Name: ____________________. Student number (matricola): ____________________.

Inferential Statistics: exam - 23/01/2020

- The exam will last two hours.
- Books and notes cannot be used during the exam.
- Non-programmable calculators are permitted.
- Any use of mobile phones is strictly prohibited during the exam.
- The back of each sheet can be used as rough paper, which will not be graded.

Part to be filled by students who got an overall grade of at least 20/30 in the three assignments

Choose at most one option:

○ I want to skip exercise 1 (which will not be graded)
○ I want to skip exercise 2 (which will not be graded)
○ I want to skip exercise 3 (which will not be graded)

If no option is chosen, all the exercises will be graded and the mark of the assignments not used.

1. (10 points)
   (a) Give the definition of convergence in probability of a sequence of random variables.
   (b) Let X be an exponential random variable with parameter $\lambda > 0$, that is with probability density function

$$f(x) = \begin{cases} \lambda \exp\{-\lambda x\} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   For every $n = 1, 2, \ldots$, define the random variable $X_n = (1 + 1/n)\, X$. Show that the sequence X_n converges in probability to X.
   (c) State the weak law of large numbers.
   (d) Does the sequence X_n defined in point b satisfy the assumptions of the weak law of large numbers? Motivate your answer.

Solution:

a) THEORY

b) WE WANT TO SHOW THAT $\forall \varepsilon > 0$

$$\lim_{m \to +\infty} P(|X_m - X| \geq \varepsilon) = 0$$

OBSERVE THAT $P(|X_m - X| \geq \varepsilon) = P(|(1+\frac{1}{m})X - X| \geq \varepsilon) =$

$= P(|\cancel{X} + \frac{1}{m}X - \cancel{X}| \geq \varepsilon) = P(|\frac{1}{m}X| \geq \varepsilon) \underset{1}{=} P(\frac{1}{m}X \geq \varepsilon) = P(X \geq m \cdot \varepsilon)$

$= 1 - P(X \leq m \cdot \varepsilon) = 1 - F(m \cdot \varepsilon)$ $\quad X \sim \mathcal{E}xp(\lambda) \Rightarrow X > 0$

WHERE F IS THE C.D.F. OF $X \sim \mathcal{E}xp(\lambda)$ I.E. $F(x) = 1 - e^{-\lambda x}$

$\Rightarrow \lim_{m \to \infty} P(|X_m - X| \geq \varepsilon) = \lim_{m \to +\infty} 1 - (1 - e^{-\lambda m \cdot \varepsilon}) = \lim_{m \to +\infty} e^{-\lambda m \varepsilon} = 0$ ☑

c) THEORY

D) THE ASSUMPTIONS OF THE WLLN ARE THAT $\{X_m\}$ MUST BE S.T.

- X_m ARE INDEPENDENT
- COMMON MEAN $\mathbb{E}[X_m]$
- FINITE VARIANCE.

IT IS ENOUGH TO SHOW THAT ONE OF THESE IS NOT SATISFIED TO CONCLUDE THAT THE ASSUMPTIONS OF THE WLLN ARE NOT SATISFIED.

COMMON MEAN? $\mathbb{E}[X_n] = \mathbb{E}\left[\left(1+\frac{1}{n}\right)X\right] = \left(1+\frac{1}{n}\right)\mathbb{E}[X] = \left(1+\frac{1}{n}\right)\frac{1}{\lambda}$

WHICH IS DIFFERENT FOR EVERY n. => ASSUMPTIONS ARE NOT SATISFIED.

2. (10 points)

Consider the statistical model $M = \{f_\theta : \theta \in \{1,2\}\}$, where

$$f_\theta(x) = \begin{cases} 2\theta x(1-x^2)^{\theta-1} & \text{if } x \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write the likelihood function of θ given one observation x_1.

(b) For which values of x_1 the inequality $L(1 \mid x_1)/L(2 \mid x_1) \geq 1$ is true? Explain how this defines the MLE $\hat{\theta}(x_1)$ for θ.

Consider the statistical model $M_1 = \{f_\theta : \theta \in (0,\infty)\}$, where f_θ is defined as before.

(c) Find the MLE $\hat{\theta}_1(s)$, given a sample $s = (x_1,\dots,x_n)$.

(d) Assume $n = 2$ and $s = (0.3, 0.4)$, what is the maximum likelihood estimate of θ?

Solution:

a) $L(\vartheta \mid x_1) = 2\vartheta x_1 (1-x_1^2)^{\vartheta-1}$ OR EQUIVALENTLY

$$\boxed{L(\vartheta \mid x_1) = \vartheta (1-x_1^2)^{\vartheta-1}}$$

b) $L(1 \mid x_1) = 1\cdot(1-x_1^2)^{1-1} = 1$

$L(2 \mid x_1) = 2(1-x_1^2)^{2-1} = 2(1-x_1^2)$

$$\Rightarrow \frac{L(1 \mid x_1)}{L(2 \mid x_1)} = \frac{1}{2(1-x_1^2)}$$

$$\frac{L(1 \mid x_1)}{L(2 \mid x_1)} \geq 1 \quad \text{IFF} \quad \frac{1}{2(1-x_1^2)} \geq 1 \quad \text{IFF} \quad x_1 \geq \frac{1}{\sqrt{2}} \ \text{OR} \ x_1 \leq -\frac{1}{\sqrt{2}}$$

SINCE x_1 IS IN $(0,1)$, THE SECOND SOLUTION CANNOT BE ACCEPTED $\Rightarrow$

$$\frac{L(1 \mid x_1)}{L(2 \mid x_1)} \geq 1 \quad \text{IF} \quad x_1 \in \left[\frac{1}{\sqrt{2}}, 1\right)$$

$$\frac{L(1 \mid x_1)}{L(2 \mid x_1)} < 1 \quad \text{IF} \quad x_1 \in \left(0, \frac{1}{\sqrt{2}}\right)$$

