2} \cos kz \frac{\partial \psi}{\partial y} \quad , \qquad H_y = i\frac{\omega\epsilon}{\gamma^2} \cos kz \frac{\partial \psi}{\partial x} \tag{3.37}$$

The function ψ is solution of

$$(\nabla_t^2 + \gamma^2)\psi = 0 \tag{3.38}$$

with the usual boundary conditions for the TE and TM modes. For each value of p, the eigenvalue γ_λ^2 determines the frequencies

$$\omega_{\lambda p} = \frac{c}{n}\sqrt{\gamma_\lambda^2 + (p\pi/d)^2} \tag{3.39}$$

and the corresponding fields. These resonant frequencies form a discrete set of values, depending on the geometry of the system. For a rectangular cavity, they are

$$\omega_{mnp} = \frac{\pi}{\sqrt{\mu\epsilon}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2} \right)^{1/2} . \tag{3.40}$$

3.3 Dielectric wave guides

Guided modes may exist also in dielectric structures (e.g. thin films and fibers) with dimensions comparable to the wavelength [2]. The optical modes in these dielectric wave guides correspond to confined propagation of electromagnetic radiation with a transverse dimension defined by the guide. We discuss the fundamental properties of guided waves in a general dielectric structure. As for the metallic ones, optical modes are presented as solution to an eigenvalue problem by solving the Maxwell's

[2] This section is based on the book by A. Yariv and P. Yeh, "*Optical Waves in Crystals*", Chap. 11.

equations subjected to the boundary condition imposed by the wave guide geometry. This approach is then applied to the slab dielectric wave guide. Both the TE and TM mode are derived. The physics of confined propagation is explained in terms of the total internal reflection of plane wave from the dielectric interfaces.

3.3.1 General properties of dielectric wave guides

We consider as before a cylindrical wave guide of a material with a refractive index n depending only on the transverse coordinates x and y (i.e. $\partial n/\partial z = 0$). Since the dielectric structure is homogeneous along the z axis, we look for solution of the form (3.3). We limit ourselves to dielectric structures which consist of piecewise homogeneous and isotropic materials, or materials with a small gradient in the distribution of the refractive index, so that the Maxwell's equation reduce to:

$$\left\{\nabla_t^2 + \left[\frac{\omega^2}{c^2} n^2(x,y) - k^2\right]\right\} \mathbf{E}(x,y) = 0 \tag{3.41}$$

In the case of piecewise homogeneous dielectric structures, Eq.(3.41) holds separately in each homogeneous region. Therefore, the field must be solved for separately in each region, and then the tangential components of the field must be matched at each interfaces. Another important boundary condition for guided modes is that the field amplitudes are zero at infinity. The axial propagation constant k must be the same throughout the guide structure in order to satisfy the boundary conditions at all points on the interfaces of those homogeneous media.

The basic problem is that of finding the solution to the eigenvalue equation (3.41) subject to the continuity conditions on the tangential components of the fields at the dielectric interfaces and the boundary conditions at infinity. Given an index profile $n^2(x,y)$, there are in general an infinite number of eigenvalues k^2, corresponding to an infinite number of modes. However, normally only a finite number of these modes are confined near the core and will propagate freely along the guide. One of the necessary conditions for a guided mode is that there is no transverse flow of energy. This requires that the fields fall off exponentially outside the guide structure. Consequently, the quantity $(\omega^2/c^2)n^2 - k^2$ must be negative in the region far away from the core. In other words, for a confined mode $k^2 > (\omega^2/c^2)n^2(\infty)$. On the other hand, if the fields vanish at infinity, the continuity of the fields requires that the field takes a maximum value at some point in the xy plane. This implies that $k^2 < (\omega^2/c^2)n^2(x,y)$ in some region of the xy plane. In the regions where this condition is satisfied, the solutions of Eq.(3.41) are oscillatory. These oscillatory solutions must be matched to the exponential solutions at the boundary of the dielectric interfaces. Therefore, only some k are legitimate eigenvalues of the confined modes.

3.3.2 A dielectric slab guide*

We consider a planar wave guide of thickness t consisting of a medium with refractive index n_2 sandwiched between two media with refractive index n_1 on one side and n_3 on the other side. Let the coordinates be chosen in such a way that the wave is propagating in the xz plane and the refractive index profile is given by

$$n(x) = \begin{cases} n_1 & x > 0 \quad \text{Region} \quad \text{I} \\ n_2 & -t < x < 0 \quad \text{Region} \quad \text{II} \\ n_3 & x < -t \quad \text{Region} \quad \text{III} \end{cases} \tag{3.42}$$

Since there is no variation of n in the y direction, Eq.(3.41) can be written separately for region I, II and III:

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \left[k_0^2 n_1^2 - k^2\right] \mathbf{E} = 0, \qquad (\text{Region} \quad \text{I}) \tag{3.43}$$

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \left[k_0^2 n_2^2 - k^2\right] \mathbf{E} = 0, \qquad (\text{Region} \quad \text{I}) \tag{3.44}$$

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \left[k_0^2 n_3^2 - k^2\right] \mathbf{E} = 0, \qquad (\text{Region} \quad \text{I}) \tag{3.45}$$

where $k_0 = \omega/c$. Let us assume that $n_2 > n_3 > n_1$. At a fixed frequency ω, the guided modes can be obtained only if

$$k_0 n_1 < k_0 n_3 < k < k_0 n_2 \tag{3.46}$$

In fact, for these values the solution is oscillating in the region II and exponential in the regions I and III. This makes it possible to have a solution $\mathbf{E}(x)$ that satisfies the boundary conditions while *decaying* exponentially in regions I and III. As we will see, the values allowed k in the propagation regime $k_0 n_3 < k < k_0 n_2$ are *discrete*. The number of confined modes depends on the thickness t, the frequency and the indexes of refraction n_1, n_2 and n_3.

A useful point of view is one of considering the wave propagation in the inner region II as that of a plane wave propagating at some angle θ to the horizontal axis z and undergoing a series of total internal reflections at the interfaces II-I and II-III. Assuming a solution in the form of $\mathbf{E} \propto \sin(k_x x + \phi) e^{-ikz}$, we obtain from (3.44) $k_x^2 + k^2 = k_0^2 n_2^2$. The propagation can be considered formally as that of a plane wave directed along the hypotenuse of a rectangular triangle, with wave number $k_o n_2$. The guiding condition $k > k_0 n_3$ leads, using $k = k_0 n_2 \cos\theta$, to $\cos\theta > n_3/n_2$. Since the angle with respect to the normal to the interface is $\theta_2 = \pi/2 - \theta$, we obtain $\theta_2 > \arcsin(n_3/n_2) = \theta_L$, where θ_L is the total-internal-reflection angle at the interface between layers II and III. Hence the condition for total-internal reflection is $\theta < \pi/2 - \theta_L$. Since $n_3 > n_2$, total reflection at the II-III interfaces guarantees total internal reflection at the I-II interface.

3.3.3 TE and TM modes in the asymmetrical dielectric wave guide*

This guide supports a finite number of confined TE modes with field components E_y, H_x and H_z, and TM modes with components H_y, E_x and E_z (see Eqs.(3.13), (3.14) and Eqs.(3.17), (3.18), where $\partial/\partial y = 0$). The field component E_y of the TE mode is

$$E_y(x) = \begin{cases} C_1 e^{-\gamma_1 x} & 0 \le x < \infty \\ A\cos(hx) + B\sin(hx) & -t \le x \le 0 \\ C_2 e^{\gamma_3(x+t)} & -\infty < x \le -t \end{cases} \tag{3.47}$$

where $\gamma_{1,3} = (k^2 - k_0^2 n_{1,3}^2)^{1/2}$ and $h = (k_0^2 n_2^2 - k^2)^{1/2}$. The acceptable solutions for E_y and $H_z = -i/(\mu_0\omega)(\partial E_y/\partial x)$ should be continuous at both $x = 0$ and $x = -t$. From them it is easy to obtain the condition

$$\tan(ht) = \frac{h(\gamma_1 + \gamma_3)}{h^2 - \gamma_1\gamma_3} \tag{3.48}$$

Fig.1 shows an example of graphical solution obtained from the condition (3.48). The parameters are $n_1 = 1$, $n_2 = 3.5$, $n_3 = 3.2$ and $k_0 t = 10$. In this case there are five TE modes, with $k_m = k_0(3.21, 3.31, 3.39, 3.45, 3.48)$. Fig.2 shows the modes TE_m as a function of the thickness $k_0 t$.

The modes TM can be obtained in a similar way. The solution of H_y is

$$H_y(x) = \begin{cases} C_1 e^{-\gamma_1 x} & 0 \le x < \infty \\ A\cos(hx) + B\sin(hx) & -t \le x \le 0 \\ C_2 e^{\gamma_3(x+t)} & -\infty < x \le -t \end{cases} \tag{3.49}$$

The continuity of H_y and $E_z = ic^2/[n^2(x)\omega](\partial H_y/\partial x)$ at both $x = 0$ and $x = -t$ leads to the condition

$$\tan(ht) = \frac{h(\gamma_1 n_3^2 + \gamma_3 n_1^2)}{h^2 n_3^2 - \gamma_1\gamma_3 n_2^2} \tag{3.50}$$

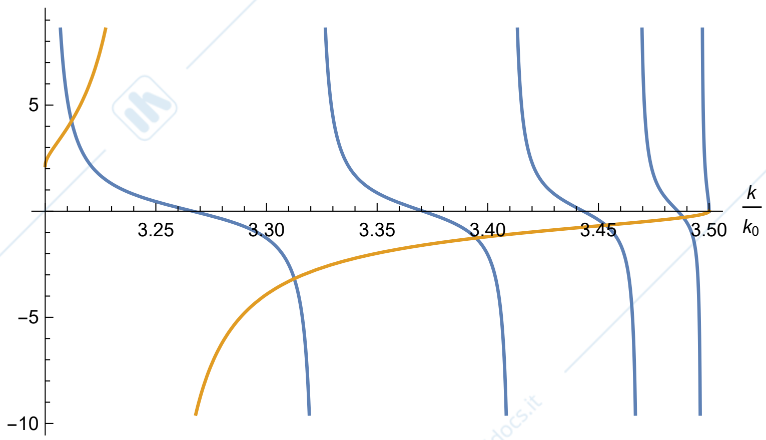

Figura 3.1: Graphical solution of (3.48). The parameters are $n_1 = 1$, $n_2 = 3.5$, $n_3 = 3.2$ and $k_0 t = 10$.

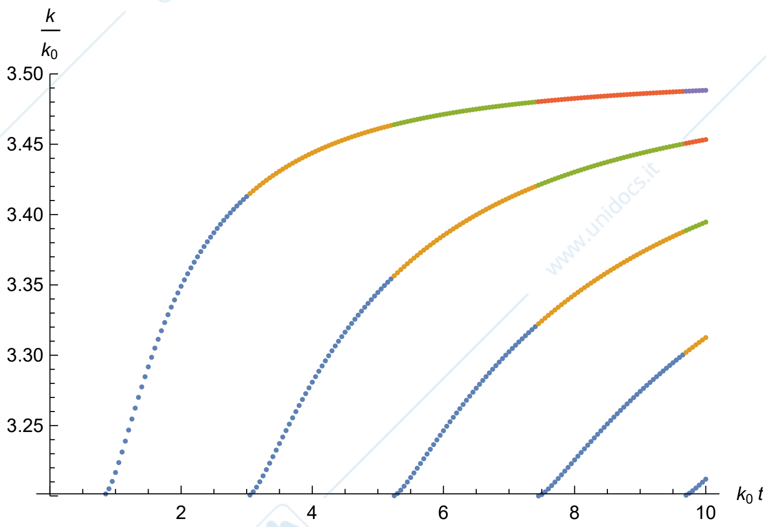

Figura 3.2: Modes TE_m as a function of the thickness $k_0 t$. The parameters are $n_1 = 1$, $n_2 = 3.5$, and $n_3 = 3.2$.

The cut-off frequency is defined by $\gamma_3 = 0$, i.e. $\omega_\lambda = ck_\lambda/n_3$. It occurs for certain values of the thickness, which for the TE and TM modes are

$$(k_0 t)_{\mathrm{TE}} = \frac{1}{\sqrt{n_2^2 - n_3^2}} \left[m\pi + \arctan \left(\frac{n_3^2 - n_1^2}{n_2^2 - n_3^2} \right)^{1/2} \right] \quad (3.51)$$

$$(k_0 t)_{\mathrm{TM}} = \frac{1}{\sqrt{n_2^2 - n_3^2}} \left[m\pi + \arctan \frac{n_2^2}{n_1^2} \left(\frac{n_3^2 - n_1^2}{n_2^2 - n_3^2} \right)^{1/2} \right] \quad (3.52)$$

where $m = 0, 1, 2, \ldots$. Notice that in the symmetric case with $n_1 = n_3$ there is no cut-off for the modes TE_0 and TM_0, whereas in the asymmetric case the cut-off thickness of the TM_m modes are larger than that of the TE_m modes, since $n_2 > n_1$.

Let's now consider the symmetric case, $n_1 = n_3$, and see what happens to the TE modes in a given wave guide (with fixed n_1, n_2 and t) as the wavelength of the light decreases gradually, assuming that the refraction indexes do not vary significantly. Since $k_0 = 2\pi/\lambda$, the effect of decreasing the wavelength is to increase the value of k_0. When

$$0 < \sqrt{n_2^2 - n_1^2} k_0 t < \pi$$

one solution of the mode condition (3.48) exists. The mode is designed as TE_0 and has a transverse parameter falling within the range

$$0 < h_1 t < \pi$$

so that it has no zero crossing the interior of the guiding layer $-t < x < 0$. When k_0 increases further so that

$$\pi < \sqrt{n_2^2 - n_1^2} k_0 t < 2\pi$$

the mode condition gives two solutions: one corresponds to a value $ht < \pi$ and is thus that of the lowest-order TE_0 mode. In the second mode

$$\pi < h_2 t < 2\pi$$

and consequently it has one zero crossing (i.e. point where $E_y = 0$) in the guiding region $-t < x < 0$. This is the so-called TE_1 mode. Both these modes correspond to the same frequency and can thus be excited simultaneously by the same input field. We notice, however, that the TE_0 mode has a larger value of γ_1, and is therefore more strongly confined to the guiding slab. We can now generalize and state that the mth mode (TE or TM) satisfies

$$m\pi < h_m t < (m+1)\pi$$

and has m zero crossings in the guiding layer.

Capitolo 4

Radiation from localized sources

We discuss the radiation emitted from localized sources and the light scattering at long wavelengths

Up to now we have been focused on the properties of the electromagnetic fields without considering how these fields are generated. Now we turn our attention to the emission of radiation by localized oscillating system of charge and current density.

4.1 Fields and radiation of a localized oscillating source

Let us consider a localized system of charges and currents oscillating at the frequency ω:

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r})e^{-i\omega t}, \qquad \mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r})e^{-i\omega t} \tag{4.1}$$

Recalling the Maxwell equation in the Lorentz gauge

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}(\mathbf{r}, t) \tag{4.2}$$

whose solution can be expressed in term of the retarded potential (1.91)

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d\mathbf{r}'\left[\frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}\right]_{\text{rit}} \tag{4.3}$$

where the squared parenthesis means that the time t' must be evaluated at the time $t' = t - |\mathbf{r} - \mathbf{r}'|/c$. Assuming the same monochromatic dependence of $\mathbf{J}(\mathbf{r}, t)$, the solution for $\mathbf{A}$ becomes

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} e^{ik|\mathbf{r}-\mathbf{r}'|} \tag{4.4}$$

where $k = \omega/c$. The magnetic and electric fields are

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.5}$$

$$\mathbf{E} = \frac{ic^2}{\omega}\nabla \times \mathbf{B} \tag{4.6}$$

Given a current distribution $\mathbf{J}(\mathbf{r})$ the fields can be, in principle, determined by calculating the integral in (4.4). In practice, this is quite complicated, and it will be useful to investigate the properties of the fields in the limit that the source of current is confined to a small region fo dimension d much smaller than the wavelength $\lambda = 2\pi/k$. There are two spatial regions of interest:

1. The near (static) zone: $d \ll r \ll \lambda$;

2. The far (radiation) zone: $d \ll \lambda \ll r$.

In the near zone $kr \ll 1$ and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4.7}$$

The field is quasi-static and propagation is neglected, decreasing as $1/r$ or even faster, so that the electric field scales at least as $1/r^2$. In the far zone ($kr \gg 1$, the exponent oscillates rapidly and it is possible to approximate

$$|\mathbf{r}-\mathbf{r}'| \approx r - \hat{\mathbf{n}}\cdot\mathbf{r}' \tag{4.8}$$

where $\hat{\mathbf{n}} = \mathbf{r}/r$. Actually, this approximation is valid for $r \gg d$, so it is valid also in the near zone. Therefore, the vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d\mathbf{r}'\mathbf{J}(\mathbf{r}')e^{-ik\hat{\mathbf{n}}\cdot\mathbf{r}'} \tag{4.9}$$

This expression shows that in the far zone the vector potential behaves as a outgoing spherical wave with an angular dependent coefficient. If $kd \ll 1$, then it is convenient to expand the exponential in power of k:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{(-ik)^n}{n!}\int d\mathbf{r}'\mathbf{J}(\mathbf{r}')(\hat{\mathbf{n}}\cdot\mathbf{r}')^n \tag{4.10}$$

Since the terms of the series decrease as $(kd)^n$, the radiation emitted from the source will come mainly from the first nonvanishing term in the expansion (4.10). The different terms correspond to the multipole contributions.

4.1.1 Electric dipole fields and radiation

If only the first term $n = 0$ in (4.10) is kept,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d\mathbf{r}'\mathbf{J}(\mathbf{r}') \tag{4.11}$$

This expression is valid everywhere outside the source, not just in the far zone. From the continuity equation $\nabla\cdot\mathbf{J} + \partial\rho/\partial t = 0$ we have, for an harmonically oscillating source,

$$\nabla\cdot\mathbf{J} = i\omega\rho. \tag{4.12}$$

With it, the integral in Eq.(4.11) can be written, by an integration by parts,

$$\int d\mathbf{r}'\mathbf{J}(\mathbf{r}') = -\int d\mathbf{r}'\mathbf{r}'\nabla'\cdot\mathbf{J} = -i\omega\int d\mathbf{r}'\mathbf{r}'\rho(\mathbf{r}') = -i\omega\mathbf{p} \tag{4.13}$$

where

$$\mathbf{p} = \int d\mathbf{r}\mathbf{r}\rho(\mathbf{r}) \tag{4.14}$$

is the *electric dipole moment.* Finally

$$\mathbf{A}(\mathbf{r}) = -i\frac{\mu_0\omega}{4\pi}\frac{e^{ikr}}{r}\mathbf{p} \tag{4.15}$$

From (4.5) and (4.6), the electric dipole fields are

$$\mathbf{B} = \frac{ck^2\mu_0}{4\pi}(\hat{\mathbf{n}}\times\mathbf{p})\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r} \tag{4.16}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{\frac{k^2}{r}(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}} + [3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)\right\}e^{ikr} \tag{4.17}$$

The magnetic field is orthogonal to the radius vector at all distances, but the electric field has components parallel and perpendicular to $\hat{\mathbf{n}}$. In the far field zone the fields take the limiting forms

$$\mathbf{B} = \frac{ck^2\mu_0}{4\pi}(\hat{\mathbf{n}}\times\mathbf{p})\frac{e^{ikr}}{r} \tag{4.18}$$

$$\mathbf{E} = \frac{k^2}{4\pi\epsilon_0}\{(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}}\}\frac{e^{ikr}}{r} = c\mathbf{B}\times\hat{\mathbf{n}} \tag{4.19}$$

The magnetic and electric fields are mutually orthogonal and decrease as $1/r$, which is typical for radiation fields.

In the near zone ($kr \ll 1$),

$$\mathbf{B} = \frac{i\omega\mu_0}{4\pi r^2}(\hat{\mathbf{n}}\times\mathbf{p}) \tag{4.20}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3}[3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p})-\mathbf{p}] \tag{4.21}$$

The electric field, a part for the oscillation in time, is just the static electric dipole field, whereas the magnetic field is a factor kr smaller than the electric dipole field. So, in the near zone the magnetic field is negligible with respect to the electric field. The magnetic field vanishes in the static limit $\omega \to 0$.

