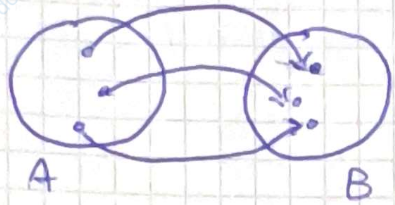
Funzione

Siano A, B 2 insiemi



Defn una funzione da A a B è una mappa / legge / relazione che associa ad ogni elemento di A uno e un solo elemento di B

scriviamo

$f : A \to B$

dominio   codominio

Funzioni Reali:

$f: A \subseteq \mathbb{R} \to \mathbb{R}$

Immagine di A tramite f.

$f(A) \subseteq \mathbb{R}$

es.

i	$f(x) = 2x+3$	$D_f = \mathbb{R}$
ii	$f(x) = \sqrt{x}$	$D: [0, +\infty)$
iii	$f(x) = \frac{1}{x}$	$D: \mathbb{R} \setminus \{0\}$
iv	$f(x) = tg\, x$	$D: \left(-\frac{\pi}{2} + 2k\pi ; \frac{\pi}{2} + 2k\pi\right), k \in \mathbb{Z}$

$f(A) = \{ y \in \mathbb{R} \mid \text{esiste } x \in A \text{ con } f(x) = y \}$

Defn. Grafico

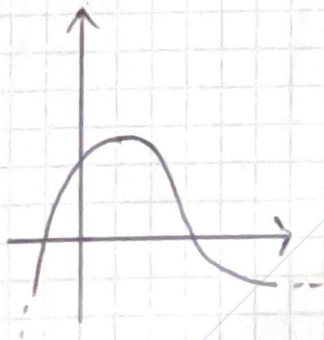
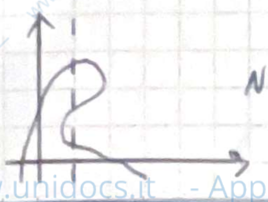


immagine: punti che sono raggiunti dal grafico

$$G(f) = \{ (x,y) \in \underset{\underset{\mathbb{R} \cdot \mathbb{R}}{\parallel}}{\mathbb{R}^2} \mid x \in \mathbb{R},\ y = f(x) \}$$



NB: NON È FUNZIONE

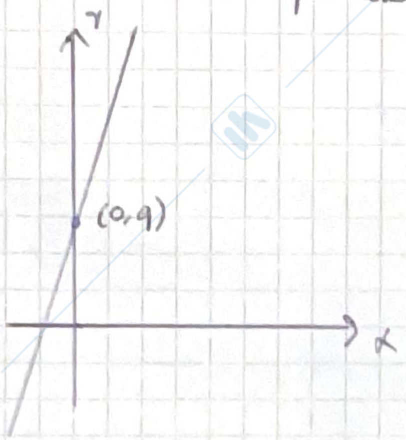
Grafico di Funzioni elementari:

1) Funzione lineare

$m \neq 0$, $q \in \mathbb{R}$ $\quad f(x) = mx + q$ $\quad f: \mathbb{R} \to \mathbb{R}$

m: pendenza

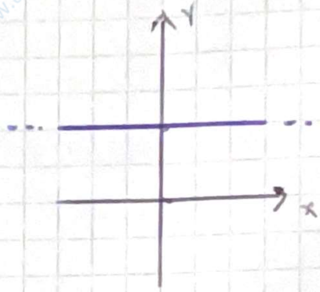
q: intersezione con asse y



- $f(0) = q$
- $f(1) = m + q$

2) $m = 0$ (nel caso 1)

$f(x) = q$



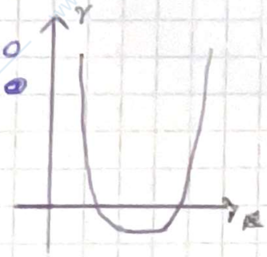
3) Funzione Quadratiche

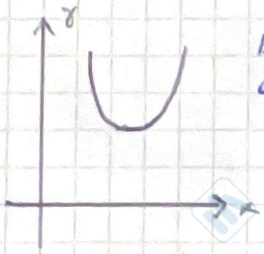
$a, b, c \neq 0 \in \mathbb{R}$ $\quad f: \mathbb{R} \to \mathbb{R}$