THIS IMPLIES THAT THE MLE IS GIVEN BY

$$\hat{\vartheta}(x_1) = \begin{cases} 1 & \text{IF } x_1 \in [1/\sqrt{2}, 1) \\ 2 & \text{IF } x_1 \in (0, 1/\sqrt{2}) \end{cases}$$

c) $L(\vartheta | x_1,\dots,x_n) = \prod_{i=1}^{n} \vartheta (1-x_i^2)^{\vartheta-1} = \vartheta^n \prod_{i=1}^{n} (1-x_i^2)^{\vartheta-1}$

$$\ell(\vartheta | x_1,\dots,x_n) = \log(L(\vartheta | x_1,\dots,x_n)) = \log(\vartheta^n) + \log\left(\prod_{i=1}^{n} (1-x_i^2)^{\vartheta-1}\right)$$

$$= n \cdot \log(\vartheta) + \sum_{i=1}^{n} \log\left((1-x_i^2)^{\vartheta-1}\right)$$

$$= n \log(\vartheta) + (\vartheta-1) \sum_{i=1}^{n} \log(1-x_i^2)$$

$$S(\vartheta | x_1,\dots,x_n) = \frac{d}{d\vartheta} \ell(\vartheta | x_1,\dots,x_n) = n \cdot \frac{1}{\vartheta} + \sum_{i=1}^{n} \log(1-x_i^2)$$

$$S(\vartheta | x_1,\dots,x_n) = 0 \iff n \cdot \frac{1}{\vartheta} + \sum_{i=1}^{n} \log(1-x_i^2) = 0$$

$$\iff \hat{\vartheta} = -\frac{n}{\sum_{i=1}^{n} \log(1-x_i^2)} \quad \text{CRITICAL POINT}$$

IS IT A MAXIMUM ?

$$\frac{d}{d\vartheta} S(\vartheta | x_1,\dots,x_n)\Big|_{\vartheta=\hat{\vartheta}} = \frac{d}{d\vartheta}\left[n \cdot \frac{1}{\vartheta} + \sum_{i=1}^{n} \log(1-x_i^2)\right]\Big|_{\vartheta=\hat{\vartheta}}$$

$$= n (-1) \frac{1}{\vartheta^2}\Big|_{\vartheta=\hat{\vartheta}} = -\frac{n}{\hat{\vartheta}^2} = -\frac{n}{\frac{n^2}{\left(\sum_{i=1}^{n} \log(1-x_i^2)\right)^2}} =$$

$$= -\frac{\left(\sum \log(1-x_i)\right)^2}{n} < 0 \Rightarrow \text{MAXIMUM}$$

$\Rightarrow \hat{\vartheta}(x_1,\dots,x_n) = -\frac{n}{\sum_{i=1}^{n} \log(1-x_i^2)}$ IS THE MLE.

d) $\hat{\vartheta}(0.3, 0.4) = -\frac{2}{\log(1-0.3^2) + \log(1-0.4^2)} = 7.444$

3. (10 points)

   (a) Let T be an estimator of $\theta \in \mathbb{R}$. Give the definition of mean-squared error of T.

   (b) Prove that the mean-squared error of an estimator coincides with the sum of its variance and the square of its bias.

Assume that the observations $s = (10.82, 10.41, 9.78, 10.95)$ are i.i.d. realizations from a normal distribution with unknown mean μ and known variance σ_0^2. You want to test the hypothesis $\mu = 10$ with level of significance $\alpha = 5\%$.

   (c) If $\sigma_0 = 0.98$, what conclusions can you draw?.

   (d) Under which values of σ_0 the observations s would allow to reject the hypothesis that $\mu = 10$? (recall that $\Phi(1.96) = 0.975$).

Solution:

a) THEORY

b) THEORY

c) $H_0: \mu = \mu_0$, $H_1: \mu \neq \mu_0$ WITH $\mu_0 = 10$

$X_1, \dots, X_4 \overset{IID}{\sim} N(\mu, \sigma_0^2)$ WITH $\sigma_0 = 0.98$ AND $\bar{x} = 10.49$

SINCE $\dfrac{|10.49-10|}{0.98/2} < 1.96$ WE CANNOT REJECT H_0 WITH A LEVEL OF SIGNIFICANCE $\alpha = 0.05$

THIS FOLLOWS FROM THIS ARGUMENT:

$$P(|\bar{X}-\mu_0| \geq |\bar{x}-\mu_0|) = P(|\bar{X}-\mu_0| \geq |10.49-10|) = P(|\bar{X}-\mu_0| \geq 0.49)$$

$$= 2P(\bar{X}-\mu_0 \geq 0.49) = 2 \cdot P\left(\frac{\bar{X}-\mu_0}{\sigma_0/\sqrt{n}} \geq \frac{0.49}{\sigma_0/\sqrt{n}}\right) = 2P\left(Z \geq \frac{0.49}{0.98/2}\right) =$$

$$= 2 \cdot (1-\Phi(1)) > 2(1-\Phi(1.96)) = 0.05.$$

d) H_0 CAN BE REJECTED WITH LEVEL OF SIGNIFICANCE $\alpha = 0.05$ IF

$$\frac{|\bar{x}-\mu_0|}{\sigma_0/\sqrt{n}} \geq 1.96 \quad \text{I.E.} \quad \frac{0.49}{\sigma_0/2} \geq 1.96 \quad \text{I.E} \quad \sigma_0 \leq \frac{1}{2}$$