The time-averaged power radiated per unit solid angle by the dipole moment $\mathbf{p}$ is

$$\frac{dP}{d\Omega} = \frac{1}{2\mu_0}\text{Re}[r^2\hat{\mathbf{n}}\cdot\mathbf{E}\times\mathbf{B}^*] \tag{4.22}$$

where $\mathbf{E}$ and $\mathbf{B}$ are given by (4.18) and (4.19). Thus we find

$$\frac{dP}{d\Omega} = \frac{ck^4}{32\pi^2\epsilon_0}|\mathbf{p}|^2\sin^2\theta \tag{4.23}$$

where θ is the angle between $\mathbf{p}$ and $\hat{\mathbf{n}}$. The total radiated power is

$$P = \frac{ck^4}{12\pi\epsilon_0}|\mathbf{p}|^2. \tag{4.24}$$

4.1.2 Magnetic dipole and electric quadrupole fields

The next order in expansion (4.10) leads to a vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\int d\mathbf{r}'\mathbf{J}(\mathbf{r}')(\hat{\mathbf{n}}\cdot\mathbf{r}') \tag{4.25}$$

where the term $1/r$ in the parenthesis comes from the expansion of $1/|\mathbf{r}-\mathbf{r}'|$ in (4.4). This vector potential can be written as the sum of two terms, one of which gives a transverse magnetic field and the other of which gives a transverse electric field. These physically distinct contributions can be separated by writing the integrand in (4.25) as the sum of a symmetric part in $\mathbf{J}$ and a part that is antisymmetric in $\mathbf{J}$ and $\mathbf{r}$:

$$\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}) = \frac{1}{2}[\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}) + \mathbf{r}(\hat{\mathbf{n}}\cdot\mathbf{J})] + \frac{1}{2}(\mathbf{r}\times\mathbf{J})\times\hat{\mathbf{n}} \tag{4.26}$$

Considering only the second term, its contribution to the vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{ik\mu_0}{4\pi}(\hat{\mathbf{n}}\times\mathbf{m})\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right) \tag{4.27}$$

where

$$\mathbf{m} = \frac{1}{2}\int d\mathbf{r}\,(\mathbf{r}\times\mathbf{J}) \tag{4.28}$$

is the magnetic dipole moment. Since the vector potential (4.27) is proportional to the magnetic field (4.16) for an electric dipole, the magnetic field for the magnetic dipole is equal to the electric field for the electric dipole, and the electric field for the magnetic dipole is equal to the magnetic field for the electric dipole, with the substitution $\mathbf{p} \to \mathbf{m}$.

The symmetric term of (4.26) is related to the electric quadrupole moment density. In fact by an integration by parts

$$\frac{1}{2}\int d\mathbf{r}[\mathbf{J}(\hat{\mathbf{n}}\cdot\mathbf{r}) + \mathbf{r}(\hat{\mathbf{n}}\cdot\mathbf{J})] = -i\frac{\omega}{2}\int d\mathbf{r}[\mathbf{r}(\hat{\mathbf{n}}\cdot\mathbf{r})]\rho(\mathbf{r})$$

where we used the continuity equation. Since the integral involves second moments of the charge density, it corresponds to an electric quadrupole source:

$$\mathbf{A}(\mathbf{r}) = -\frac{ck^2\mu_0}{8\pi}\frac{e^{ikr}}{r}\left(1-\frac{1}{ikr}\right)\int d\mathbf{r}'\,\mathbf{r}'(\hat{\mathbf{n}}\cdot\mathbf{r}')\rho(\mathbf{r}') \tag{4.29}$$

In the far zone, the fields are

$$\mathbf{B} = ik\hat{\mathbf{n}}\times\mathbf{A} \tag{4.30}$$

$$\mathbf{E} = ick(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}. \tag{4.31}$$

As a consequence, the magnetic field is

$$\mathbf{B}(\mathbf{r}) = -i\frac{ck^3\mu_0}{8\pi}\frac{e^{ikr}}{r}\int d\mathbf{r}'(\hat{\mathbf{n}}\times\mathbf{r}')(\hat{\mathbf{n}}\cdot\mathbf{r}')\rho(\mathbf{r}') \tag{4.32}$$

Introducing the quadrupole momentum tensor,

$$Q_{\alpha\beta} = \int d\mathbf{r}(3r_\alpha r_\beta - r^2\delta_{\alpha\beta})\rho(\mathbf{r})$$

we can write the integral in (4.32) as

$$\int d\mathbf{r}'(\hat{\mathbf{n}}\times\mathbf{r}')(\hat{\mathbf{n}}\cdot\mathbf{r}')\rho(\mathbf{r}') = \frac{1}{3}\hat{\mathbf{n}}\times\mathbf{Q}(\hat{\mathbf{n}})$$

where the vector $\mathbf{Q}(\hat{\mathbf{n}})$ has components $Q_\alpha = \sum_\beta Q_{\alpha\beta}n_\beta$, which depends in direction and magnitude both on the observation direction and on the source properties. With these definitions,

$$\mathbf{B}(\mathbf{r}) = -i\frac{ck^3\mu_0}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{n}}\times\mathbf{Q}(\hat{\mathbf{n}}). \tag{4.33}$$

The time-averaged power radiated per unit solid angle by the quadrupole moment vector $\mathbf{Q}(\hat{\mathbf{n}})$ is

$$\frac{dP}{d\Omega} = \frac{ck^6}{1152\pi^2\epsilon_0}|[\hat{\mathbf{n}}\times\mathbf{Q}(\hat{\mathbf{n}})]\times\hat{\mathbf{n}}|^2 \tag{4.34}$$

It is possible to demonstrate [see Jackson, pag.402] that the total radiated power is

$$P = \frac{ck^6}{1440\pi\epsilon_0}\sum_{\alpha\beta}|Q_{\alpha\beta}|^2. \tag{4.35}$$

For a given quadrupole moment, the radiated power scales as ω^6, contrarily to the dipole radiation which scales as ω^4. A simple example of quadrupole radiation source is provided by a spherical oscillating charge distribution. In this case the out-of-diagonal elements of $Q_{\alpha\beta}$ vanish and the diagonal elements are $Q_{11} = Q_{22} = -Q_{33}/2 = -Q_0/2$ and the angular distribution of the emitted radiation is

$$\frac{dP}{d\Omega} = \frac{ck^6}{512\pi^2\epsilon_0}Q_0^2\sin^2\theta\cos^2\theta. \tag{4.36}$$

This distribution presents four lobes, with maxima at $\theta = \pi/4$ and $3\pi/4$. The total radiated power is

$$P = \frac{ck^6}{960\pi\epsilon_0}Q_0^2. \tag{4.37}$$

4.2 Scattering at long wavelengths

The scattering of electromagnetic waves by systems with dimension small compared with a wavelength is an important subject. In such interaction it is convenient to think of the incident fields as inducing electric and magnetic multipoles that oscillate in definite phase relationship with the incident wave and radiate energy in directions other than the direction of incidence. If the wavelength of the radiation is long compared to the size of the scatterer, only the lowest multipoles, usually electric and magnetic dipoles, are important. We assume a plane monochromatic incident wave with polarization vector $\hat{\mathbf{e}}_0$ and direction $\hat{\mathbf{n}}_0$:

$$\mathbf{E}_{\rm inc} = \hat{\mathbf{e}}_0 E_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{r}}, \qquad \mathbf{B}_{\rm inc} = \frac{1}{c}\hat{\mathbf{n}}_0\times\mathbf{E}_{\rm inc} \tag{4.38}$$

This fields induce electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$ in the small scatterer and these dipoles radiate energy in all directions. Far away from the scatterer, the scattered fields are found from (4.19) and (4.18)

$$\mathbf{E}_{\rm sc} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left\{(\hat{\mathbf{n}}\times\mathbf{p})\times\hat{\mathbf{n}} - \frac{1}{c}(\hat{\mathbf{n}}\times\mathbf{m})\right\} \tag{4.39}$$

$$\mathbf{B}_{\rm sc} = \frac{1}{c}\hat{\mathbf{n}}\times\mathbf{E}_{\rm sc} \tag{4.40}$$

where $\hat{\mathbf{n}}$ is the unit vector in the direction of observation and r is the distance away from the scatterer. We define the differential cross section $d\sigma/d\Omega$ as the ratio between the power $dP/d\Omega$ radiated in the direction $\hat{\mathbf{n}}$ with polarization $\hat{\mathbf{e}}$, per unit solid angle, and the incident flux I_{inc} in the direction $\hat{\mathbf{n}}_0$ with polarization $\hat{\mathbf{e}}_0$:

$$\frac{d\sigma}{d\Omega} = r^2 \frac{|\hat{\mathbf{e}}^* \cdot \mathbf{E}_{\text{sc}}|^2}{|\hat{\mathbf{e}}_0^* \cdot \mathbf{E}_{\text{inc}}|^2} = \frac{k^4}{(4\pi\epsilon_0)^2 E_0^2} \left| \hat{\mathbf{e}}^* \cdot \mathbf{p} - \frac{1}{c}(\hat{\mathbf{n}} \times \mathbf{e}^*) \cdot \mathbf{m} \right|^2 \tag{4.41}$$

The dependence on $\hat{\mathbf{n}}_0$ and $\hat{\mathbf{e}}_0$ is implicitly contained in the dipole moments $\mathbf{p}$ and $\mathbf{m}$. The variation of the scattering cross section with the wavenumber as k^4 (or in the wavelength as $1/\lambda^4$) is known as Rayleigh's law, and it is the reason of the blue sky, since in the visible spectrum the red is scattered least and the violet most. Light received away from the direction of the incident beam is more heavily weighted in high frequency (blue) components than the spectral distribution of the incident beam, while the transmitted beam becomes increasingly red in its spectral composition, as well as diminishing in overall intensity.

A small dielectric sphere with radius a and with relative dielectric constant ϵ has an induce dielectric dipole moment

$$\mathbf{p} = 4\pi\epsilon_0 \left(\frac{\epsilon - 1}{\epsilon + 2} \right) a^3 \mathbf{E}_{\text{inc}} \tag{4.42}$$

and there is no magnetic dipole moment. The differential cross section is

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 |\hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0|^2. \tag{4.43}$$

The scattered radiation is linearly polarized in the plane defined by the dipole moment direction $\hat{\mathbf{e}}_0$ and the unit vector $\hat{\mathbf{n}}$. Typically the incident radiation is not polarized. The cross section (4.43) is averaged over the initial polarization $\hat{\mathbf{e}}_0$ for a fixed choice of $\hat{\mathbf{e}}$. The scattering plane is defined by the vector $\hat{\mathbf{n}}_0$ and $\hat{\mathbf{n}}$. By choosing two polarization vectors parallel and perpendicular to the scattering plane, the differential cross sections for scattering with polarization parallel and perpendicular to the scattering plane, averaged over the initial polarization, are

$$\frac{d\sigma_\parallel}{d\Omega} = \frac{k^4 a^6}{2} \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 \cos^2\theta, \qquad \frac{d\sigma_\perp}{d\Omega} = \frac{k^4 a^6}{2} \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 \tag{4.44}$$

where θ is the angle between $\hat{\mathbf{n}}_0$ and $\hat{\mathbf{n}}$. The differential cross section summed over scattered polarization is

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_\parallel}{d\Omega} + \frac{d\sigma_\perp}{d\Omega} = k^4 a^6 \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 \frac{1 + \cos^2\theta}{2} \tag{4.45}$$

and the total scattering cross section is

$$\sigma = \frac{8\pi}{3} k^4 a^6 \left| \frac{\epsilon - 1}{\epsilon + 2} \right|^2 . \tag{4.46}$$

Notice that at $\theta = \pi/2$ the scattered light is completely linearly polarized perpendicular to the scattering plane.

Up to now we have considered the scattering from a single dipole moment. In the presence of many scatterers with positions $\mathbf{r}_j$, their moment will contain a factor $\exp[ik\hat{\mathbf{n}}_0 \cdot \mathbf{r}_j]$. Furthermore, the fields radiated at a large distance will contain a phase factor $\exp[-ik\hat{\mathbf{n}} \cdot \mathbf{r}_j]$. Assuming that all the scatterers are identical, the cross section (4.41) will be multiplied by the factor form

$$F(\mathbf{q}) = \left| \sum_j e^{i\mathbf{q}\cdot\mathbf{r}_j} \right|^2 \tag{4.47}$$

where $\mathbf{q} = k(\hat{\mathbf{n}}_0 - \hat{\mathbf{n}})$ is the scattered wavenumber. If $\mathcal{N}$ scatterers are randomly distributed, then $F(\mathbf{q}) = \mathcal{N}$, whereas it is $F(\mathbf{q}) = \mathcal{N}^2$ if $\mathbf{q} = 0$ (forward direction) or if the scatteres are regularly distributed in some specific direction. In the first case we talk about *incoherent scattering*.

An important issue is the scattering from a gas where the atoms or molecules are randomly distributed. In this case the scattering is given by the fluctuations of the dielectric constant $\delta\epsilon(\mathbf{r}) \ll \epsilon_0$. Assuming the following relation between $\mathbf{D}$ and $\mathbf{E}$:

$$\mathbf{D}(\mathbf{r}) = [\epsilon_0 + \delta\epsilon(\mathbf{r})]\mathbf{E}(\mathbf{r})$$

it is possible to see that the resulting differential cross section is analog to that of Eq.(4.41), where instead of the electric dipole $\mathbf{p}$ it appears a diffusion amplitude proportional to $\mathbf{D} - \epsilon_0\mathbf{E} \approx (\delta\epsilon(\mathbf{r})/\epsilon_0)\mathbf{D}^{(0)}$, where $\mathbf{D}^{(0)}$ represents the zero order of a perturbative series. The final result is

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} |\hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0|^2 \left| \int d\mathbf{r} \frac{\delta\epsilon(\mathbf{r})}{\epsilon_0} e^{i\mathbf{q}\cdot\mathbf{r}} \right|^2 . \tag{4.48}$$

If the single molecules have a dipole moment $\mathbf{p}_j = \epsilon_0 \gamma_m \mathbf{E}_{inc}(\mathbf{r}_j)$ with polarizability γ_m, then we can write

$$\delta\epsilon(\mathbf{r}) = \epsilon_0 \sum_j \gamma_m \delta(\mathbf{r} - \mathbf{r}_j) \tag{4.49}$$

and

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} |\gamma_m|^2 |\hat{\mathbf{e}}^* \cdot \hat{\mathbf{e}}_0|^2 F(\mathbf{q}). \tag{4.50}$$

If the $\mathcal{N}$ molecules are randomly distributed, then $F(\mathbf{q}) \sim \mathcal{N}$ and the total scattering cross section is $\sigma = \mathcal{N}\sigma_1$ where

$$\sigma_1 = \frac{k^4}{6\pi}|\gamma_m|^2 \tag{4.51}$$

is the cross section for a single molecule. Since in a dilute gas $\epsilon = \epsilon_0(1+N\gamma_m)$ where N is the number of molecules per volume, then

$$\sigma_1 = \frac{k^4}{6\pi N^2}|n^2 - 1|^2 \tag{4.52}$$

The cross section (4.52) is the scattered power per molecule and per unit incident flux. Crossing a thickness dx of the gas the relative loss of flux is $N\sigma_1 dx$, so that the incident intensity becomes $I(x) = I_0 \exp(-\alpha x)$, where

$$\alpha = N\sigma_1 = \frac{2k^4}{3\pi N}|n-1|^2 \tag{4.53}$$

is the absorption coefficient (where we assumed $n^2 - 1 \sim 2(n-1)$). This expression represents the Rayleigh's scattering from randomly distributed molecules in a gas. Notice that the absorption coefficient depends on $1/N$, and decreases with an increasing density. Assuming $n - 1 \approx 2.9 \times 10^{-4}$ and $N \approx 2.7 \times 10^{19}$ molecules/cm^3, the attenuation length α^{-1} is 30, 77 and 188 km at $\lambda = 410$nm (violet), $\lambda = 520$nm (green) and $\lambda = 650$nm (red). The blueness of the sky, the waveness of the winter sun, and the ease of sunburning at midday in summer are all consequences of Rayleigh scattering in the atmosphere.

Capitolo 5

Theory of Relativity

A summary of the theory of the special relativity, specially focused on the electromagnetic fields.

5.1 Foundations of Theory of Relativity

5.1.1 The principle of relativity

In Newtonian mechanics, the Galileo's principle of relativity is that the laws of mechanics are identical in all inertial systems of reference under Galileo transformations. Suppose there are two inertial reference frames K and K' with K' moving at a velocity $\mathbf{V}$ relative to K. Then

$$\mathbf{r} = \mathbf{r}' + \mathbf{V}t', \qquad t = t'. \tag{5.1}$$

Note that time is absolute in classical mechanics. It can be shown, however, that Maxwell's equations do not satisfy Galileo's principle of relativity under Galileo transformation. There were three possible ways to solve this problem:

1. The Maxwell's equations were wrong. The proper theory of electromagnetism was invariant under Galileo transformations.

2. Galileo's principle of relativity only applied to classical mechanics. Electromagnetism is not mechanics.

3. There exists a general principle of relativity for both classical mechanics and electromagnetism.

Einstein did some really hard thinking and chose (3). He then proposed the following two postulates, based on experiments done by other people and lots of his own thinking:

1. *The principle of relativity: all the laws of nature are identical in all inertial systems of reference.*

2. *The constancy of the speed of light. The speed of light (c) is independent of the motion of its source; its numerical value is* $c = 2.998\,10^8$ *m/s.*

It can be immediately shown that time being absolute is not consistent with Einsteins principle of relativity. From Galileo's principle of relativity, the velocity transforms like $\mathbf{v} = \mathbf{v}' + \mathbf{V}$. This equation directly follows from that Δt being invariant. However, this leads to that $\mathbf{v}$ can be larger than c, not consistent with Einstein's second postulate.

5.1.2 Intervals in spacetime

Definition of interval

For convenience of presentation, we will first introduce few concepts.

- *Event:* An event is described by the place it occurred and the time when it occurred.

We also introduce a fictitious four-dimensional space, marked by three space coordinates and one time coordinate. For an idealized particle, an event is defined by three coordinates and the time when the event occurs. We consider two inertial reference frames K and K', with parallel axes. Suppose K' moves relative to K with V along the x-axis. Now we define two events in K system:

- *Event 1*: sending out a light signal from (x_1, y_1, z_1) at t_1.

- *Event 2*: the arrival of the signal at (x_2, y_2, z_2) at t_2.

The signal traveled $c\Delta t$, or $(\Delta x^2 + \Delta y^2 + \Delta z^2)^{1/2}$, so we have

$$c^2 \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = 0$$

Because of the constancy of light speed c, we have in K' system

$$c^2 \Delta t'^2 - (\Delta x'^2 + \Delta y'^2 + \Delta z'^2) = 0$$

If (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) are the coordinates of any two events, we define Δs by

$$\Delta s = (c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2)^{1/2} \tag{5.2}$$

and call it the *interval* between these two events.

The invariance of interval

From the previous analysis, we reach an important conclusion that if $\Delta s = 0$ in K, then $\Delta s' = 0$ in any K'. To find the relationship between Δs and $\Delta s'$ for $\Delta s \neq 0$, we consider two events infinitely close to each other. In this case, the interval ds is

$$ds = (c^2dt^2 - dx^2 - dy^2 - dz^2)^{1/2}$$

The form of ds allows us to regard it as the distance between two world points in the fictitious four-dimensional space. This space, called Minkowski space (axes: (ct, x, y, z), is pseudo-Euclidean. [if Euclidean, the distance would be $dl = (c^2dt^2 + dx^2 + dy^2 + dz^2)^{1/2}$]. Now the question is: What is the relationship between ds in K and ds' in K' in general (valid for $ds \neq 0$)? We have two constraints: (a) If $ds = 0$, then $ds' = 0$ and (b) ds and ds' are infinitesimal of the same order. From the homogeneity of space and time, these two conditions imply that

$$ds^2 = ds'^2 \tag{5.3}$$

The interval between two events is invariant under transformation from one inertial frame to another. This is the mathematical formulation of the invariance of c. From the invariance of ds, we can immediately reach the following conclusion: If a particle moves with $|v| < c$ in K, then $|v'| < c$ in all other K', because $ds^2 = c^2dt^2 - dx^2 = (c^2 - v^2)dt^2$ is an invariant.

Space-like and time-like intervals

With the invariance of ds, time is no longer absolute. Statements like two events occur simultaneously do not necessarily hold if we transform to another reference frame. Lets now discuss this type of problem. Our first question is, if two events occur at two different times in K ($\Delta t \neq 0$), can we find a reference frame K' in which $\Delta t' = 0$? Suppose we can find a K' so that $\Delta t' = 0$. From the invariance of Δs^2, we have

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = -\Delta x'^2 < 0$$

Hence if $\Delta s^2 < 0$ (or if Δs is imaginary) it is possible to find a reference frame where $\Delta t' = 0$. Imaginary intervals are said to be *space-like*. For space-like intervals, the concepts of "simultaneous", "earlier", and "later" are relative. Note that $c^2\Delta t^2 < \Delta x^2$ means that the two events are so separated that any signal can propagate from one point to the other point within Δt.