$f(x) = ax^2 + bx + c$

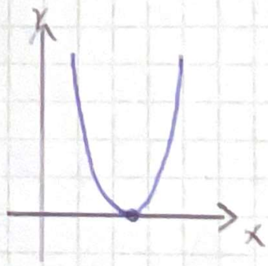
parabola
- verso l'alto $a > 0$
- verso il basso $a < 0$

$\Delta > 0$, $a > 0$



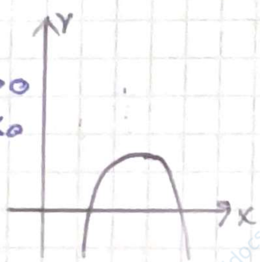


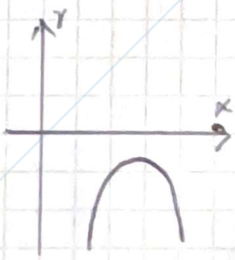
$\Delta < 0$, $a > 0$



$\Delta = 0$, $a > 0$

$\Delta > 0$, $a < 0$



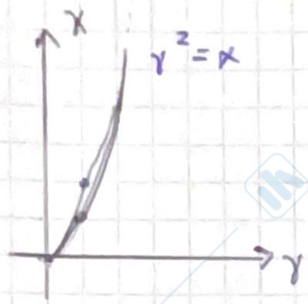


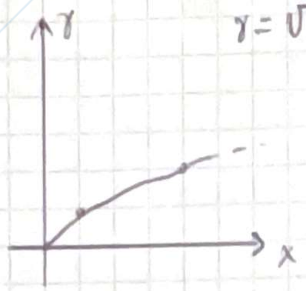
$\Delta < 0$, $a < 0$

4) Funzione radice quadrata

$f(x) = \sqrt{x}$ $\qquad f: [0, +\infty) \to [0, +\infty)$

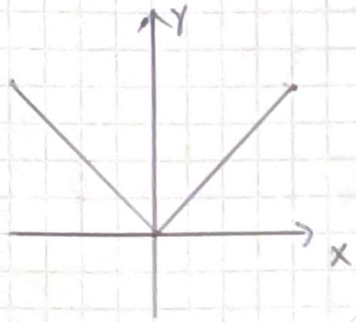
$(x, y) \in G(f) \iff x \in [0, +\infty) \text{ e } y \in [0, +\infty) \iff y^2 = x$





5) Funzione Modulo

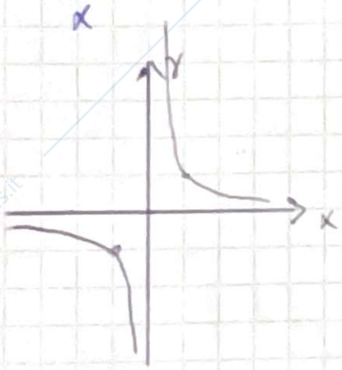
$f: \mathbb{R} \to \mathbb{R} \qquad f(x) = |x|$



$\begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$

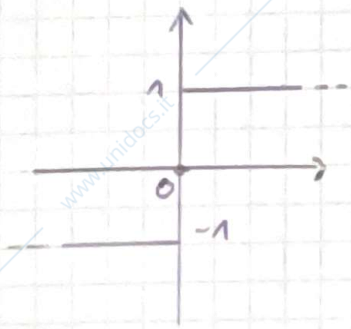
6) Funzione Iperbole

$f(x) = \frac{1}{x} \qquad f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \quad (\mathbb{R} \setminus \{0\})$