Following the previous question, it is natural to ask another one: if two events occur at two different times in K ($\Delta t \neq 0$), what is the condition for ($\Delta t' \neq 0$) in all K'? From the invariance of Δs^2, we have $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 - \Delta x'^2$. The minimum value for $c^2\Delta t'^2$ to take is when $\Delta x'^2 = 0$, and

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 > 0$$

Hence if $\Delta s^2 > 0$ (or if Δs is real), it is not possible to find a K' so that $\Delta t' = 0$. Note that $c^2\Delta t^2 > \Delta x^2$. Real intervals are said to be *time-like*. Note that for time-like intervals, it is possible to connect two events using a signal with propagation speed less than c, since $\Delta x/\Delta t < c$. For time-like intervals, the concepts of "future" and "past" are absolute. To see this, let's assume the interval between event 1 and event 2 is time-like, then

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 - \Delta x'^2 > 0$$

Solving for $\Delta t'$ from this equation, we have

$$\Delta t' = \pm|\Delta t|\left[1 - \frac{\Delta x^2 - \Delta x'^2}{c^2\Delta t^2}\right]^{1/2}$$

Which sign should we take for $\Delta t'$? To see this, we know that $\Delta t' \to \Delta t$ as $V \to 0$, where V is the velocity of K' relative to K. Hence

$$\Delta t' = |\Delta t|\left[1 - \frac{\Delta x^2 - \Delta x'^2}{c^2\Delta t^2}\right]^{1/2}$$

Therefore for time-like intervals, we must have: (a) If $\Delta t > 0$, then $\Delta t' > 0$ in all K'. (b) if $\Delta t < 0$, then $\Delta t' < 0$ in all K'. That is, the concepts of "future" or "past" are absolute for time-like intervals.

The concept of time-like or space-like intervals are important if two events are causally related. For event 1 to be the reason of event 2, event 1 must occur "before" event 2 in all reference frames; i.e., event 1 is in the "absolute past" of event 2. Therefore the interval must be time-like. On the other hand, if event 1 is the reason of event 2, a signal has to propagate from event 1 to event 2. The propagation speed of signal is then $|\Delta x/\Delta t|$, which is less than c, since the interval is time-like. This is the same statement as that c is the maximum speed of propagation of interaction. Finally we point out that because ds is an absolute concept, the time-like or space-like property of an interval is also absolute.

5.1.3 Proper time

Since time is not absolute, it is convenient to introduce the concept of "proper time". Proper time is the time read by a clock moving with the object; it is normally denoted by τ. Lets now derive the relationship between dt observed in laboratory frame K and the proper time $d\tau$ of the object. In the laboratory frame K, the object moves with a constant velocity v. Let the inertial reference frame moving with the object be called K'. From the invariance of ds^2 and $dx' = 0$ (because the object is at rest in K'), we have

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2$$

From this equation,

$$d\tau = ds/c = dt(1 - dx^2/c^2 dt^2)^{1/2} = dt(1 - v^2/c^2)^{1/2} = \frac{dt}{\gamma}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and $\beta = v/c$. Since $\beta < 1$ and $\gamma \geq 1$, then

$$d\tau \leq dt$$

Or by integrating this equation, we have

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} \frac{dt}{\gamma} \leq t_2 - t_1.$$

i.e. *moving clocks go more slowly than those at rest.*

5.1.4 The Lorentz transformation

We know that the Galileo transformations (5.1) are not correct in general. Now time is not absolute, how do we make coordinate transformations? Again we consider two inertial reference frames K and K', where K' moves relative to K at $\mathbf{V}$. For an event with coordinate $(t, \mathbf{r})$, what is the corresponding coordinate $(t', \mathbf{r}')$ in K'? First, for simplicity, we assume that the two coordinate systems have the same origin; i.e., $t = 0$, $\mathbf{r} = 0$ in K corresponds to the point $t' = 0$, $\mathbf{r}' = 0$ in K'. Second, the basic requirement we have is that the transformation should not change any Δs, the distance between any two points in Minkowski space. The Δs in K is

$$\Delta s^2 = c^2 \Delta t^2 - \Delta r^2$$

To make sure that changing coordinate system does not change Δs^2, we make use of the fact that $\cosh^2\theta - \sinh^2\theta = 1$ for any θ. Therefore, we can assume $c\Delta t = \Delta s \cosh\theta$ and $\Delta r = \Delta s \sinh\theta$ in K and $c\Delta t' = \Delta s \cosh\theta'$ and $\Delta r' = \Delta s \sinh\theta'$ in K'. In this ways Δs is always invariant in any coordinate system. Now we try to find out what is the relation of coordinates between K and K'. Lets assume $\theta = \theta' + \chi$. Then

$$\begin{aligned} c\Delta t &= \Delta s \cosh(\theta' + \chi) = \Delta s \cosh(\theta')\cosh(\chi) + \Delta s \sinh(\theta')\sinh(\chi) \\ &= c\Delta t' \cosh(\chi) + \Delta r' \sinh(\chi) \\ \Delta r &= \Delta s \sinh(\theta' + \chi) = \Delta s \sinh(\theta')\cosh(\chi) + \Delta s \cosh(\theta')\sinh(\chi) \\ &= \Delta r' \cosh(\chi) + c\Delta t' \sinh(\chi) \end{aligned}$$

To obtain the coordinate transformation, let's consider the interval between any event and the origin of the coordinate system. Hence the transformations are

$$r = r' \cosh\chi + ct' \sinh\chi \tag{5.4}$$

$$ct = r' \sinh\chi + ct' \cosh\chi \tag{5.5}$$

Starting from the origin of the K' system, we have $r' = 0$ so that

$$\begin{aligned} r &= ct' \sinh\chi \\ ct &= ct' \cosh\chi \end{aligned}$$

We thus have $\beta = V/c = r/ct = \tanh\chi$. Since $\cosh^2\chi - \sinh^2\chi = 1$, we obtain

$$\cosh\chi = \gamma \tag{5.6}$$

$$\sinh\chi = \beta\gamma \tag{5.7}$$

where $\gamma = (1 - \beta^2)^{-1/2}$. Assuming that $\mathbf{V}$ is in the direction of $\hat{\mathbf{x}}$ i.e. K' moves relative to K with $\mathbf{V} = V\hat{\mathbf{x}}$, so that $y' = y$ and $z' = z$, we obtain

$$x = \gamma(x' + c\beta t') \tag{5.8}$$

$$ct = \gamma(ct' + \beta x') \tag{5.9}$$

$$y = y' \tag{5.10}$$

$$z = z' \tag{5.11}$$

These are the *Lorentz transformations*. The inverse transformation can be obtained by noting that K moves relative to K' with velocity $-\mathbf{V}$:

$$x' = \gamma(x - c\beta t) \tag{5.12}$$

$$ct' = \gamma(ct - \beta x) \tag{5.13}$$

They reduce to the Galileo's transformations in the limit $\beta \ll 1$, or $\gamma \approx 1$.

From the Lorentz transformation, we derive the proper length, which is the length of an object in an inertial reference frame in which it is at rest. We denote the proper length by l_0. Suppose we have a rod parallel to the x'-axis; it is at rest in K' system. In K', the length of the rod is $l_0 = |x'_2 - x'_1|$. In K, to measure the length of the rod we have x_1 at t_1 and x_2 at t_2, and $t_1 = t_2$. From the inverse Lorentz transformation, we have

$$x'_1 = \gamma(x_1 - c\beta t_1)$$

$$x'_2 = \gamma(x_2 - c\beta t_2)$$

Noting that $t_1 = t_2$, we have

$$x'_2 - x'_1 = \gamma(x_2 - x_1)$$

so

$$l = \frac{l_0}{\gamma} = l_0\sqrt{1-\beta^2} \tag{5.14}$$

Since $l < l_0$, this is called *Lorentz contraction*.

Using the Lorentz transformation, we can re-derive the equation for proper time. Suppose a clock to be at rest in K', two events occurred at x' at t'_1 and t'_2. Then $\Delta\tau = t'_2 - t'_1$ and in K

$$t_1 = \gamma(t'_1 + \beta x'/c)$$

$$t_2 = \gamma(t'_2 + \beta x'/c)$$

therefore,

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\beta^2}},$$

the same as we obtained before.

5.1.5 Transformation of velocities

To completely determine the state of a particle, we need both its space coordinates and velocity. From the Lorentz transformation, we can easily derive the formulas for the transformation of velocities. Suppose a particle has a velocity $\mathbf{v}$ in K and $\mathbf{v}'$ in K' where K' moves with $V = c\beta$ relative to K along the x axis. The differentiation of the Lorentz transformation gives

$$dx = \gamma(dx' + c\beta dt') \tag{5.15}$$

$$dt = \gamma(dt' + \beta dx'/c) \tag{5.16}$$

$$dy = dy' \tag{5.17}$$

$$dz = dz' \tag{5.18}$$

The velocity transformation can be obtained easily as $\mathbf{v} = d\mathbf{r}/dt$, $\mathbf{v}' = d\mathbf{r}'/dt'$. The resulting velocity transformations are

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} \tag{5.19}$$

$$v_y = \frac{v'_y}{\gamma(1 + v'_x V/c^2)} \tag{5.20}$$

$$v_z = \frac{v'_z}{\gamma(1 + v'_x V/c^2)} \tag{5.21}$$

In case of $c \to \infty$, the above equations recovers the familiar Galileo's velocity transformation.

5.1.6 Vectors and Tensors in Spacetime

Minkowski once said: *Space of itself, and time of itself will sink into mere shadows, and only a kind of union between them shall survive.* Indeed, from the previous introduction, in the framework of the special theory of relativity time and space are put on equal footing: both time and space are not absolute and are transformed together from one reference system to another reference system.

Four-vectors

It is convenient to discuss time and space together in terms of vectors in a pseudo-Euclidean Minkowski space. For example, the coordinates of an event (ct, x, y, z) can be considered as the components of a radius four-vector, $x_0 = ct$, $x_1 = x$,

$x_2 = y$ and $x_3 = z$ The radius four-vector satisfies the Lorentz transformation when changing reference system from K to K'. With this representation, the Lorentz transformation can be written as

$$x'_0 = \gamma(x_0 - \beta x_1) \tag{5.22}$$

$$x'_1 = \gamma(x_1 - \beta x_0) \tag{5.23}$$

$$x'_2 = x_2 \tag{5.24}$$

$$x'_3 = x_3 \tag{5.25}$$

We model four-vectors after the radius four-vector.

- Definition: any set of four quantities A_0, A_1, A_2, A_3, which transform like the radius four-vector x_α under the change from K to K' is called a four-vector.

This definition essentially means that when changing from K to K', all four-vectors are transformed using Lorentz transformation:

$$A'_0 = \gamma(A_0 - \beta A_1) \tag{5.26}$$

$$A'_1 = \gamma(A_1 - \beta A_0) \tag{5.27}$$

$$A'_2 = A_2 \tag{5.28}$$

$$A'_3 = A_3 \tag{5.29}$$

The invariance with respect a transformation from K to K', based on the Einstein's second postulate, it is expressed by the invariance of the 'magnitude' of the four-vector:

$$(A_0)^2 - (A_1)^2 - (A_2)^2 - (A_3)^2$$

To more conveniently represent the dot-product between four-vectors, we introduce two kinds of components of four-vectors:

- The contravariant components, denoted by A^α
- The covariant components, denoted by A_α

so that A^α and A_α are related by

$$A_0 = A^0, \quad A_1 = -A^1, \quad A_2 = -A^2, \quad A_3 = -A^3$$

so the square magnitude of a four-vector A^α can now be compactly written as

$$\sum A_\alpha A^\alpha = A_0 A^0 + A_1 A^1 + A_2 A^2 + A_3 A^3 = A_\alpha A^\alpha = A^\alpha A_\alpha$$

The dot-product between two four-vectors A^α and B^α is $A^\alpha B_\alpha = A_\alpha B^\alpha$. To convert between the contravariant and the covariant components, one needs to use the metric coefficients $g^{\alpha\beta}$. From the definitions of contra- and co-variant components above, we can see that for the four-vectors in Minkowski space,

$$(g^{\alpha\beta}) = (g_{\alpha\beta}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{5.30}$$

Therefore we can convert between A^α and A_α by

$$A_\alpha = g_{\alpha\beta} A^\beta, \qquad \text{or} \qquad A^\alpha = g^{\alpha\beta} A_\beta$$

Four-scalars

- Definition: we define a four-scalar an invariant under the transformation of coordinates, here the Lorentz transformation.

Because they are invariant under Lorentz transformations, the four-scalars play important roles in the study of special relativity and electrodynamics. For example, the dot product between two four-vectors is a four-scalar. Note that there is no free index in this term. Example: the interval $s^2 = x_\alpha x^\alpha$ is a four-scalar.

Four-tensors

- Definition: A second-order four-tensor is a set of 16 quantities $F^{\alpha\beta}$, which under coordinate transformations, transform like the products of components of two four-vectors.

A four-tensor can be written in different forms: $F^{\alpha\beta}$, $F_{\alpha\beta}$, $F_\alpha^{\ \beta}$ and $F^\alpha_{\ \beta}$. Different positions of indices represent different kind of components of a four-tensor. In case of a four-vector, there are two different kinds of components (co- and contra-variant components); in the case of a four-tensor, there are four different kinds of components. Again one uses the metric tensor $g_{\alpha\beta}$ to convert between different kinds of representations of a four-tensor. For instance,

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$$

We have the following simple rules: (a) raising or lowering index $(1, 2, 3)$ changes the sign, (b) raising or lowering index (0) does not change the sign. From four-vectors

A^α and B^α, we know $A^\alpha A_\alpha$ and $A^\alpha B_\alpha$ are four-scalars; i.e., they are invariants under Lorentz transformations. The trick to construct 4-scalars from 4-vectors/tensors is that all indices are dummy indices; there is no free index in the final expression. Similarly, we can also construct four-scalars from 4-tensors. For example, for $F^{\alpha\beta}$, we know that $F^{\alpha\beta}F_{\alpha\beta}$ is an invariant. Also it is possible to construct four-scalars from four-vectors and four-tensors, like $A^\alpha B^\beta F_{\alpha\beta}$ is a four-scalar, since there is no free index in the final expression. This operation is called *contraction*.

Basic four-vector/tensor differential calculus

Lets now introduce four-vector calculus. The four-gradient of a scalar ϕ is the four-vector

$$\frac{\partial\phi}{\partial x^\alpha} = \left(\frac{1}{c}\frac{\partial\phi}{\partial t}, \boldsymbol{\nabla}\phi\right) \tag{5.31}$$

From the location of the index, we know they are the covariant components of a four-vector:

$$\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \boldsymbol{\nabla}\right) \tag{5.32}$$

Similarly, you have the contra-variant components of a four-vector,

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial x^0}, -\boldsymbol{\nabla}\right) \tag{5.33}$$

The 4-divergence of a 4-vector $A^\alpha = (A_0, \mathbf{A})$ is a four-scalar,

$$\frac{\partial A^\alpha}{\partial x^\alpha} = \partial_\alpha A^\alpha = \frac{1}{c}\frac{\partial A^0}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} \tag{5.34}$$

It is a scalar under coordinate transformation.

5.1.7 Four-velocity and four-acceleration

Using the knowledge of four vectors, we can construct the four-velocity by $u^\alpha = dx^\alpha/d\tau$, where $d\tau = dt/\gamma$

$$u^0 = \frac{cdt}{dt}\gamma = \gamma c \tag{5.35}$$

$$u^1 = \frac{dx}{dt}\gamma = \gamma v_x \tag{5.36}$$

$$u^2 = \frac{dy}{dt}\gamma = \gamma v_y \tag{5.37}$$

$$u^3 = \frac{dz}{dt}\gamma = \gamma v_z, \tag{5.38}$$

Therefore, the four-velocity is

$$u^\alpha = (\gamma c, \gamma \mathbf{v}) \tag{5.39}$$

The contraction of the four-velocity is, noting that $d\tau = ds/c$,

$$u^\alpha u_\alpha = \gamma^2(c^2 - v^2) = c^2 \tag{5.40}$$

which is a scalar and an invariant. One can construct the four-acceleration similarly by $w^\alpha = du^\alpha/d\tau$. One can easily show that $u_\alpha w^\alpha = 0$, so that the four-acceleration is always 'perpendicular' to the four-velocity.

5.2 Relativistic Dynamics

5.2.1 A brief review of Lagrangian mechanics

In classical mechanics, the action is defined by the integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt.$$

with $L(q, \dot{q}, t)$ the Lagrangian. The *principle of least action* says that the path from $q(t_1) = q^{(1)}$ to $q(t_2) = q^{(2)}$ minimizes the action S, i.e. $\delta S = 0$. To obtain the equations of motion from the principle of least action, note that if $q = q(t)$ is the actual path, then for a given variation of $q(t) + \delta q(t)$ the change in S is

$$\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta\dot{q}, t)dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0$$

with the constraints $\delta q(t_1) = 0$ and $\delta q(t_2) = 0$. Keeping only the first-order terms in δS, we have

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} \right) dt = 0$$

Since $\delta\dot{q} = (d/dt)\delta q$, by an integration per parts,

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0$$

and δS must vanish for all δq, so that

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$

These equations are called *Lagrange's equations.* If we know L of a system, then we can write down its equations of motion. Note that the Lagrangian is not uniquely determined for a given system. For if we have two Lagrangians $L'(q, \dot{q}, t)$ and $L(q, \dot{q}, t)$, and

$$L' = L + \frac{df(q,t)}{dt}$$

then δS and $\delta S'$ would lead to the same motion equations.

Let's find the Lagrangian for a free particle in an inertial reference system. Note that time is homogeneous so L does not depend on t. Space is homogeneous so L does not depend on $\mathbf{r}$. Space is isotropic, therefore L must not depend on the direction of $\mathbf{v}$. In conclusion, $L = L(v^2)$. Applying the Lagrange's equation, noting that $\partial L/\partial v$ must be a function of v only, the equation of motion is $\dot{v} = 0$. This equation states that a free body in an inertial reference frame moves with a constant velocity. Of course, this velocity can be zero, which means the free body is at rest all the time. This is the laws of *inertia.* To find the form of $L(v^2)$, we need to use Galileo's principle of relativity. Consider two inertial reference frames K and K', with K in motion with $\delta\mathbf{V}$ relative to K'. Then $\mathbf{v}' = \mathbf{v} + \delta\mathbf{V}$. Because of Galileo's relativity principle, $L' = L(v'^2) = L(v^2 + 2\mathbf{v}\cdot\delta\mathbf{V})$, the leading terms of L' are

$$L(v'^2) = L(v^2) + \frac{\partial L}{\partial v^2} 2\mathbf{v}\cdot\delta\mathbf{V}$$

L' and L should lead to the same equations of motion, therefore

$$L' = L + \frac{df(\mathbf{r},t)}{dt}.$$

Hence

$$\frac{\partial L}{\partial v^2} 2\mathbf{v}\cdot\delta\mathbf{V} = \frac{df(\mathbf{r},t)}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f(\mathbf{r},t)$$

and

$$\frac{\partial L}{\partial v^2} 2\delta\mathbf{V} = \boldsymbol{\nabla} f(\mathbf{r},t)$$

Since the LHS is a function of v only, while the RHS is a function of t and $\mathbf{r}$ only, then

$$\frac{\partial L}{\partial v^2} = \alpha$$

where α is some constant. Hence

$$L = \alpha v^2 = \frac{m}{2} v^2$$

where we set $\alpha = m/2$ where m is the mass of the particle. The Lagrangian for a particle moving in a given external field can be obtained by adding to $L = mv^2/2$ a quantity describing the interaction between the field and the particle,

$$L = \frac{m}{2}v^2 - U(\mathbf{r}, t)$$

so that the equation of motion is

$$m\frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla} U.$$

Using Lagrangian mechanics, it is very convenient to discuss conservation properties. Here we discuss the conserved quantities from the isotropy and homogeneity of space and time. If time is homogeneous, then for a closed system $L = L(q_i, \dot{q}_i)$. The total time derivative of L is

$$\frac{d}{dt}L = \dot{q}_i\frac{\partial L}{\partial q_i} + \frac{d\dot{q}_i}{dt}\frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d\dot{q}_i}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right)$$

so that

$$\frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$$

We define total energy E of the system by

$$E = \dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L.$$

Therefore, if time is homogeneous, E is a constant. If space is homogeneous i.e., if we replace $\mathbf{r}_i$ by $\mathbf{r}_i + \delta\mathbf{R}$, then

$$\delta L = \sum_i \frac{\partial L}{\partial \mathbf{r}_i}\cdot\delta\mathbf{R} = 0$$

therefore

$$\sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0$$

From Lagrange's equation,

$$\sum_i \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \mathbf{v}_i} = 0$$

If space is homogeneous, for a closed system,

$$\mathbf{P} = \sum_i \frac{\partial L}{\partial \mathbf{v}_i}$$

is constant. This is called the momentum of the system.

5.2.2 Relativistic action for a free particle

We will now apply what we have introduced to obtain the relativistic action for a free particle,

$$S = \int_a^b dS.$$

The relativistic action must satisfy: (a) dS is invariant under Lorentz transformation, or differed by a $df(x^\alpha)$ when transformed from K' to K; (b) It should recover the non-relativistic dynamics for $\beta \ll 1$.

From the previous section, we know that for a free particle, the event interval ds is an invariant. So we try

$$S = \int_a^b \alpha ds.$$

where α is a four-scalar. Let's see whether S can recover non-relativistic dynamics if $c \to \infty$. If it can, then it suits our need and can be used as the relativistic action for a free particle. To recover the non-relativistic dynamics, we first write S in the normal 3D form, i.e. in the $S = \int Ldt$ form. Because $ds = cd\tau = cdt\sqrt{1 - v^2/c^2}$, we write

$$S = \int_a^b \alpha c\sqrt{1 - v^2/c^2}dt.$$

so that the Lagrangian is

$$L = \alpha c\sqrt{1 - \frac{v^2}{c^2}}$$

in case of $c \to \infty$ we obtain

$$L \approx \alpha c - \alpha\frac{v^2}{2c}$$

Therefore, if $\alpha = -mc$, then S satisfies the two constraints. The relativistic action for a free particle is then

$$S = -mc\int_a^b ds \tag{5.41}$$

and the corresponding Lagrangian is

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} = -\frac{mc^2}{\gamma}. \tag{5.42}$$

5.2.3 Energy and momentum

The momentum of a particle is given by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$$

so, using (5.42),

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m\mathbf{v} \tag{5.43}$$

The energy of a particle is defined from L by

$$E = \mathbf{v} \cdot \frac{\partial L}{\partial \mathbf{v}} - L = \mathbf{v} \cdot \mathbf{p} - L \tag{5.44}$$

Using (5.43) we have

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 \tag{5.45}$$

If $v \to 0$ then $E = mc^2$, the rest energy of the particle. If $c \to \infty$, $E \approx mc^2 + (m/2)v^2$. This definition of energy leads to that mass is not a conserved quantity anymore in relativistic mechanics. Suppose we have a body with mass m, consisting of n particles. In classical mechanics, we have $m = \sum_i m_i$. In relativistic mechanics, however, the energy of a body at least contains: a) the rest energies of its constituent particles $\sum_i m_i c^2$, (b) the kinetic energy of particles, c) the interaction energy etc.. Therefore, $mc^2 \neq \sum_i m_i c^2$ and $m \neq \sum_i m_i$. Only the law of conservation of energy holds.

Energy and momentum are closely related. Let consider the four-velocity we introduced $u^\alpha = (\gamma c, \gamma \mathbf{v})$. Multiplying u^α by m, a four-scalar, we have the four-momentum vector

$$p^\alpha = mu^\alpha = (\gamma mc, \gamma m\mathbf{v}) = (E/c, \mathbf{p})$$

Since p^α is a four-vector, we can apply the knowledge we know about four-vectors. First, we can immediately obtain their transformation equation, since any four-vector transforms like the radius four-vector x^α i.e.

$$E' = \gamma(E - c\beta p_x) \tag{5.46}$$

$$p'_x = \gamma(p_x - \beta E/c) \tag{5.47}$$

$$p'_y = p_y \tag{5.48}$$

$$p'_z = p_z \tag{5.49}$$

Also the dot product of a four-vector with itself is a Lorentz invariant. Since four-momentum $p^\alpha = mu^\alpha$, we immediately have

$$p_\alpha p^\alpha = m^2c^2$$

Substituting $p^\alpha = (E/c, \mathbf{p})$ and $p_\alpha = (E/c, -\mathbf{p})$,

$$E^2/c^2 - p^2 = m^2c^2$$

or

$$E^2 = m^2c^4 + c^2p^2, \quad \text{and} \quad \mathbf{p} = \frac{E}{c^2}\mathbf{v} \tag{5.50}$$

This is the energy-momentum relation. If $v \to c$, both p and E become infinite unless $m \to 0$. Actually, for a photon ($m = 0$), we have $p = E/c$.

5.3 Charges in a Given Electromagnetic Field

5.3.1 Four-potentials of a field

As in classical mechanics, the total action for a particle in a given field is

$$S = S_p + S_{pf} = -mc\int ds + S_{pf}$$

Here S_p is the action for the particle, and S_{pf} is the action characterizing the interaction. Experiments suggest that the interaction is determined by the charge q, and the four-vector potential A^α characterizing the field. To make the integral S_{pf} a Lorentz invariant, we can construct

$$S_{pf} = -q\int A^\alpha dx_\alpha$$

where $A^\alpha = (\phi/c, \mathbf{A})$, where ϕ and $\mathbf{A}$ are the scalar and vectorial electromagnetic potentials. Therefore the total action is

$$S = \int_a^b (-mcds - qA^\alpha dx_\alpha) \tag{5.51}$$

In 3D coordinates,

$$S_{pf} = \int_a^b (-q\phi dt + q\mathbf{A}\cdot d\mathbf{r}) = q\int_a^b (\mathbf{A}\cdot\mathbf{v} - \phi)dt$$

Hence, the total Lagrangian is

$$L = -\frac{mc^2}{\gamma} + q(\mathbf{A}\cdot\mathbf{v} - \phi) \tag{5.52}$$

5.3.2 Equations of motion of a charge in a field

We derive here the equations of motion of a charge. For simplicity, we assume that the charge is small, so its effects on the field can be neglected. The field potential does not depend on the position and velocity of the charge. The Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$

From the total Lagrangian

$$\frac{\partial L}{\partial \mathbf{r}} = \boldsymbol{\nabla} L = q\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{v}) - q\boldsymbol{\nabla}\phi$$

From the vectorial identity

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{v}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}+\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{v})+\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}+\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})$$

On the other hand,

$$\frac{\partial L}{\partial \mathbf{v}} = \mathbf{p} + q\mathbf{A}. \tag{5.53}$$

Therefore the Lagrangian equation gives

$$\frac{d}{dt}(\mathbf{p}+q\mathbf{A}) = q[(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) - \boldsymbol{\nabla}\phi] \tag{5.54}$$

However

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}$$

so that

$$\frac{d\mathbf{p}}{dt} = q\left[-\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A})\right] = q[\mathbf{E}+\mathbf{v}\times\mathbf{B}] \tag{5.55}$$

This is the usual Lorentz force felt by a particle in a given field. Notice that the Lorentz gauge

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \tag{5.56}$$

can be written in covariant form as

$$\partial_\alpha A^\alpha = 0. \tag{5.57}$$

The wave equations for ϕ and $\mathbf{A}$ in the Lorentz gauge,

$$\begin{aligned}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi &= \frac{\rho}{\epsilon_0}\\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} &= \mu_0\mathbf{J}\end{aligned}$$

can be written in a covariant form for A^α:

$$\Box A^\alpha = \mu_0 J^\alpha \tag{5.58}$$

where $\Box = \partial_\alpha \partial^\alpha$ and

$$A^\alpha = \left(\frac{\phi}{c}, \mathbf{A}\right), \qquad J^\alpha = (c\rho, \mathbf{J}). \tag{5.59}$$

The continuity equation can be written as

$$\partial_\alpha J^\alpha = 0. \tag{5.60}$$

5.3.3 Hamiltonian of a charge in an electromagnetic field

From the Lagrangian (5.52) and the canonical momentum (5.53) we can obtain the energy (5.44) (and so the Hamiltonian) for a particle with mass m and charge q in an electromagnetic field. In fact, from (5.44), (5.52) and (5.53) we have

$$E = \mathbf{v} \cdot \frac{\partial L}{\partial \mathbf{v}} - L = mc^2\gamma + q\phi \tag{5.61}$$

Expressing the energy as a function of the canonical momentum

$$\mathbf{P} = \partial L/\partial \mathbf{v} = m\gamma\mathbf{v} + q\mathbf{A} \tag{5.62}$$

$$(E - q\phi)^2 - c^2(\mathbf{P} - q\mathbf{A})^2 = m^2c^4\gamma^2 - m^2\gamma^2c^2v^2 = m^2c^4\gamma^2(1-\beta^2) = m^2c^4$$

so that

$$E = \sqrt{m^2c^4 + c^2(\mathbf{P} - q\mathbf{A})^2} + q\phi = H(\mathbf{Q}, \mathbf{P}) \tag{5.63}$$

in the non relativistic limit $\beta \ll 1$

$$H(\mathbf{Q}, \mathbf{P}) = \frac{1}{2m}(\mathbf{P} - q\mathbf{A})^2 + q\phi + mc^2 \tag{5.64}$$

The Hamiltonian in an electromagnetic field can be obtained adding the energy potential $q\phi$ and substituting in the kinetic energy the kinetic momentum $\mathbf{p} = m\gamma\mathbf{v}$ by $\mathbf{P} - q\mathbf{A}$.

5.3.4 The electromagnetic field tensor

To obtain the Lorentz equation of motion in covariant form, we consider the action in four-dimensional form, Eq. (5.51). The principle of least action leads to

$$\delta S = \delta \int_a^b (-mcds - qA^\alpha dx_\alpha) = 0$$

Recall that since $ds^2 = dx^\alpha dx_\alpha$, then $\delta(ds) = dx^\alpha \delta(dx_\alpha)/ds = (u^\alpha/c)\delta(dx_\alpha)$. Therefore

$$\delta S = \int_a^b [-mu^\alpha \delta(dx_\alpha) - qA^\alpha \delta(dx_\alpha) - q\delta A^\alpha dx_\alpha] = 0$$

Integrating per parts, it becomes

$$\delta S = \int_a^b [mdu^\alpha \delta x_\alpha + qdA^\alpha \delta x_\alpha - q\delta A^\alpha dx_\alpha] = 0$$

(since $\delta x_\alpha(a) = 0$ and $\delta x_\alpha(b) = 0$). Noting that $dA^\alpha = (\partial A^\alpha/\partial x_\beta)dx_\beta$ and $\delta A^\alpha = (\partial A^\alpha/\partial x_\beta)\delta x_\beta$, then

$$\begin{aligned}
\delta S &= \int_a^b \left[mdu^\alpha \delta x_\alpha + q\left(\frac{\partial A^\alpha}{\partial x_\beta}\right) dx_\beta \delta x_\alpha - q\left(\frac{\partial A^\alpha}{\partial x_\beta}\right) dx_\alpha \delta x_\beta\right] \\
&= \int_a^b \left[mdu^\alpha \delta x_\alpha + q\left(\frac{\partial A^\alpha}{\partial x_\beta}\right) dx_\beta \delta x_\alpha - q\left(\frac{\partial A^\beta}{\partial x_\alpha}\right) dx_\beta \delta x_\alpha\right] \\
&= \int_a^b \left[m\frac{du^\alpha}{d\tau} - q\left(\frac{\partial A^\beta}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_\beta}\right) u_\beta\right] d\tau \delta x_\alpha
\end{aligned}$$

If we define electromagnetic field tensor $F^{\alpha\beta}$ by

$$F^{\alpha\beta} = \frac{\partial A^\beta}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{5.65}$$

Then the principle of least action yields

$$m\frac{du^\alpha}{d\tau} = qF^{\alpha\beta} u_\beta \tag{5.66}$$

The tensor $F^{\alpha\beta}$ is antisymmetric, $F^{\alpha\beta} = -F^{\beta\alpha}$. Since

$$\partial^\alpha = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \qquad A^\alpha = \left(\frac{\phi}{c}, \mathbf{A}\right)$$

its components are

$$\begin{aligned}
F^{0i} &= \partial^0 A^i - \partial^i A^0 = \frac{1}{c}\left(\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i}\right) = -\frac{E_i}{c} \quad (i = 1,2,3) \\
F^{ij} &= \partial^i A^j - \partial^j A^i = \left(-\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j}\right) = -B_k \quad (i,j,k = 1,2,3\,\text{cyclic})
\end{aligned}$$

As a matrix

$$(F^{\alpha\beta}) = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \tag{5.67}$$

and

$$(F_{\alpha\beta}) = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix} \tag{5.68}$$

Using $F^{\alpha\beta}$, we obtain the components of the covariant equations of motion (5.66),

$$m\frac{du^0}{d\tau} = qF^{0i}u_i \tag{5.69}$$

$$m\frac{du^i}{d\tau} = qF^{i\beta}u_\beta = qF^{i0}u_0 + qF^{ij}u_j + qF^{ik}u_k \tag{5.70}$$

Since $u^\alpha = (\gamma c, \gamma\mathbf{v})$, $u_\alpha = (\gamma c, -\gamma\mathbf{v})$, $d\tau = dt/\gamma$, then

$$mc^2\frac{d\gamma}{dt} = q\mathbf{v}\cdot\mathbf{E} \tag{5.71}$$

$$m\frac{d(\gamma\mathbf{v})}{dt} = q[\mathbf{E} + \mathbf{v}\times\mathbf{B}] \tag{5.72}$$

i.e. the equation for the energy $E = mc^2\gamma$ and the Lorentz force.

The two non-homogeneous Maxwell equations

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} &= \mu_0\mathbf{J} \end{aligned}$$

can be written in a covariant form as

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta \tag{5.73}$$

Instead, the two homogeneous Maxwell equations

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} &= 0 \end{aligned}$$

become the four equations

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{5.74}$$

where α, β and γ take the values $(1,2,3)$, $(0,1,2)$, $(0,2,3)$ and $(0,3,1)$.

5.3.5 Lorentz transformation of the EM fields

Rules of transformation

The Lorentz transform of potentials ϕ and $\mathbf{A}$ can be obtained easily, because $A^\alpha = (\phi/c, \mathbf{A})$ is a four-vector:

$$\phi' = \gamma(\phi - c\beta A_x) \tag{5.75}$$
$$A'_x = \gamma(A_x - \beta\phi/c) \tag{5.76}$$
$$A'_y = A_y \tag{5.77}$$
$$A'_z = A_z \tag{5.78}$$

The transformation of $\mathbf{E}$ and $\mathbf{B}$ can be obtained from the transformation of components of $F^{\alpha\beta}$. In general, this can be done more easily using matrices. If we take the vector

$$x = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

and the matrix for the Lorentz transformation ('boost') along x^1

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then $x'^\alpha = \Lambda^\alpha_\beta x^\beta$ can be written in matricial notation as

$$x' = \Lambda x$$

Then the transformation

$$F'^{\alpha\beta} = \Lambda^\alpha_\gamma \Lambda^\beta_\delta F^{\gamma\delta}$$

can be written in matricial notation as

$$F' = \Lambda F \tilde{\Lambda}$$

where $\tilde{\Lambda}$ is the transposed of Λ. Explicitly, we obtain

$$E'_x = E_x, \quad E'_y = \gamma(E_y - c\beta B_z), \quad E'_z = \gamma(E_z + c\beta B_y) \tag{5.79}$$
$$B'_x = B_x, \quad B'_y = \gamma(B_y + \beta E_z/c), \quad B'_z = \gamma(B_z - \beta E_y/c) \tag{5.80}$$

where $c\beta = V$ is the velocity of K' with respect to K. Notice that if $\mathbf{B}' = 0$ in K', then $\mathbf{B} = \mathbf{V} \times \mathbf{E}/c^2$ in K. If $\mathbf{E}' = 0$ in K', then $\mathbf{E} = \mathbf{B} \times \mathbf{V}$ in K. In both cases, $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular in K. This point can be obtained more clearly by considering the invariants of $\mathbf{E}$ and $\mathbf{B}$.

Lorentz invariants of electromagnetic fields

We can obtain two important and useful invariants by constructing four-scalars. First, we can easily see that $F^{\alpha\beta}F_{\beta\alpha} = -F^{\alpha\beta}F_{\alpha\beta}$ is an invariant

$$F^{\alpha\beta}F_{\alpha\beta} = -\frac{2}{c^2}(E_x^2 + E_y^2 + E_z^2) + 2(B_x^2 + B_y^2 + B_z^2)$$

i.e.

$$c^2B^2 - E^2 = \text{invariant} \tag{5.81}$$

We can construct the second invariant using the Levi-Civita pseudo-tensor $\epsilon^{\alpha\beta\mu\nu}$ [1], taking $F_{\alpha\beta}\epsilon^{\beta\alpha\mu\nu}F_{\nu\mu}$ as invariant. Since $F_{\alpha\beta}$ and $\epsilon^{\alpha\beta\mu\nu}$ are anti-symmetric by interchanging any pair of indices, then

$$\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} = \text{invariant}$$

which leads to

$$\mathbf{E} \cdot \mathbf{B} = \text{invariant} \tag{5.82}$$

Using these two invariants, we draw the following conclusions:

1. If $E > cB$ in K, then $E > cB$ in all other K', and vice versa;
2. If $\mathbf{E} \perp \mathbf{B}$ in K, then we can find a K' so that $\mathbf{E} = 0$ if $c^2B^2 > E^2$ or $\mathbf{B} = 0$ if $c^2B^2 < E^2$.
3. If $\mathbf{E} \perp \mathbf{B}$ in K, then $\mathbf{E}' \perp \mathbf{B}'$ in all other K' if $\mathbf{E}' \neq 0$ and $\mathbf{B}' \neq 0$.

5.3.6 The covariant form of the strain Maxwell 4-tensor

From the Lorentz force in covariant form (5.66) we can define the density Lorentz force f^α:

$$f^\alpha = F^{\alpha\beta}J_\beta \tag{5.83}$$

[1] The Levi-Civita pseudo-tensor $\epsilon^{\alpha\beta\mu\nu}$ is a tensor of rank four, completely anti-symmetric, equal to $+1$ for $\alpha = 0$, $\beta = 1$, $\mu = 2$, $\nu = 3$ and the even permutations, equal to -1 for the odd permutations and equal to 0 if two or more indexes are equal.

where $J_\beta = qn_0u_\beta$, where n_0 is the invariant number density (number of particles in a proper volume). Then, using the Maxwell equation (5.73),

$$\mu_0 f^\alpha = F^{\alpha\beta}\partial^\mu F_{\mu\beta} = \partial^\mu(F^{\alpha\beta}F_{\mu\beta}) - F_{\mu\beta}\partial^\mu(F^{\alpha\beta})$$

The second term can be written as

$$F_{\mu\beta}\partial^\mu(F^{\alpha\beta}) = \frac{1}{2}\left[F_{\mu\beta}\partial^\mu(F^{\alpha\beta}) + F_{\beta\mu}\partial^\beta(F^{\alpha\mu})\right] = \frac{F_{\mu\beta}}{2}\left[\partial^\mu(F^{\alpha\beta}) + \partial^\beta(F^{\mu\alpha})\right]$$

where we interchanged the indices β and μ and used the anti-symmetry property of $F^{\alpha\beta}$. Then, using (5.74),

$$F_{\mu\beta}\partial^\mu(F^{\alpha\beta}) = -\frac{F_{\mu\beta}}{2}\partial^\alpha(F^{\beta\mu}) = \frac{F_{\mu\beta}}{2}\partial^\alpha(F^{\mu\beta}) = \frac{1}{4}\partial^\alpha(F_{\mu\beta}F^{\mu\beta})$$

Using this expression in the density force equation,

$$\mu_0 f^\alpha = \partial^\mu(F_{\mu\beta}F^{\alpha\beta}) - \frac{1}{4}\partial^\alpha(F_{\mu\beta}F^{\mu\beta}) = -\mu_0\partial_\beta T^{\alpha\beta}$$

where $T^{\alpha\beta}$ is the symmetric strain tensor:

$$T^{\alpha\beta} = \frac{1}{\mu_0}\left[g^{\alpha\gamma}F_{\gamma\delta}F^{\delta\beta} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right] \tag{5.84}$$

From this expression we recover the conservation laws for the energy and the momentum of the electromagnetic field. The components are:

$$T^{00} = \frac{1}{2\mu_0}(E^2/c^2 + B^2) \tag{5.85}$$

$$T^{0i} = \frac{1}{\mu_0 c}(\mathbf{E}\times\mathbf{B})_i \tag{5.86}$$

$$T^{ij} = -\epsilon_0\left[E_iE_j + c^2B_iB_j - \frac{1}{2}\delta_{ij}(E^2 + c^2B^2)\right]. \tag{5.87}$$

i.e. T^{00} is the electromagnetic energy density, cT^{0i} are the components of the Poynting vector $\mathbf{S}$ (and so T^{0i}/c are the components of the momentum density of the electromagnetic field $\mathbf{g} = \mathbf{S}/c^2$) and $-T^{ij}$ are the components of the strain Maxwell's tensor [see Eq.(1.55)].

5.4 Fields of a uniformly moving charge*

As an example of the transformation rules, we use them to obtain the electromagnetic fields of a uniformly moving charge. Since the velocity of the charge is constant, we analyze the problem in two steps:

- Calculate the field in the reference system K' where the charge is at rest.
- Make a coordinate transformation to the observation reference system K.

In K' system, let a static point charge q be located at the origin $x' = y' = z' = 0$. The potentials of the charge q in K' is

$$\phi' = \frac{q}{4\pi\epsilon_0 r'}, \qquad \mathbf{A}' = 0$$

We know how to transformation ϕ' and $\mathbf{A}'$ to ϕ and $\mathbf{A}$ in K:

$$\phi = \gamma(\phi' + c\beta A'_x) = \frac{\phi'}{\sqrt{1-\beta^2}} \tag{5.88}$$

$$A_x = \gamma(A'_x + \beta\phi'/c) = \frac{(\beta/c)\phi'}{\sqrt{1-\beta^2}} \tag{5.89}$$

$$A_y = 0 \tag{5.90}$$

$$A_z = 0 \tag{5.91}$$

We need one more step to express ϕ and $\mathbf{A}$ in terms of $\mathbf{r}$ instead of r'. To do this, we know that $r' = \sqrt{x'^2 + y'^2 + z'^2}$ and

$$x' = \frac{x - c\beta t}{\sqrt{1-\beta^2}} \tag{5.92}$$

$$y' = y \tag{5.93}$$

$$z' = z \tag{5.94}$$

Therefore

$$r' = \sqrt{\frac{(x-c\beta t)^2}{1-\beta^2} + y^2 + z^2}$$

By substitution of r' in the equations for ϕ and $\mathbf{A}$:

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{(x-c\beta t)^2 + (y^2+z^2)(1-\beta^2)}} = \frac{q}{4\pi\epsilon_0 r^*} \tag{5.95}$$

$$A_x = (\beta/c)\phi \tag{5.96}$$

where

$$r^* = \sqrt{(x-c\beta t)^2 + (y^2+z^2)(1-\beta^2)} \tag{5.97}$$

Now we can calculate the fields $\mathbf{E}$ and $\mathbf{B}$. We have

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\phi - \frac{\beta}{c}\frac{\partial\phi}{\partial t}\hat{\mathbf{x}}$$

Explicitly, we have

$$\frac{\partial \phi}{\partial x} = -\frac{q}{4\pi\epsilon_0 r^{*3}}(x - c\beta t) \tag{5.98}$$

$$\frac{\partial \phi}{\partial y} = -\frac{q}{4\pi\epsilon_0 r^{*3}}(1-\beta^2)y \tag{5.99}$$

$$\frac{\partial \phi}{\partial z} = -\frac{q}{4\pi\epsilon_0 r^{*3}}(1-\beta^2)z \tag{5.100}$$

$$\frac{\partial \phi}{\partial t} = -\frac{q}{4\pi\epsilon_0 r^{*3}}(x - c\beta t)(-c\beta) \tag{5.101}$$

By substitution we obtain

$$E_x = \frac{q}{4\pi\epsilon_0 r^{*3}\gamma^2}(x - c\beta t) \tag{5.102}$$

$$E_y = \frac{q}{4\pi\epsilon_0 r^{*3}\gamma^2}y \tag{5.103}$$

$$E_z = \frac{q}{4\pi\epsilon_0 r^{*3}\gamma^2}z \tag{5.104}$$

To find the magnetic field $\mathbf{B}$, we note that $\mathbf{B'} = \mathbf{0}$ in K', hence in K system $\mathbf{B} = \mathbf{v} \times \mathbf{E}/c^2$. Lets now discuss the electric field $\mathbf{E}$. Defining $\mathbf{R} = (x - c\beta t)\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and the angle θ between $\mathbf{R}$ and the x-axis, then

$$r^* = R\sqrt{\cos^2\theta + (1-\beta^2)\sin^2\theta} = R\sqrt{1 - \beta^2\sin^2\theta}$$

so the electric field can be written as

$$\mathbf{E} = \frac{q\mathbf{R}}{4\pi\epsilon_0 R^3}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} \tag{5.105}$$

For a fixed distance R, the value of the electric field maximize at $\theta = \pi/2$ (perpendicular direction),

$$E_\perp = \frac{q}{4\pi\epsilon_0 R^2}\gamma$$

and minimizes at parallel directions ($\theta = 0$ or π),

$$E_\parallel = \frac{q}{4\pi\epsilon_0 R^2}\frac{1}{\gamma^2}$$

So as velocity increases, $E_\parallel$ decreases and $E_\perp$ increases, i.e. the field is contracted in the direction of motion. As $\beta \to 1$, the denominator $1-\beta^2\sin^2\theta$ tends to zero within an angle $\Delta\theta < \arcsin(1/\beta)$. So when $\gamma \geq 1$ an observer see a maximum electric field when the particle passes at the minimal distance $y = b$, polarized along the y axis and a magnetic field which, for non relativistic velocities, is $\mathbf{B} = (\mu_0/4\pi)q[\mathbf{v}\times\mathbf{R}]/R^3$, according to the Biot-Savart law. When $\gamma \gg 1$, the transverse field increases during a transit time $\Delta t \sim b/(\gamma c\beta)$.

Capitolo 6

Motion of a charged particle in assigned static and uniform fields

We describe the motion of a charge in static and uniform electric and magnetic fields.

6.1 Static electromagnetic fields

If the potentials do not have time dependence, then

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi \tag{6.1}$$

Therefore, ϕ determines $\mathbf{E}$, while $\mathbf{A}$ determines $\mathbf{B}$ for static electromagnetic fields. In case of static fields, we can add to ϕ an arbitrary constant. Usually, we choose $\phi(\infty) = 0$. Also it is obvious that $\mathbf{E}$ and $\mathbf{B}$ fields are completely decoupled in case of static fields.

For a closed system in a static EM field, the energy is conserved,

$$E = mc^2\gamma + q\phi$$

If the field is constant and uniform, then simple forms of $\mathbf{A}$ and ϕ can be obtained. From $\mathbf{E} = -\boldsymbol{\nabla}\phi$,

$$\phi = -\mathbf{E} \cdot \mathbf{r} \tag{6.2}$$

From $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, we can choose $\mathbf{A}$ to be

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{6.3}$$

In fact,

$$\begin{aligned} -\nabla\phi &= \nabla(\mathbf{E}\cdot\mathbf{r}) = \mathbf{E}\cdot\nabla(\mathbf{r}) = \mathbf{E} \\ \nabla\times\mathbf{A} &= \frac{1}{2}\nabla\times(\mathbf{B}\times\mathbf{r}) = \frac{1}{2}[\mathbf{B}(\nabla\cdot\mathbf{r}) - (\mathbf{B}\cdot\nabla)\mathbf{r})] = \mathbf{B} \end{aligned}$$

since $\nabla\cdot\mathbf{r} = 3$ and $(\mathbf{B}\cdot\nabla)\mathbf{r} = \mathbf{B}$.

6.2 Motion in uniform static electric field

First, let us consider the motion in a uniform static $\mathbf{E}$ field. The equation of motion is

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} \tag{6.4}$$

Assuming $\mathbf{E} = E\hat{\mathbf{x}}$, $p_x(0) = 0$, $p_y(0) = p_0 = mc\gamma_0\beta_0$ and $p_z(0) = 0$, then $p_x = qEt$, $p_y = p_0$ and $p_z = 0$. The kinetic energy of the particle is

$$E_k = mc^2\gamma = \sqrt{m^2c^4 + c^2p^2} = \sqrt{m^2c^4 + c^2p_0^2 + (qcEt)^2} = \sqrt{E_0^2 + (qcEt)^2} \tag{6.5}$$

where $E_0 = \sqrt{m^2c^4 + c^2p_0^2} = mc^2\gamma_0$ is the energy at $t = 0$. To find the trajectory of the particle, we need to solve

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \frac{\mathbf{p}}{m\gamma} = \frac{c^2\mathbf{p}}{E_k} \tag{6.6}$$

The components of the equation are

$$\frac{dx}{dt} = \frac{qc^2Et}{\sqrt{E_0^2 + (qcEt)^2}} \tag{6.7}$$

$$\frac{dy}{dt} = \frac{p_0c^2}{\sqrt{E_0^2 + (qcEt)^2}} \tag{6.8}$$

Note that as $t \to \infty$, $v_x \to c$ and $v_y \to 0$. By integrating dx/dt and dy/dt assuming $x(0) = y(0) = 0$,

$$x(t) = \frac{1}{qE}\left[\sqrt{E_0^2 + (qcEt)^2} - E_0\right] \tag{6.9}$$

$$y(t) = \frac{p_0c}{qE}\sinh^{-1}\left(\frac{qEct}{E_0}\right) \tag{6.10}$$

To obtain the trajectory, let invert $y = y(t)$ to obtain $t = t(y)$ and substitute in $x(t)$:

$$x = \frac{E_0}{qE}\left[\cosh\left(\frac{qEy}{cp_0}\right) - 1\right]. \tag{6.11}$$

This is an equation of the "catenary". In the non relativistic limit, $p_0 \ll mc$, the solution is $x \sim (a/2)t^2$ (where $a = qE/m$ is the acceleration) and $y \sim (p_0/m)t$. For $t \to \infty$, $x(t) \sim ct - (E_0/qE)$ and $y(t) \sim (cp_0/qE)\ln(2qEct/E_0)$. The accelerations are

$$a_x = \frac{a_0}{[1+\bar{t}^2]^{3/2}} \tag{6.12}$$

$$a_y = -\frac{a_0\beta_0\bar{t}}{[1+\bar{t}^2]^{3/2}} \tag{6.13}$$

where $a_0 = qEc^2/E_0$ and $\bar{t} = (qE/E_0)ct$. In fig.1 we plot v_x/c, v_y/v_0 (above) and a_x/a_0 and $a_y/(a_0\beta_0)$ versus the scaled time $\bar{t}$. Notice that, since $v_y \to 0$ as $t \to \infty$, it must exist a 'transverse' acceleration a_y also if in the non relativistic limit the motion is uniform along y. This acceleration along y is a relativistic effect.

6.3 Motion in uniform static magnetic field

Now let us consider the motion of a particle in a uniform static **B** field, assuming $\mathbf{B} = B\hat{\mathbf{z}}$. There is no work done by **B**, so the energy $E = mc^2\gamma$ is constant. The equations of motion are

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v}\times\mathbf{B} \tag{6.14}$$

Since $\mathbf{p} = m\gamma\mathbf{v}$ and γ is constant,

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m\gamma}\mathbf{v}\times\mathbf{B} = \mathbf{v}\times\mathbf{\Omega} \tag{6.15}$$

where $\mathbf{\Omega} = (q/m\gamma)\mathbf{B} = (qB/m\gamma)\hat{\mathbf{z}}$ is the precession frequency. The components of the acceleration are

$$\dot{v}_x = \Omega v_y \qquad \dot{v}_y = -\Omega v_x, \qquad \dot{v}_z = 0 \tag{6.16}$$

from which $v_z = v_{z0}$. To solve for **v**, let $\hat{v} = v_x + iv_y$, so that

$$\dot{\hat{v}} = \dot{v}_x + i\dot{v}_y = \Omega v_y - i\Omega v_x = -i\Omega\hat{v}$$

from which $\hat{v}(t) = \hat{v}_0 e^{-i\Omega t}$. Here $\hat{v}_0 = v_{\perp 0}e^{-i\alpha}$ is a complex constant, with α constant. From $\hat{v} = v_{\perp 0}e^{-i(\Omega t+\alpha)} = v_x + iv_y$, we obtain

$$v_x = v_{\perp 0}\cos(\Omega t+\alpha) \tag{6.17}$$

$$v_y = -v_{\perp 0}\sin(\Omega t+\alpha) \tag{6.18}$$

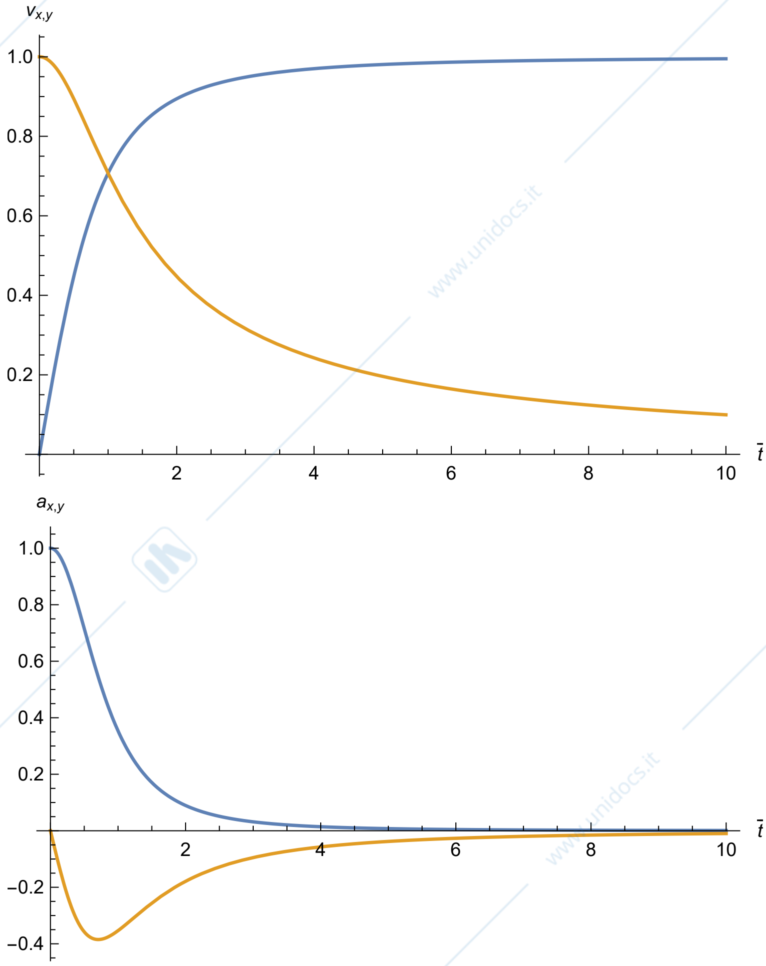

Figura 6.1:

It is clear that $v_\perp = \sqrt{v_x^2 + v_y^2} = v_{\perp 0}$ and Ω is the angular frequency. The trajectory can be obtain by integrating $d\mathbf{r}/dt = \mathbf{v}$, which gives

$$x = x_0 + \rho \sin(\Omega t + \alpha) \tag{6.19}$$

$$y = y_0 + \rho \cos(\Omega t + \alpha) \tag{6.20}$$

$$z = z_0 + v_{z0} t \tag{6.21}$$

where $\rho = v_{\perp 0}/\Omega$ is the gyro-radius. In case of $v \ll c$, $\gamma \sim 1$ and $\Omega \sim qB/m$ (Larmor frequency). The motion is circular in the plane perpendicular to $\mathbf{B}$, with a radius ρ and versus which depends on the sign of the charge q. If we add the uniform motion along the field, we have an helix with pitch $(2\pi/\Omega)v_{z0} = (2\pi m\gamma/qB)v_{z0}$.

6.4 Motion in uniform static parallel electric and magnetic field

Let's now consider the motion of a charge in a static uniform and parallel $\mathbf{E} = E\hat{\mathbf{z}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$ field. The kinetic energy varies in time due to the presence of the electric field:

$$E_k = \sqrt{E_0^2 + (qcEt)^2} = E_0\sqrt{1 + (qcEt/E_0)^2} \tag{6.22}$$

where E_0 is the initial energy. Since

$$\frac{dE_k}{dt} = q\mathbf{v} \cdot \mathbf{E} = qEv_z \rightarrow z(t) = \frac{E_k - E_0}{qE}$$

with $z(0) = 0$. The equations for the transverse momentum are

$$\frac{dp_x}{dt} = qBv_y = \frac{qc^2B}{E_k}p_y \tag{6.23}$$

$$\frac{dp_y}{dt} = -qBv_x = -\frac{qc^2B}{E_k}p_x \tag{6.24}$$

We still have $p_\perp = \sqrt{p_x^2 + p_y^2}$ constant. We let $\hat{p} = p_x + ip_y = p_\perp e^{-i\phi}$, with equation

$$\frac{d\hat{p}}{dt} = -i\frac{qc^2B}{E_k}\hat{p} = -i\dot{\phi}\hat{p}$$

so that

$$\dot{\phi} = \frac{qc^2B}{E_k} = \frac{cB}{E}\frac{qcE/E_0}{\sqrt{1 + (qcEt/E_0)^2}}$$

By integrating

$$\phi(t) = \frac{cB}{E} \sinh^{-1}\left(\frac{qcEt}{E_0}\right) \tag{6.25}$$

with $\phi(0) = 0$. The inverse yields

$$t = \frac{E_0}{qcE} \sinh\left(\frac{E\phi}{cB}\right). \tag{6.26}$$

The trajectory is parametrized by ϕ, with $dt = (E_k/qc^2B)d\phi$:

$$p_x + ip_y = \frac{E_k}{c^2}\frac{d}{dt}(x+iy) = qB\frac{d}{d\phi}(x+iy)$$

so that

$$x(\phi) = \frac{p_\perp}{qB}\sin(\phi), \quad y(\phi) = \frac{p_\perp}{qB}(\cos(\phi) - 1)$$

with $x(0) = 0$ and $y(0) = 0$. We can express also $z(t)$ as a function of ϕ:

$$z(\phi) = \frac{E_0}{qE}\left[\cosh\left(\frac{E\phi}{cB}\right) - 1\right] \tag{6.27}$$

the trajectory is an helix, with $\rho = p_\perp/qB$: the motion is circular in the plane perpendicular to $\mathbf{E}$ and $\mathbf{B}$ and is accelerated in the parallel direction. The pitch of the helix increases at each period,

$$z(t) = \frac{E_0}{qE}\left[\sqrt{1 + (qcEt/E_0)^2} - 1\right]$$

and

$$\dot{z}(t) = \frac{qcE}{E_0}\frac{ct}{\sqrt{1 + (qcEt/E_0)^2}} \to c$$

for $t \to \infty$, whereas $\dot{\phi} \to 0$: the helix stretches an the rotation slow down when the time increases. The transverse velocity must decrease because the longitudinal velocity is increasing. In the non relativistic limit $\phi = \Omega t$ and $\dot{z} = (qE/m)t$, the motion is an helix with radius $\rho = v_\perp/\Omega$ and pitch increasing with time.

6.5 Motion in uniform static perpendicular electric and magnetic field

Let's now consider the motion of a charge in a static uniform $\mathbf{E}$ and $\mathbf{B}$ with $\mathbf{E} \perp \mathbf{B}$.

We consider first the non relativistic limit $\beta \ll 1$, and take $\mathbf{E} = E\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$. For a non relativistic particle $\mathbf{p} = m\mathbf{v}$, the component of the equations are:

$$m\dot{v}_x = qv_y B \tag{6.28}$$
$$m\dot{v}_y = qE - qv_x B \tag{6.29}$$
$$m\dot{v}_z = 0 \tag{6.30}$$

To solve for $\mathbf{v}$, let again $\hat{v} = v_x + iv_y$, so that

$$\frac{d\hat{v}}{dt} + i\Omega\hat{v} = i\frac{qE}{m}$$

where $\Omega = qB/m$. The general solution of the equation is

$$\hat{v} = \hat{v}_0 e^{-i\Omega t} + \frac{qE}{m\Omega}$$

or letting $\hat{v}_0 = v_\perp e^{-i\alpha}$,

$$v_x = v_\perp \cos(\Omega t + \alpha) + \frac{E}{B} \tag{6.31}$$
$$v_y = -v_\perp \sin(\Omega t + \alpha) \tag{6.32}$$

The x component of the velocity is the sum of an oscillating term and a drift term $\bar{v}_x = E/B$. The motion in the (x, y) plane is the well-known $\mathbf{E} \times \mathbf{B}$ drift. In vector form, the drift velocity is $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$. To find the trajectory, we integrate v_x, v_y and v_z:

$$x = x_0 + \rho \sin(\Omega t + \alpha) + v_E t \tag{6.33}$$
$$y = y_0 + \rho[\cos(\Omega t + \alpha) - \cos\alpha] \tag{6.34}$$
$$z = z_0 + v_{z0} t \tag{6.35}$$

where $\rho = v_\perp/\Omega$ and $v_E = E/B$ is the $\mathbf{E} \times \mathbf{B}$ drift velocity.

Let's now consider the general relativistic case. The motion equations are

$$\frac{d\mathcal{E}}{dt} = q\mathbf{v} \cdot \mathbf{E} \tag{6.36}$$
$$\frac{d\mathbf{p}}{dt} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \tag{6.37}$$

These show that the particle energy $\mathcal{E}$ is not constant. As a consequence, it is not possible to obtain a simple relation for the velocity, as it has been done in the presence of the static magnetic field only. But with a suitable Lorentz transformation, it is possible to modify the motion equations. Let consider a transformation to an inertial frame K' moving with respect to K with the drift velocity $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$. The Lorentz force acting on the particle in K' is

$$\frac{d\mathbf{p}'}{dt} = q[\mathbf{E}' + \mathbf{v}' \times \mathbf{B}'] \tag{6.38}$$

Choosing $\mathbf{E} = E\hat{\mathbf{y}}$ and $\mathbf{B} = B\hat{\mathbf{z}}$, then $\mathbf{v}_E = (E/B)\hat{\mathbf{x}}$, with $E < cB$ (since it must be $v_E < c$) The transformed fields are

$$E'_x = 0, \quad E'_y = \gamma(E - v_E B) = 0, \quad E'_z = 0 \tag{6.39}$$

$$B'_x = 0, \quad B'_y = 0, \quad B'_z = \gamma(B - v_E E/c^2) = \frac{B}{\gamma} \tag{6.40}$$

where

$$\gamma = (1 - v_E^2/c^2)^{-1/2} = \sqrt{\frac{B^2}{B^2 - E^2/c^2}}$$

In K' the only field acting on the particle is a magnetic field $\mathbf{B}'$ in the same direction of $\mathbf{B}$ but with a module less by a factor γ^{-1}. Thus the motion in K' is the same as that considered in sec.6.3, namely a spiraling around the lines of force. As viewed from the original coordinate system, this gyration is accompanied by a uniform drift $\mathbf{v}_E$ perpendicular to $\mathbf{E}$ and $\mathbf{B}$. The drift can be understood qualitatively by noting that a particle which starts gyrating around $\mathbf{B}$ is accelerated by the electric field, gains energy, and so moves in a path with larger radius for roughly half of its cycle. On the other half, the electric field decelerates it, causing it to lose energy and so move in a tighter arc. The combination of arcs produces a translation perpendicular to $\mathbf{E}$ and $\mathbf{B}$. The direction of the drift is independent of the sign of the charge of the particle. The drift velocity $\mathbf{v}_E$ has physical meaning only if it is less that the velocity of light c, i.e. if $E < cB$. If $E > cB$, the electric field is so strong that the particle is continuously accelerated in the direction of $\mathbf{E}$ and its average energy continue to increase with time. To see this we consider a Lorentz transformation from K to K'' moving with a velocity $\mathbf{u} = c^2\mathbf{E} \times \mathbf{B}/E^2$ relative to K. In this frame the electric and magnetic field are

$$E''_x = 0, \quad E''_y = \gamma(E - uB) = \frac{E}{\gamma}, \quad E''_z = 0 \tag{6.41}$$

$$B''_x = 0, \quad B''_y = 0, \quad B''_z = \gamma(B - u_E E/c^2) = 0 \tag{6.42}$$

with

$$\gamma = (1 - u^2/c^2)^{-1/2} = \sqrt{\frac{E^2}{E^2 - c^2E^2}}$$

Thus in the system K'' the particle is acted on by a purely electrostatic field which causes hyperbolic motion with ever increasing velocity.

The fact that a particle can move through crossed **E** and **B** field with the uniform velocity $v_E = E/B$ provides the possibility of selecting charged particles according to velocity. If a beam of particles having a spread in velocities is normally incident on a region containing uniform crossed electric and magnetic fields, only those particles with velocities equal to E/B will travel without deflection. Suitable entrance and exit slits will then allow only a very narrow band of velocities around E/B to be transmitted, the resolution depending on the geometry, the velocities desired and the field strengths.

Capitolo 7

Radiation by a moving charge

We discuss how a moving charge generates electromagnetic fields.

7.1 Liénard-Wiechert potentials for a point charge

Let's consider the radiation emitted by a moving charge. We start from the retarded potential [see Eqs.(1.90) and (1.91)]

$$\phi(\mathbf{x},t) = \frac{1}{4\pi\epsilon_0}\int d\mathbf{x}'\left[\frac{\rho(\mathbf{x}',\tau)}{|\mathbf{x}-\mathbf{x}'|}\right]_{\rm ret} \tag{7.1}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d\mathbf{x}'\left[\frac{\mathbf{J}(\mathbf{x}',\tau)}{|\mathbf{x}-\mathbf{x}'|}\right]_{\rm ret} \tag{7.2}$$

where the squared parenthesis means that the time τ must be evaluated at the retarded time $\tau = t - |\mathbf{x}-\mathbf{x}'|/c$. We consider a charge q with trajectory $\mathbf{r}(t)$ and velocity $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$,

$$\rho(\mathbf{x},t) = q\delta[\mathbf{x}-\mathbf{r}(t)], \qquad \mathbf{J}(\mathbf{x},t) = q\mathbf{v}(t)\delta[\mathbf{x}-\mathbf{r}(t)]. \tag{7.3}$$

Since we can write

$$\rho(\mathbf{x}',\tau) = \int d\tau'\delta(\tau'-\tau)\rho(\mathbf{x}',\tau')$$

then

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\int d\mathbf{x}'\int d\tau'\left[\frac{\delta[\mathbf{x}'-\mathbf{r}(\tau')]\delta(\tau'-\tau)}{|\mathbf{x}-\mathbf{x}'|}\right]_{\rm ret} \tag{7.4}$$

$$= \frac{q}{4\pi\epsilon_0}\int d\tau'\frac{\delta[\tau'-t+R(\tau')/c]}{R(\tau')} \tag{7.5}$$

where $R(\tau') = |\mathbf{x} - \mathbf{r}(\tau')|$. Changing the integration variable from τ' to $\tau'' = \tau' - t + R(\tau')/c$,

$$d\tau'' = d\tau' + \frac{1}{c}\frac{\partial R(\tau')}{\partial \tau'}d\tau'. \tag{7.6}$$

Since

$$\frac{\partial R(\tau')}{\partial \tau'} = \frac{\partial}{\partial \tau'}|\mathbf{x} - \mathbf{r}(\tau')| = -\mathbf{n}\cdot\mathbf{v}(\tau') \tag{7.7}$$

where $\mathbf{n} = \mathbf{R}/R$, then

$$d\tau'' = [1 - \mathbf{n}\cdot\boldsymbol{\beta}(\tau')]d\tau'. \tag{7.8}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ and

$$\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(1 - \mathbf{n}\cdot\boldsymbol{\beta})R}\right]_{\tau = t - R(\tau)/c} \tag{7.9}$$

In a similar way we can show that

$$\mathbf{A}(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\boldsymbol{\beta}}{(1 - \mathbf{n}\cdot\boldsymbol{\beta})R}\right]_{\tau = t - R(\tau)/c} \tag{7.10}$$

These are the *Liénard-Wiechert potentials* for a point charge. In these expressions, the quantities $\boldsymbol{\beta}$, $\mathbf{n}$ and R are evaluated at the retarded time $\tau = t - R(\tau)/c$. From the expressions of ϕ and $\mathbf{A}$ we can obtain the electric and magnetic field of a moving charge by straightforward derivation with respect to t and $\mathbf{x}$. Before proceeding in the calculations of the fields, we obtain first the covariant form of the Liènard-Wiechert potentials.

7.2 Covariant expression of the Liénard-Wiechert potentials

7.2.1 Covariant form of the retarded potentials.

First, we obtain the solution of the wave equation in terms of the covariant Green function. The wave equation in the covariant form is

$$\Box A^\alpha = \mu_0 J^\alpha \tag{7.11}$$

where $\Box = \partial_\alpha\partial^\alpha$ and

$$A^\alpha = \left(\frac{\phi}{c}, \mathbf{A}\right), \qquad J^\alpha = (c\rho, \mathbf{J}). \tag{7.12}$$

We solve Eq.(7.11) in terms of the Green function $D(x - x')$,

$$A^\alpha(x) \;=\; \mu_0 \int d^4x' D(x-x')J^\alpha(x') \tag{7.13}$$

where

$$\Box_x D(z) = \delta^{(4)}(z), \tag{7.14}$$

where $z = x - x' = (x_0 - x_0', \mathbf{x} - \mathbf{x}')$ and $\delta^{(4)}(z) = \delta(z_0)\delta(\mathbf{z})$ is the four-dimensional delta function. The Green function can be written using the Fourier transform as

$$D(z) = \frac{1}{(2\pi)^4}\int d^4k \tilde{D}(k) e^{-ik\cdot z}$$

where $k \cdot z = k_0 z_0 - \mathbf{k}\cdot\mathbf{z}$. From (7.14) we obtain $\tilde{D}(k) = -1/(k\cdot k)$ and

$$D(z) = -\frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot z}}{k\cdot k} = -\frac{1}{(2\pi)^4}\int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2}$$

where $\kappa = |\mathbf{k}|$. The integrand function has two poles at $k_0 = \pm\kappa$. We can deform the integration axis considering k_0 as a complex variable and choosing two different paths parallel to the $\mathrm{Re}\, k_0$ axis, passing above (path r) or below (path a) the two poles. Then, depending if z_0 is positive or negative, we close the path by a big half-circle in the lower or upper half-plane (as the exponential $e^{(\mathrm{Im} k_0) z_0}$ must tend to zero when the radius of the half-circle tends to infinity). For the path r, the integral in k_0 is zero for $z_0 < 0$, since there are no singularities inside the closed path, and different from zero for $z_0 > 0$, which can be calculated by the residue theorem:

$$\oint_r dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} = H(z_0)(-2\pi i)\mathrm{Res}\left[\frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2}\right] = -H(z_0)\frac{2\pi}{\kappa}\sin(\kappa z_0)$$

where $H(x)$ is the Heavyside (or step) function, $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. Instead, for the path a the integral in k_0 is zero for $z_0 > 0$ and different from zero for $z_0 < 0$;

$$\oint_a dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} = H(-z_0)(2\pi i)\mathrm{Res}\left[\frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2}\right] = H(-z_0)\frac{2\pi}{\kappa}\sin(\kappa z_0)$$

Hence, the Green functions with the two different paths r or a are

$$D_{r,a}(z) = \pm\frac{H(\pm z_0)}{2\pi^2 R}\int_0^\infty d\kappa\, \sin(\kappa R)\sin(\kappa z_0)$$

where $R = |\mathbf{z}| = |\mathbf{x} - \mathbf{x}'|$. The integral in κ can be expressed in terms of delta functions:

$$\int_0^\infty d\kappa\, \sin(\kappa R)\sin(\kappa z_0) = \frac{\pi}{2}[\delta(z_0 - R) - \delta(z_0 + R)]$$

Since $R > 0$, we obtain

$$D_r(z) = \frac{H(z_0)}{4\pi R}\delta(z_0 - R) \quad (7.15)$$
$$D_a(z) = \frac{H(-z_0)}{4\pi R}\delta(z_0 + R). \quad (7.16)$$

They can be put in covariant form using the identity[1]

$$\delta(z^2) = \delta(z_0^2 - R^2) = \delta[(z_0 - R)(z_0 + R)] = \frac{1}{2R}[\delta(z_0 - R) + \delta(z_0 + R)].$$

So, calling D_r and D_a the *retarded* and *advanced* invariant Green function, we write:

$$D_r(x - x') = \frac{1}{2\pi}H(x_0 - x'_0)\delta[(x - x')^2], \quad (7.17)$$
$$D_a(x - x') = \frac{1}{2\pi}H(x'_0 - x_0)\delta[(x - x')^2], \quad (7.18)$$

The step function shows that the retarded (advanced) Green function is different from zero only on the forward (backward) light cone of the source point. Notice that the invariance of the step function is guaranteed by the invariance of the delta function $\delta[(x - x')^2]$. Hence, the covariant form of the retarded potential is

$$A^\alpha(x) = \mu_0 \int d^4x' D_r(x - x')J^\alpha(x') \quad (7.19)$$

7.2.2 Four-current for a point charge

Assuming a point charge with (7.3), we can write

$$\rho(\mathbf{x}, t) = q \int dt' \delta(t' - t)\delta[\mathbf{x} - \mathbf{r}(t')] \quad (7.20)$$
$$\mathbf{J}(\mathbf{x}, t) = q \int dt' \delta(t' - t)\mathbf{v}(t')\delta[\mathbf{x} - \mathbf{r}(t')]. \quad (7.21)$$

In covariant form, introducing the proper time $d\tau = dt'/\gamma$, the four-current is

$$J^\alpha = cq \int d\tau u^\alpha(\tau)\delta^{(4)}[x - r(\tau)] \quad (7.22)$$

where $r^\alpha(\tau) = [c\tau, \mathbf{r}(\tau)]$ is the charge's four-vector coordinate as a function of the charge's proper time τ and $u^\alpha = (\gamma c, \gamma \mathbf{v})$ is the charge's four-velocity.

[1] From the rule $\delta[f(x)] = \sum_i \delta(x - x_i)/|f'(x_i)|$, where x_i are the zeros of the function $f(x)$.

7.2.3 Covariant form of the Liénard-Wiechert potentials

We are now ready to find the covariant form of the Lìenard-Wiechert potentials: using A^α from (7.19) with the retarded Green function (7.17) and J^α from (7.22):

$$\begin{aligned} A^\alpha(x) &= \frac{cq\mu_0}{2\pi}\int d^4x' H(x_0-x'_0)\delta[(x-x')^2]\int d\tau u^\alpha(\tau)\delta^{(4)}[x'-r(\tau)] \\ &= \frac{cq\mu_0}{2\pi}\int d\tau H[x_0-r_0(\tau)]u^\alpha(\tau)\delta\{[x-r(\tau)]^2\} \end{aligned} \tag{7.23}$$

where in the second step we integrated over d^4x'. The remaining integral over the charge's proper time $d\tau$ gives a contribution only at $\tau=\tau_0$, where τ_0 is determined by the light-cone condition $[x-r(\tau_0)]^2=0$ (i.e. $\tau_0=t-R(\tau_0)/c$), and the retardation requirement, $x_0>r_0(\tau_0)$. Since

$$\frac{d}{d\tau}[x-r(\tau)]^2 = -2[x-r(\tau)]_\beta u^\beta(\tau) = -2(x-r)\cdot u$$

then, integrating over $d\tau$ we obtain

$$A^\alpha(x) = \frac{1}{4\pi\epsilon_0 c}\left[\frac{qu^\alpha(\tau)}{u(\tau)\cdot[x-r(\tau)]}\right]_{\tau=\tau_0}. \tag{7.24}$$

The significance of the conditions imposed by the step and delta functions in the integral (7.23) is shown in fig.7.2.3: the Green function is different from zero only on the backward light cone of the observation point at $x=0$. The world line of the particle $r(\tau)$ intersects the light cone at only two points, one in the future and one in the past, later than τ_0. The particle's event in $r(\tau_0)$ in the past is the only part of the word line that contributes to the field at $x=0$. It expresses the causal connection between the particle and the observer.

In noncovariant form the light-cone constraint implies $x_0-r_0(\tau_0)=R(\tau_0)$ (where $R(\tau)=|\mathbf{x}-\mathbf{r}(\tau)|$) and $u\cdot(x-r)=u_0(x_0-r_0)-\mathbf{u}\cdot(\mathbf{x}-\mathbf{r})=c\gamma R(1-\mathbf{n}\cdot\boldsymbol{\beta})$, so that

$$A^\alpha = (\phi/c,\mathbf{A}) = \frac{1}{4\pi\epsilon_0 c}\left[q\frac{(1,\boldsymbol{\beta})}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})}\right]_{\text{ret}}. \tag{7.25}$$

which coincides with (7.9) and (7.10).

7.3 Electric and magnetic field from a moving charge

We have obtained the scalar and vectorial potentials (and the covariant form of A^α) of a moving charge. Of course, the relativistic effect relies in the factor $(1-\mathbf{n}\cdot\boldsymbol{\beta})$ in the denominators.

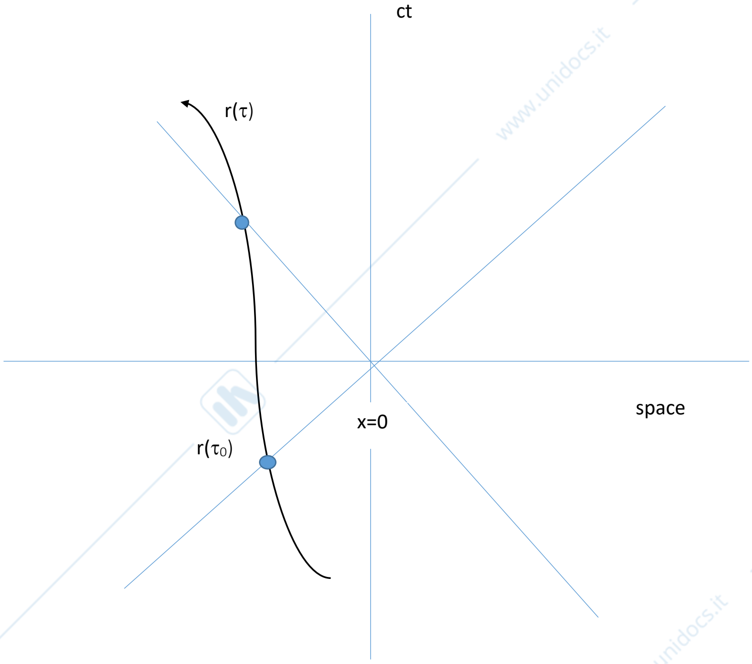

Figura 7.1:

It is important to obtain now the electric and magnetic fields **E** and **B**, which are the experimentally measurable quantities. There are two alternative ways to obtain them:

1. Calculating the fields from the potential (7.9) and (7.10) using the relations

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A},$$

   taking into account that the quantities R, $\mathbf{n}$ and $\boldsymbol{\beta}$ depend on time and position also via the retarded time $\tau = t - R(\tau)/c$. This makes the calculation rather long and cumbersome, and it is reported in sec.7.3.2.

2. More elegantly, using the covariant form of A^α to obtain the electromagnetic tensor $F^{\alpha\beta}$, whose components are related to **E** and **B**. This method is a bit shorter, and it will be presented in the following.

7.3.1 Covariant derivation of the fields from the Liénard-Wiechert potentials

We return to the expression of A^α as an integral in $d\tau$:

$$A^\alpha = \frac{qc\mu_0}{2\pi} \int d\tau H[x_0 - r_0(\tau)] u^\alpha(\tau) \delta\{[x - r(\tau)]^2\} \tag{7.26}$$

to obtain

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

The derivative ∂^α acts on H and δ. The derivative of H gives a delta function $\delta(x_0 - r_0(\tau))$ which makes the delta function $\delta\{[x - r(\tau)]^2\} \to \delta(-R^2)$ which does not give contribution except when $R \neq 0$. Hence, excluding $R = 0$, the derivative of H does not give any contribution and

$$\partial^\alpha A^\beta = \frac{qc\mu_0}{2\pi} \int d\tau H[x_0 - r_0(\tau)] u^\beta(\tau) \partial^\alpha \delta\{[x - r(\tau)]^2\}$$

If $f = [x - r(\tau)]^2$, then

$$\partial^\alpha \delta[f] = \frac{d\delta[f]}{df} \partial^\alpha f = \partial^\alpha f \frac{d\tau}{df} \frac{d\delta[f]}{d\tau}$$

but $df/d\tau = -2[x - r(\tau)] \cdot u$ and $\partial^\alpha f = 2[x - r(\tau)]^\alpha$, so that

$$\partial^\alpha \delta[f] = -\frac{[x - r(\tau)]^\alpha}{u \cdot [x - r(\tau)]} \frac{d\delta[f]}{d\tau}$$

which when inserted in the integral and by an integration by parts, yields

$$\partial^\alpha A^\beta = \frac{qc\mu_0}{2\pi} \int d\tau H[x_0 - r_0(\tau)] \frac{d}{d\tau} \left\{ \frac{[x - r(\tau)]^\alpha u^\beta(\tau)}{u(\tau) \cdot [x - r(\tau)]} \right\} \delta\{[x - r(\tau)]^2\}$$

This equation has the same form of the equation for A^α in (7.26) with the substitution

$$u^\alpha(\tau) \to \frac{d}{d\tau} \left\{ \frac{[x - r(\tau)]^\alpha u(\tau)^\beta}{u(\tau) \cdot [x - r(\tau)]} \right\}$$

From the corresponding retarded solution (7.24) we obtain

$$\partial^\alpha A^\beta = \frac{\mu_0 c}{4\pi} \left\{ \frac{q}{u \cdot (x - r)} \frac{d}{d\tau} \left[\frac{(x - r)^\alpha u^\beta}{u \cdot (x - r)} \right] \right\}_{\tau = \tau_0} . \tag{7.27}$$

and

$$F^{\alpha\beta} = \frac{\mu_0 cq}{4\pi} \left\{ \frac{1}{u \cdot (x - r)} \frac{d}{d\tau} \left[\frac{(x - r)^\alpha u^\beta - (x - r)^\beta u^\alpha}{u \cdot (x - r)} \right] \right\}_{\tau = \tau_0} . \tag{7.28}$$

This is the solution in covariant form. To obtain **E** and **B** we must calculate the derivative with respect to τ. We have the following relations:

$$\frac{d}{d\tau}[u \cdot (x - r)] = -u_\alpha u^\alpha + (x - r)_\alpha w^\alpha = -c^2 + (x - r) \cdot w \tag{7.29}$$

and

$$\frac{d}{d\tau} \left[\frac{(x - r)^\alpha u^\beta - (x - r)^\beta u^\alpha}{u \cdot (x - r)} \right] = -\frac{-c^2 + (x - r) \cdot w}{[u \cdot (x - r)]^2} [(x - r)^\alpha u^\beta - (x - r)^\beta u^\alpha]$$
$$+ \frac{1}{u \cdot (x - r)} \left[-u^\alpha u^\beta + u^\beta u^\alpha + (x - r)^\alpha w^\beta - (x - r)^\beta w^\alpha \right]$$

where $w^\alpha = du^\alpha / d\tau$ is the charge's four-acceleration. With these relations,

$$\begin{aligned} F^{\alpha\beta} &= \frac{\mu_0 cq}{4\pi} \left\{ \frac{c^2 - (x - r) \cdot w}{[u \cdot (x - r)]^3} [(x - r)^\alpha u^\beta - (x - r)^\beta u^\alpha] \right. \\ &+ \left. \frac{1}{[u \cdot (x - r)]^2} \left[(x - r)^\alpha w^\beta - (x - r)^\beta w^\alpha \right] \right\}_{\tau = \tau_0} \end{aligned}$$

We can separate the terms which depend on the velocity u only from those which depend on the acceleration w:

$$\begin{aligned} F^{\alpha\beta}(u) &= \frac{\mu_0 cq}{4\pi} \left\{ \frac{c^2}{[u \cdot (x - r)]^3} [(x - r)^\alpha u^\beta - (x - r)^\beta u^\alpha] \right\}_{\tau = \tau_0} \\ F^{\alpha\beta}(u, w) &= \frac{\mu_0 cq}{4\pi} \\ &\times \left\{ -\frac{(x - r) \cdot w}{[u \cdot (x - r)]^3} [(x - r)^\alpha u^\beta - (x - r)^\beta u^\alpha] + \frac{(x - r)^\alpha w^\beta - (x - r)^\beta w^\alpha}{[u \cdot (x - r)]^2} \right\}_{\tau = \tau_0} . \end{aligned}$$

We remember that

$$u \cdot (x-r) = u_\alpha (x-r)^\alpha = \gamma c R(1 - \mathbf{n} \cdot \boldsymbol{\beta})$$

Let's consider first the 'velocity' field: the components of the electric field are

$$\begin{aligned} E_i &= cF^{i0}(u) = \frac{q}{4\pi\epsilon_0} \left\{ \frac{c^2}{[u \cdot (x-r)]^3} [(x^i - r^i)u^0 - (x^0 - r^0)u^i] \right\}_{\tau=\tau_0} \\ &= \frac{q}{4\pi\epsilon_0} \left\{ \frac{n_i - \beta_i}{\gamma^2 R^2 (1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right\}_{\text{ret}} \end{aligned}$$

whereas these of the magnetic fields are

$$\begin{aligned} B_k &= -F^{ij}(u) = \frac{\mu_0 c q}{4\pi} \left\{ \frac{c^2}{[u \cdot (x-r)]^3} [(x^i - r^i)u^j - (x^j - r^j)u^i] \right\}_{\tau=\tau_0} \\ &= -\frac{\mu_0 c q}{4\pi} \left\{ \frac{n_i \beta_j - n_j \beta_i}{\gamma^2 R^2 (1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right\}_{\text{ret}} = -\frac{\mu_0 c q}{4\pi} \left\{ \frac{(\mathbf{n} \times \boldsymbol{\beta})_k}{\gamma^2 R^2 (1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right\}_{\text{ret}} \end{aligned}$$

Therefore:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \left[\frac{\mathbf{n} - \boldsymbol{\beta}}{\gamma^2 R^2 (1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right]_{\text{ret}}, \qquad \mathbf{B} = \frac{1}{c} [\mathbf{n} \times \mathbf{E}]_{\text{ret}}. \tag{7.30}$$

Let's now obtain the electric and magnetic fields from the part depending on the acceleration w^α. The four acceleration is

$$w^\alpha = \left(c\frac{d\gamma}{d\tau}, c\frac{d(\gamma\boldsymbol{\beta})}{d\tau} \right) = c\gamma(\dot{\gamma}, \gamma\dot{\boldsymbol{\beta}} + \dot{\gamma}\boldsymbol{\beta})$$

where the dot stays for a derivative with respect to t. Since $\dot{\gamma} = \gamma^3(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})$, then

$$w^\alpha = c\gamma^2(\gamma^2(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}), \dot{\boldsymbol{\beta}} + \gamma^2(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})\boldsymbol{\beta})$$

and

$$(x-r) \cdot w = cR\gamma^4(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})[1 - \mathbf{n} \cdot \boldsymbol{\beta}] - cR\gamma^2(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})$$

Using these relations, the electric field components are

$$\begin{aligned} E_i &= cF^{i0}(u,w) = \frac{q}{4\pi\epsilon_0} \\ &\times \left\{ -\frac{(x-r) \cdot w}{[u \cdot (x-r)]^3} [(x-r)^i u^0 - (x-r)^0 u^i] + \frac{(x-r)^i w^0 - (x-r)^0 w^i}{[u \cdot (x-r)]^2} \right\}_{\tau=\tau_0} \\ &= \frac{\mu_0 c q}{4\pi} \left\{ \frac{(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})(n_i - \beta_i) + (1 - \mathbf{n} \cdot \boldsymbol{\beta})\dot{\beta}_i}{R(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right\}_{\text{ret}} \end{aligned}$$

a similar calculation gives the components of the magnetic field. In summary, we obtain collecting both the 'velocity' and 'acceleration' fields:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2 R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret} + \frac{q}{4\pi\epsilon_0 c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret} \quad (7.31)$$

$$\mathbf{B} = \frac{1}{c}\left[\mathbf{n}\times\mathbf{E}\right]_{\rm ret}. \quad (7.32)$$

7.3.2 Straightforward derivation of the fields from the Liénard-Wiechert potentials *

We calculate now the fields directly from the potential (7.9) and (7.10) using the relations

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi, \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \quad (7.33)$$

taking into account that the quantities $R = |\mathbf{x}-\mathbf{r}|$, $\hat{\mathbf{n}} = \mathbf{R}/R$ and $\boldsymbol{\beta}$ depend on time and position also via the retarded time $\tau = t - R(\tau)/c$. We need the following relations:

a)

$$\frac{\partial\tau}{\partial t} = 1 - \frac{1}{c}\frac{\partial R}{\partial\tau}\frac{\partial\tau}{\partial t} = 1 + \hat{\mathbf{n}}\cdot\boldsymbol{\beta}\frac{\partial\tau}{\partial t}$$

so that

$$\frac{\partial\tau}{\partial t} = \frac{1}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} \quad (7.34)$$

b)

$$\boldsymbol{\nabla}\tau = -\frac{1}{c}\left(\boldsymbol{\nabla}R + \frac{\partial R}{\partial\tau}\boldsymbol{\nabla}\tau\right) = -\frac{1}{c}\boldsymbol{\nabla}R + \hat{\mathbf{n}}\cdot\boldsymbol{\beta}\boldsymbol{\nabla}\tau = -\frac{\hat{\mathbf{n}}}{c} + \hat{\mathbf{n}}\cdot\boldsymbol{\beta}\boldsymbol{\nabla}\tau$$

so that

$$\boldsymbol{\nabla}\tau = -\frac{\hat{\mathbf{n}}/c}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} \quad (7.35)$$

Using (7.9) and (7.10) the electric field is

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}\boldsymbol{\nabla}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})] - \frac{q}{4\pi\epsilon_0 c}\frac{\partial}{\partial t}\frac{\boldsymbol{\beta}}{R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}\boldsymbol{\nabla}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})] \\
&- \frac{q}{4\pi\epsilon_0 c}\left\{\frac{1}{R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})}\frac{\partial\boldsymbol{\beta}}{\partial t} + \boldsymbol{\beta}\frac{\partial}{\partial t}\frac{1}{R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})}\right\} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}\boldsymbol{\nabla}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})] \\
&- \frac{q}{4\pi\epsilon_0 c}\left\{\frac{\dot{\boldsymbol{\beta}}}{R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})}\frac{\partial\tau}{\partial t} - \frac{\boldsymbol{\beta}}{R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}\frac{\partial}{\partial t}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})]\right\}
\end{aligned}$$

Now we need to calculate:

$$a) \qquad \boldsymbol{\nabla}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})] \tag{7.36}$$

$$b) \qquad \frac{\partial}{\partial t}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})] \tag{7.37}$$

a)

$$\begin{aligned}
\boldsymbol{\nabla}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})] &= \boldsymbol{\nabla}R - \boldsymbol{\nabla}(R\hat{\mathbf{n}}\cdot\boldsymbol{\beta}) = -c\boldsymbol{\nabla}\tau - \boldsymbol{\nabla}(\mathbf{R}\cdot\boldsymbol{\beta}) \\
&= \frac{\hat{\mathbf{n}}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} - \boldsymbol{\nabla}\boldsymbol{\beta}\cdot\mathbf{R} - \boldsymbol{\nabla}\mathbf{R}\cdot\boldsymbol{\beta} \\
&= \frac{\hat{\mathbf{n}}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} - \dot{\boldsymbol{\beta}}\cdot\mathbf{R}\boldsymbol{\nabla}\tau - \boldsymbol{\beta} + c\beta^2\boldsymbol{\nabla}\tau \\
&= \frac{\hat{\mathbf{n}}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} + \dot{\boldsymbol{\beta}}\cdot\mathbf{R}\frac{\hat{\mathbf{n}}/c}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} - \boldsymbol{\beta} - \beta^2\frac{\hat{\mathbf{n}}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} \\
&= \frac{\hat{\mathbf{n}}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}\left[1-\beta^2+\frac{\dot{\boldsymbol{\beta}}\cdot\mathbf{R}}{c}\right] - \boldsymbol{\beta}
\end{aligned}$$

where $\mathbf{R} = R\hat{\mathbf{n}}$ and we used $R = c(t-\tau)$.

b)

$$\begin{aligned}
\frac{\partial}{\partial t}[R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})] &= \frac{\partial R}{\partial t} - \frac{\partial}{\partial t}(\mathbf{R}\cdot\boldsymbol{\beta}) = \frac{\partial R}{\partial \tau}\frac{\partial \tau}{\partial t} - \left(\frac{\partial\boldsymbol{\beta}}{\partial t}\cdot\mathbf{R} + \frac{\partial\mathbf{R}}{\partial t}\cdot\boldsymbol{\beta}\right) \\
&= -\frac{c(\hat{\mathbf{n}}\cdot\boldsymbol{\beta})}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} - \left(\frac{\dot{\boldsymbol{\beta}}\cdot\mathbf{R}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} - \frac{c\beta^2}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}\right)
\end{aligned}$$

with these expressions,

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{1}{R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}\left\{\frac{\hat{\mathbf{n}}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}\left[1-\beta^2+\frac{\dot{\boldsymbol{\beta}}\cdot\mathbf{R}}{c}\right]-\boldsymbol{\beta}\right\} \\
&- \frac{q}{4\pi\epsilon_0 c}\left\{\frac{\dot{\boldsymbol{\beta}}}{R(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2}\right\} \\
&+ \frac{q}{4\pi\epsilon_0 c}\frac{\boldsymbol{\beta}}{R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}\left\{-c(\hat{\mathbf{n}}\cdot\boldsymbol{\beta})-\dot{\boldsymbol{\beta}}\cdot\mathbf{R}+c\beta^2\right\} \\
&= \frac{q}{4\pi\epsilon_0 R^2}\frac{1}{\gamma^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}\left\{\hat{\mathbf{n}}-\boldsymbol{\beta}\right\} \\
&+ \frac{q}{4\pi\epsilon_0 c R}\frac{1}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}\left\{\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]\right\}
\end{aligned}$$

which coincides with Eq.(7.31). About the magnetic field,

$$\begin{aligned}
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = \frac{q}{4\pi\epsilon_0 c}\boldsymbol{\nabla}\times\left[\frac{\boldsymbol{\beta}}{(1-\mathbf{n}\cdot\boldsymbol{\beta})R}\right] \\
&= \frac{q}{4\pi\epsilon_0 c}\left\{\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})R}\boldsymbol{\nabla}\times\boldsymbol{\beta}-\boldsymbol{\beta}\times\boldsymbol{\nabla}\left[\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})R}\right]\right\} \\
&= \frac{q}{4\pi\epsilon_0 c}\left\{\frac{1}{(1-\mathbf{n}\cdot\boldsymbol{\beta})R}\boldsymbol{\nabla}\tau\times\dot{\boldsymbol{\beta}}+\frac{\boldsymbol{\beta}}{R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}\times\left[\hat{\mathbf{n}}\left(1-\beta^2+\frac{\dot{\boldsymbol{\beta}}\cdot\mathbf{R}}{c}\right)\right]\right\} \\
&= \frac{q}{4\pi\epsilon_0 c}\hat{\mathbf{n}}\times\left\{-\frac{\boldsymbol{\beta}}{\gamma^2R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}-\frac{\dot{\boldsymbol{\beta}}}{cR(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}-\frac{\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\hat{\mathbf{n}})}{cR(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}\right\} \\
&= \frac{q}{4\pi\epsilon_0 c}\hat{\mathbf{n}}\times\left\{\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2R^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3}-\frac{\dot{\boldsymbol{\beta}}(1-\mathbf{n}\cdot\boldsymbol{\beta})+\boldsymbol{\beta}(\dot{\boldsymbol{\beta}}\cdot\hat{\mathbf{n}})}{cR(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right\} \\
&= \frac{1}{c}(\mathbf{n}\times\mathbf{E}).
\end{aligned}$$

7.4 Velocity and radiation fields

The important result (7.31) and (7.32) deserves a detailed discussion.

7.4.1 Velocity field

The velocity field (7.30) is essential a static field decreasing as $1/R^2$ and it does not radiate. This because, if we consider the flux through a spherical surface with radius R around the source, the Poynting vector goes as $1/R^4$ and the flux decreases as $1/R^2$, so it becomes negligible far from the source. This is in contrast with the second term of (7.31), where the electric field decreases as $1/R$ and the flux through the sphere of radius R does not decrease far from the source and transport energy to infinity. This term is the *radiation field*, which propagates.

The velocity field must be the same as that obtained in sec. 5.4 as the field from a uniformly moving charge:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{n}}{\gamma^2 R^2 (1-\beta^2 \sin^2\theta)^{3/2}}$$

where θ is the angle between $\mathbf{n}$ and $\boldsymbol{\beta}$. It was obtained by a Lorentz transformation of the static Coulomb field. The apparently different form of the two expressions is due to the fact that (7.30) is evaluated at the retarded time. The proof of their equivalence is reported in the next section.

Equivalence of the velocity field (7.30) and the field from an uniformly moving charge*

We demonstrate that the field (7.30)

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \left[\frac{\mathbf{n}-\boldsymbol{\beta}}{\gamma^2 R^2 (1-\mathbf{n}\cdot\boldsymbol{\beta})^3} \right]_{\text{ret}} \tag{7.38}$$

for an uniformly moving charge is the same as that obtained by Lorentz transformation of the static Coulomb field, as calculated in sec. 5.4:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{n}}{\gamma^2 R^2 (1-\beta^2 \sin^2\theta)^{3/2}} \tag{7.39}$$

where θ is the angle between $\mathbf{n}$ and $\boldsymbol{\beta}$, which is supposed uniform and directed along the x-axis. In (7.38) the quantities n and R are evaluated at the retarded time $t - R/c$, whereas in (7.39) they are evaluated at the time t. The different positions of the charge for the two fields are shown in fig.7.4.1 as the points P for (7.39) and P' for (7.38). Their distances from the observation point O are R and R' and the angle between R and R' and the velocity directed along the x-axis are θ and θ'. Considering the parallel and transverse components x and y, the field (7.38) is

$$E'_x = \frac{q}{4\pi\epsilon_0\gamma^2} \frac{R'(\cos\theta' - \beta)}{R'^3(1-\beta\cos\theta')^3} \tag{7.40}$$

$$E'_y = \frac{q}{4\pi\epsilon_0\gamma^2} \frac{R'\sin\theta'}{R'^3(1-\beta\cos\theta')^3} \tag{7.41}$$

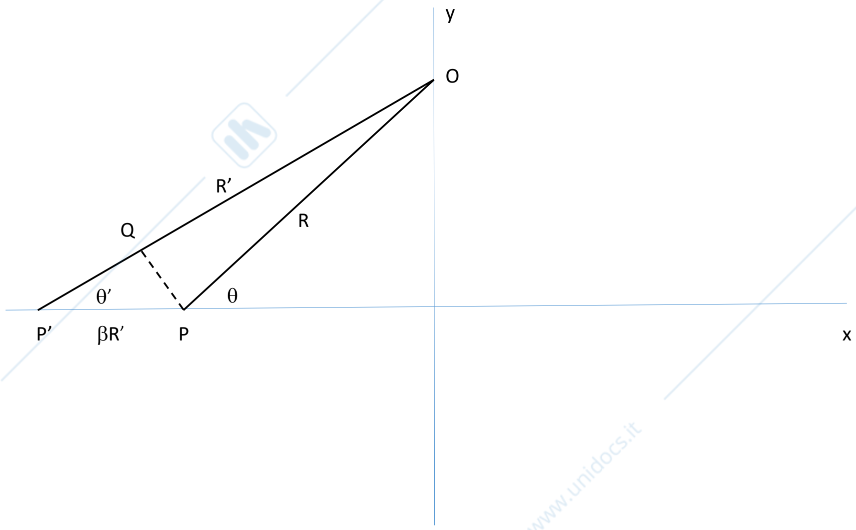

Figura 7.2: Different positions P for (7.39) and P' for (7.38).

whereas the field (7.39) is

$$E_x = \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\cos\theta}{R^2(1-\beta^2\sin^2\theta)^{3/2}} \tag{7.42}$$

$$E_y = \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\sin\theta}{R^2(1-\beta^2\sin^2\theta)^{3/2}} \tag{7.43}$$

From the fig.7.4.1 $R\sin\theta = R'\sin\theta'$. The distance between P' and P is $\beta R'$ (equal to the distance traveled by the charge in the retarded time R'/c). The distance $P'Q$ is $\beta R'\cos\theta'$. Therefore the distance QO is $R'(1-\beta\cos\theta')$. Since $PQ = \beta R'\sin\theta' = \beta R\sin\theta$ and $(QO)^2 + (PQ)^2 = R^2$, we have $R'(1-\beta\cos\theta') = R(1-\beta^2\sin^2\theta)^{1/2}$. Furthermore, we have $R'\cos\theta' = \beta R' + R\cos\theta$. By substituting them in E'_x and E'_y,

$$E'_x = \frac{q}{4\pi\epsilon_0\gamma^2}\frac{R\cos\theta}{R^3(1-\beta^2\sin^2\theta)^{3/2}} = E_x \tag{7.44}$$

$$E'_y = \frac{q}{4\pi\epsilon_0\gamma^2}\frac{R\sin\theta}{R^3(1-\beta^2\sin^2\theta)^{3/2}} = E_y \tag{7.45}$$

7.4.2 Radiation field

The second term of (7.31) is the radiation field

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]}{R(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret} \tag{7.46}$$

It is proportional to the charge acceleration and it vanishes for an uniform motion. Since it decreases as $1/R$, it transports radiated energy to infinite. In general, we can say that radiation is generated by accelerated charges only.

For a non relativistic motion, the radiation field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})}{R}\right]_{\rm ret} \tag{7.47}$$

If we remember the field emitted by an electric dipole oscillating at the frequency $\omega = ck$ at long wavelength [see Chap. 4, Eq.(4.39)], we see the relationship between the two expressions in a case of an oscillating charge with position $\mathbf{r}(t) = \mathbf{r}_0\cos(\omega t)$. Its acceleration is $\mathbf{a} = -\omega^2\mathbf{r}$ so that, taking into account the retarded time $\tau = t - R/c$ defining the electric dipole as $\mathbf{p} = q\mathbf{r}$, gives

$$\mathbf{E} = \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikR}}{R}[(\mathbf{n}\times\mathbf{p})\times\mathbf{n}]e^{-i\omega t} \tag{7.48}$$

in agreement with the Eq.(4.39).

7.4.3 Radiated energy.

The Poynting vector for the fields (7.46) and (7.32) is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = c\epsilon_0|\mathbf{E}|^2\mathbf{n} = \frac{q^2}{16\pi^2 c\epsilon_0}\left[\frac{|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]|^2}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^6}\right]_{\rm ret}\mathbf{n} \tag{7.49}$$

where we used $c^2 = (\epsilon_0\mu_0)^{-1}$. The Poynting vector is the amount of energy passing through the unit area in the direction $\mathbf{n}$ in the unit time. Therefore, the energy radiated from t_1 to t_2 is

$$\Delta\mathcal{E} = \int_{t_1}^{t_2}\mathbf{S}\cdot\mathbf{n}\,dt$$

Since the fields must be evaluated at the retarded time $\tau = t - R(\tau)/c$, changing the integration time from t to τ with $dt = (1-\mathbf{n}\cdot\boldsymbol{\beta})d\tau$,

$$\Delta\mathcal{E} = \int_{\tau_1}^{\tau_2}\mathbf{S}\cdot\mathbf{n}\,(1-\mathbf{n}\cdot\boldsymbol{\beta})\,d\tau.$$

The energy radiated in $d\tau$ through a surface $d\Sigma = R^2 d\Omega$, where $d\Omega = \sin\theta d\theta d\phi$ is the solid angle, is

$$d\mathcal{E} = \mathbf{S}\cdot\mathbf{n}\,(1-\mathbf{n}\cdot\boldsymbol{\beta})R^2 d\Omega d\tau.$$

The radiated power per solid angle is

$$\frac{dP}{d\Omega} = \mathbf{S}\cdot\mathbf{n}(1-\mathbf{n}\cdot\boldsymbol{\beta})R^2 \tag{7.50}$$

With (7.49), the radiated power per solid angle is

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 c\epsilon_0}\frac{|\mathbf{n}\times[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} \tag{7.51}$$

We can see that there are two types of relativistic effects present. One is the specific spatial relationship between the charge's velocity and acceleration vectors, which will determine the detailed angular distribution. The other is a general, relativistic effect arising from the transformation from the rest frame of the particle to the observer's frame and manifesting itself by the presence of the factor $(1-\mathbf{n}\cdot\boldsymbol{\beta})$ in the denominator of (7.51). For ultra-relativistic particles the latter dominates the whole angular distribution.

7.4.4 The radiation of a slow charge

For a non relativistic charge, $\beta \ll 1$ and

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 c\epsilon_0}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2 = \frac{q^2}{16\pi^2 c\epsilon_0}\dot{\beta}^2\sin^2\Theta \tag{7.52}$$

where Θ is the angle between $\mathbf{n}$ and $\dot{\boldsymbol{\beta}}$. This is the well-know result that the power is maximally emitted in the direction perpendicular to the acceleration, with the typical $\sin^2\Theta$ distribution. The radiation is polarized in the plane containing $\mathbf{n}$ and $\dot{\boldsymbol{\beta}}$. The total power is obtained integrating over all solid angle[2],

$$P = \frac{q^2}{6\pi\epsilon_0 c^3}a^2 \tag{7.53}$$

where $a = c\dot{\beta}$ is the charge's acceleration. This is the *Larmor*'s formula, which states that the power radiated by a non relativistic point charge is proportional to the square of its acceleration.

[2] The integration over the solid angle gives $\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta = 8\pi/3$.

7.5 The radiation of a relativistic charge.

For a fast particle with $\beta \sim 1$, we cannot ignore $\boldsymbol{\beta}$ or $\dot{\boldsymbol{\beta}}$ in Equation (7.51) and things in general are complicated. Here we first discuss two special cases where $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are parallel or perpendicular.

7.5.1 Parallel velocity and acceleration.

If $\boldsymbol{\beta}$ is parallel to $\dot{\boldsymbol{\beta}}$, then Equation (7.51) becomes

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c}\frac{|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^5} \tag{7.54}$$

If θ is the angle between $\mathbf{n}$ and $\boldsymbol{\beta}$,

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c}\frac{\dot{\beta}^2\sin^2\theta}{(1-\beta\cos\theta)^5} \tag{7.55}$$

Note that the angular distribution of the radiated power is independent of the sign of the acceleration (i.e. whether the particle is accelerating or de-accelerating). For instance, when the particle hits a target, it radiates ('*bremsstrahlung*' or '*braking*' radiation).

For $\beta \ll 1$ this is the Larmor's result. But as $\beta \to 1$, the angular distribution is tipped forward more and more and increases in magnitude. Since $1-\beta^2 = 1/\gamma^2$, for $\gamma \gg 1$

$$\beta \sim 1 - \frac{1}{2\gamma^2}.$$

Furthermore, if $\theta \ll 1$ then $\cos\theta \sim 1-\theta^2/2$ and

$$1-\beta\cos\theta \sim \frac{1}{2\gamma^2}(1+\gamma^2\theta^2) \tag{7.56}$$

so that

$$\frac{dP}{d\Omega} \approx \frac{2q^2}{\pi^2\epsilon_0 c}\dot{\beta}^2\gamma^8\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^5} \tag{7.57}$$

The natural angular width is evidently $\theta \sim 1/\gamma$, which is typical of the relativistic radiation patterns, regardless of the vectorial relation between $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$. The angular distribution is shown in Fig.7.3 for $\beta = 0$ (non relativistic case) and $\beta = 0.8$. The acceleration and the velocity are directed along the horizontal axis.

The total power for the case of parallel velocity and acceleration is

$$P_{\parallel} = \frac{q^2\dot{\beta}^2}{16\pi^2\epsilon_0 c}\int_0^{2\pi}d\phi\int_0^{\pi}\frac{\sin^3\theta d\theta}{(1-\beta\cos\theta)^5} = \frac{q^2}{6\pi\epsilon_0 c}\dot{\beta}^2\gamma^6 \tag{7.58}$$

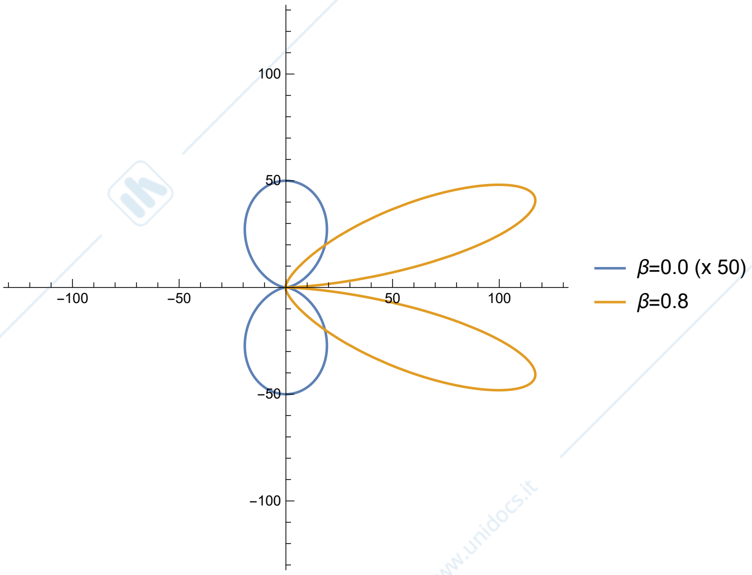

Figura 7.3: Angular dependence of Eq.(7.55) for $\beta \ll 1$ (non relativistic case) and $\beta = 0.8$ ($\gamma = 1.66$), the non relativistic one having been multiplied by a factor 50.

Sometimes it is convenient to express the radiation power in terms of the force felt by the particle. For parallel velocity and acceleration, the force is

$$\mathbf{F} = \dot{\mathbf{p}} = mc\gamma^3\dot{\boldsymbol{\beta}}$$

where we used the relation $\dot{\gamma} = \gamma^3\beta\dot{\beta}$. Therefore in terms of force $\mathbf{F}$, the total radiation power for parallel velocity and acceleration is

$$P_{\parallel} = \frac{q^2}{6\pi\epsilon_0 c}\left(\frac{F}{mc}\right)^2. \tag{7.59}$$

For a given applied force the radiated power is independent of γ for a given force. Furthermore, it depends inversely on the square of the mass of the particle involved. Consequently these radiative effects are largest for electrons. In a linear accelerator the radiation loss will be unimportant for typical energy gain $d\mathcal{E}/dx = F$ less than 10 MeV/m. In fact the ratio of power radiated to power supplied by the external source is:

$$\frac{P_{\parallel}}{(d\mathcal{E}/dt)} = \frac{2}{3}\left[\frac{r_e}{mc^2}\right](d\mathcal{E}/dx). \tag{7.60}$$

where we approximated $d\mathcal{E}/dx \approx (1/c)d\mathcal{E}/dt$, $r_e = e^2/(4\pi\epsilon_0 mc^2) = 2.8 \times 10^{-15}$m is the *classical electron radius*[3] and $mc^2 \sim 0.511$ MeV for electrons. For $d\mathcal{E}/dx \leq 10$ MeV/m this ratio is less than 10^{-14} !

7.5.2 Perpendicular velocity and acceleration.

We now consider the case where $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are perpendicular. Taking a coordinate system where $\boldsymbol{\beta} = \beta\hat{\mathbf{z}}$ and $\dot{\boldsymbol{\beta}} = \dot{\beta}\hat{\mathbf{x}}$, then in polar coordinates (R, θ, ϕ) we have $\mathbf{n}\cdot\boldsymbol{\beta} = \beta\cos\theta$, $\mathbf{n}\cdot\dot{\boldsymbol{\beta}} = \dot{\beta}\sin\theta\cos\phi$ and

$$\begin{aligned}\mathbf{n} \times [(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}] &= (\mathbf{n}\cdot\dot{\boldsymbol{\beta}})(\mathbf{n}-\boldsymbol{\beta}) - [\mathbf{n}\cdot(\mathbf{n}-\boldsymbol{\beta})]\dot{\boldsymbol{\beta}} \\ &= \dot{\beta}\sin\theta\cos\phi\,(\mathbf{n}-\boldsymbol{\beta}) - (1-\beta\cos\theta)\,\dot{\boldsymbol{\beta}}.\end{aligned}$$

From that it is straightforward to obtain:

$$|\mathbf{n} \times [(\mathbf{n}-\boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]|^2 = \dot{\beta}^2[(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi]$$

and

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 c\epsilon_0}\frac{\dot{\beta}^2[(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi]}{(1-\beta\cos\theta)^5} \tag{7.61}$$

The angular distribution is shown in Fig.7.4 for $\beta = 0$ (non relativistic case) and $\beta = 0.8$. Integrating (7.61) over the all solid angle we obtain the total power

$$P_\perp = \frac{q^2}{6\pi c\epsilon_0}\dot{\beta}^2\gamma^4 \tag{7.62}$$

This results was first obtained by *Liénard* in 1898. Since the applied force $\mathbf{F} = \dot{\mathbf{p}}$ in this case is perpendicular to the velocity, then the energy $mc^2\gamma$ is constant, and $\dot{\mathbf{p}} = mc\gamma\dot{\boldsymbol{\beta}}$. The radiation power in terms of the force felt by the particle is

$$P_\perp = \frac{q^2\gamma^2}{6\pi\epsilon_0 c}\left(\frac{F}{mc}\right)^2. \tag{7.63}$$

Therefore for a given force, the radiation increases as the energy squared.

The above results for $P_\parallel$ and $P_\perp$ are important for the design of accelerators. There are at least two types of accelerators: linear accelerators ($\mathbf{v} \parallel \dot{\mathbf{v}}$) and circular accelerators ($\mathbf{v} \perp \dot{\mathbf{v}}$). For a given force, $P_\perp = \gamma^2 P_\parallel$. Therefore the radiation power loss in circular accelerators increases much more when the particle energy increases.

[3]Remember that $r_e = \alpha(\lambda_c/2\pi) = \alpha^2 a_0$, where $\lambda_c = h/mc$ is the Compton wavelength, $\alpha = e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$ is the fine-structure constant and $a_0 = 4\pi\epsilon_0\hbar^2/me^2$ is the Bohr radius.

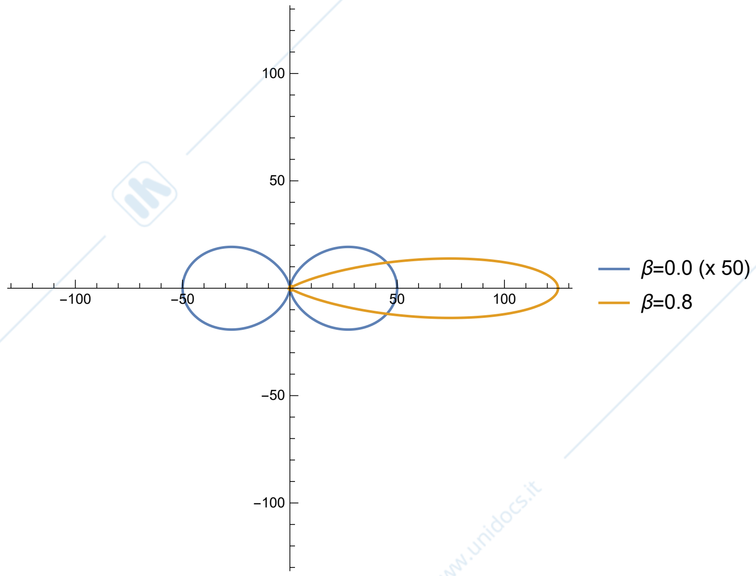

Figura 7.4: Angular dependence of Eq.(7.61) for $\beta \ll 1$ (non relativistic case) and $\beta = 0.8$ ($\gamma = 1.66$), the non relativistic one having been multiplied by a factor 50.

So, in principle it is more efficient using linear accelerators to accelerate particles to very high energy. On the other hand, circular accelerators have a very important application as source of the high-frequency radiation which is called *synchrotron radiation*. We will study the spectrum of this kind of radiation in the next section. Before that, we calculate first the power radiated by a charge in arbitrary motion.

7.5.3 Radiation from a charge in arbitrary motion

The electric and magnetic field satisfy the superposition principle. Since

$$\mathbf{E} \propto \mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}],$$

if we write $\dot{\boldsymbol{\beta}} = \dot{\boldsymbol{\beta}}_{\parallel} + \dot{\boldsymbol{\beta}}_{\perp}$, where $\parallel$ and $\perp$ are with respect to $\boldsymbol{\beta}$, then we can write

$$\mathbf{E}(\dot{\boldsymbol{\beta}}) = \mathbf{E}(\dot{\boldsymbol{\beta}}_{\parallel}) + \mathbf{E}(\dot{\boldsymbol{\beta}}_{\perp}) = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}$$

and

$$E^2 = E_{\parallel}^2 + E_{\perp}^2 + 2\mathbf{E}_{\parallel} \cdot \mathbf{E}_{\perp}$$

and for (7.51)

$$\frac{dP}{d\Omega} = \frac{dP_{\parallel}}{d\Omega} + \frac{dP_{\perp}}{d\Omega} + \left(\frac{q^2}{4\pi^2 c\epsilon_0}\right)^2 \frac{\{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}_{\parallel}]\} \cdot \{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}_{\perp}]\}}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^{10}}$$

It is possible to demonstrate that when integrated over all solid angle the last term vanishes, so that

$$P = P_{\parallel} + P_{\perp} = \frac{q^2}{6\pi c\epsilon_0}\gamma^4\left[\gamma^2\dot{\beta}_{\parallel}^2 + \dot{\beta}_{\perp}^2\right] \tag{7.64}$$

More generally, it is possible to show that the total radiated power can be written in a Lorentz invariant form:

$$P = -\frac{q^2}{6\pi c\epsilon_0}\frac{1}{m^2}\left(\frac{dp_\alpha}{d\tau}\frac{dp^\alpha}{d\tau}\right) \tag{7.65}$$

which, when expressed in terms of the velocity and the acceleration $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$, takes the form:

$$P = \frac{q^2}{6\pi c\epsilon_0}\gamma^6[\dot{\beta}^2 - |\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}|^2] \tag{7.66}$$

Since $|\boldsymbol{\beta}\times(\dot{\boldsymbol{\beta}}_{\parallel}+\dot{\boldsymbol{\beta}}_{\perp})|^2 = \beta^2\dot{\beta}_{\perp}^2$ and $\dot{\beta}^2 = \dot{\beta}_{\parallel}^2 + \dot{\beta}_{\perp}^2$, (7.66) is equal to (7.64).

7.6 Frequency distribution of the radiation by a charge in a ultra relativistic motion

Since the radiation power emitted by an accelerated charge in a extremely relativistic motion, $\gamma \gg 1$, is much larger for transverse acceleration rather than for longitudinal acceleration, we can neglect the power due to the longitudinal component of the acceleration and assume that the power depends on the instantaneous curvature radius ρ of the charge trajectory, given by

$$\rho = \frac{v^2}{\dot{v}_\perp} \approx \frac{c^2}{\dot{v}_\perp}$$

The radiation is emitted in a narrow cone along the instantaneous velocity vector of the charge. For instance, a charge in a circular motion emits radiation along the tangent to the conference in all the directions. An observer along the tangent direction will detect a radiation pulse of very short duration in the short time the electron travels along the circumference arc subtended by the radiation, with an angle $\theta \sim 1/\gamma$. The time duration of the pulse depends on the charge 's energy. In fact, radiation is continuously emitted in an interval time Δt during which the charge travels on the circumference arc $d = \rho\theta \sim \rho/\gamma$. Supposing the charge emits a rectangular wave of length L, its time duration will be $L = D - d$, where

$$D = c\Delta t = \frac{cd}{v} = \frac{\rho}{\gamma\beta}$$

is the space traveled by the light's front in the time Δt during which the charge travels through the arc $d = \rho/\gamma$. The light pulse ends after the distance d. Therefore, the pulse length is

$$L = D - d = \left(\frac{c}{v} - 1\right) d = \left(\frac{1}{\beta} - 1\right) \frac{\rho}{\gamma} \approx \frac{\rho}{2\gamma^3}$$

where the last expression holds for $\gamma \gg 1$. Therefore the duration of the light pulse is

$$\delta T \approx \frac{\rho}{2c\gamma^3} \tag{7.67}$$

and, for the Fourier theorem, the maximum frequency of the radiation spectrum is

$$\omega_c \sim \frac{2\pi}{\delta T} \sim \frac{2\pi c\gamma^3}{\rho} \sim \omega_0\gamma^3 \tag{7.68}$$

where $\omega_0 \sim 2\pi c/\rho$ is the rotation frequency for a circular motion with radius ρ. The frequency $\omega \sim \omega_0\gamma^3$ is called *critical frequency of the synchrotron light*. It means that a relativistic charged particle (for instance an electron) emits a broad spectrum of frequencies, up to γ^3 times the fundamental rotation frequency ω_0. Therefore, the synchrotron radiation spectrum is very wide, covering many frequency decades, dropping exponentially to zero for $\omega > \omega_c$. For instance, a synchrotron with $\rho \sim 30$m has $\omega_0 \sim (2\pi)10$MHz. For an electron energy $E = 5$GeV, then $\gamma \sim 10^4$ and $\omega_c = (2\pi) \times 10^{19}$ Hz, i.e. X-rays with energy up to of about 50 keV.

7.7 Radiation from an undulator

It exists a method to avoid the broad spectrum of the radiation emitted by an electron in a circular motion, as in a synchrotron light source. It consists of forcing the electron to move along an oscillating trajectory, making an angle with respect to the axis smaller than the angle $1/\gamma$ of the emitted radiation cone. In this way, other than collimated in the forward direction, the radiation is also quasi-monochromatic. In order to determine qualitatively the spectral features of the *undulator radiation*, let consider the oscillating motion due to a transverse magnetic field, varying sinusoidally along the propagation axis z:

$$\mathbf{B}(z) = B_w \cos(k_w z)\hat{\mathbf{y}} \tag{7.69}$$

where $\lambda_w = 2\pi/k_w$ is the undulator period. It is easy to see that the electron transverse velocity is

$$\mathbf{v}_\perp = c\beta_x\hat{\mathbf{x}} = -\frac{cK}{\gamma}\sin(k_w z)\hat{\mathbf{x}} \tag{7.70}$$

where K is the undulator parameter:

$$K = \frac{eB_w}{mc^2 k_w} \sim 0.93\, B_0[\mathrm{T}]\, \lambda_w[\mathrm{cm}]. \tag{7.71}$$

If $K < 1$, the angle the trajectory makes on the z-axis, $v_x/v_z \sim \beta_x \sim K/\gamma$, is smaller than the light cone angle $1/\gamma$, and the light radiated by the electron while it pass through the undulator remains in phase over different periods λ_w. On the contrary, if $K \gg 1$ the radiation bursts emitted during each undulator period are phase-uncorrelated, and each burst of light emitted during each undulator period has a duration equivalent to that by a single bending magnet. Hence for $K \gg 1$ the spectrum of the emitted radiation is very broad and similar to that of a synchrotron source. In the following we focus on the first regime only, i.e. on the *undulator radiation* obtained for $K < 1$.

Let now evaluate the time delay of the light accumulated during the passage of the electron through an undulator period. Since the electron travels at velocity v_z and the light at the velocity c, the time delay between the electron and the light is

$$\Delta T = \frac{\lambda_w}{v_z} - \frac{\lambda_w \cos\theta}{c}$$

where θ is the angle of observation with respect to the z axis. The radiation remains in phase with that one emitted in the previous undulator periods only if $\Delta T = \lambda/c$. This yields a formula for the resonant wavelength:

$$\lambda_r = \lambda_w \left(\frac{1}{\beta_z} - \cos\theta \right) \approx \frac{\lambda_w}{2\gamma^2} \left(1 + \frac{K^2}{2} + \gamma^2\theta^2 \right) \tag{7.72}$$

where we assumed $\beta_z \sim 1 - 1/2\gamma_z^2$, $\gamma = \gamma_z^2(1 + K^2/2)$ and $\cos\theta \sim 1 - \theta^2/2$. After $L_w = N_w \lambda_w$, the observer will receive a train of pulses with a duration $N_w \lambda_w / c$, so that from the Fourier's theorem the relative width of the spectrum is

$$\frac{\Delta\omega}{\omega} \sim \frac{1}{2N_w} \tag{7.73}$$

The exact expression on-axis is

$$\frac{dI}{d\Omega d\omega} = 2N_w^2 \gamma^2 K^2 \mathrm{sinc}^2 \left[\pi N_w \left(\frac{\omega - \omega_r}{\omega_r} \right) \right] \tag{7.74}$$

where $\omega_r = 2\pi/\lambda_r$ is the resonant frequency and $\mathrm{sinc}(x) = \sin(x)/x$.

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