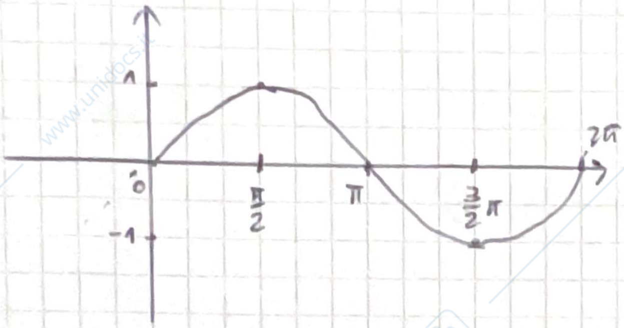


7) Funzione segno

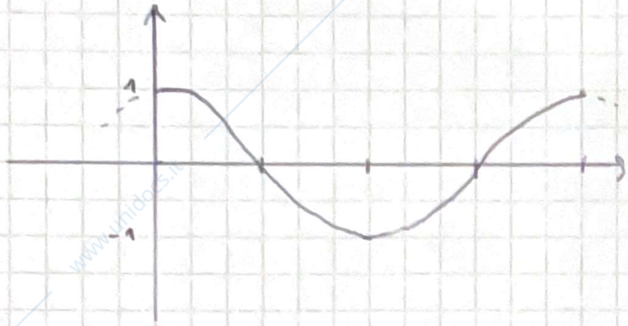
$f(x) = \text{segn}(x) \qquad f: \mathbb{R} \to \{0, -1, 1\}$
signum
sign



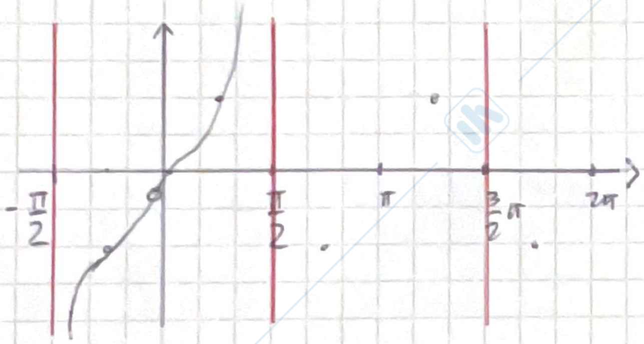
8) $f(x) = \sin x$ $\qquad f: \mathbb{R} \to [-1, 1]$



9) $f(x) = \cos x$ $\qquad f: \mathbb{R} \to [-1, 1]$



10) $f(x) = tg(x)$ $\qquad f: \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\} \to \mathbb{R} \quad , k \in \mathbb{Z}$



11) altre funzioni

- parte intera $\qquad f(x) = [x] \qquad f: \mathbb{R} \to \mathbb{Z}$

  $$\overset{!}{=} \max \{ k \in \mathbb{Z} \mid k \leq x \}$$

- floor $\quad$ arrotondamento verso il basso
- ceil $\quad$ arrotondamento verso l'alto
- mantissa $\quad \{\pi\} = x - [x]$

Operazioni con le funzioni

date $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$

1) $(f+g)(x) = f(x) + g(x)$
2) $(f \cdot g)(x) = f(x)\, g(x)$
3) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\qquad \frac{f}{g} : A \setminus \{x \in A \mid g(x) = 0\} \to \mathbb{R}$

4) Composizione di funzioni

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, ""] & f(x) \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$g \circ f \qquad f : A \to \mathbb{R}$

$g \circ f(x) = g(f(x))$

es.

$f(x) = \frac{1}{x-5} \qquad g(x) = \frac{1}{\sqrt{x}} \qquad ?\ g \circ f,\ f \circ g$

$D_f : \mathbb{R} - \{5\}$

$D_g : (0, +\infty)$

- $(f \circ g)(x) = \frac{\sqrt{x}}{1 - 5\sqrt{x}}$ $\quad$ c.e. $x \geq 0$, $x \neq \frac{1}{25}$, $x \neq 0$
- $(g \circ f)(x) = \frac{1}{\sqrt{\frac{1}{x-5}}}$ $\quad D = (5, +\infty)$

Funzione Identità

$I : \mathbb{R} \to \mathbb{R} \qquad I(x) = x \qquad$ elemento neutro operazione composizione

data f, una funzione generica definita in $\mathbb{R}$:

$f \circ I = I \circ f = f$

oss

$\circ$ - non è commutativa $\qquad f \circ g \neq g \circ f$

Traslazioni

dato $c \in \mathbb{R}$, $c \neq 0$ e $g(x) = x + c$

1) $(f \circ g)(x) = f(x+c)$

$G(f \circ g)$ si ottiene da $G(f) \begin{cases} \text{a dx} & \text{se } x < 0 \\ \text{a sx} & \text{se } x > 0 \end{cases}$

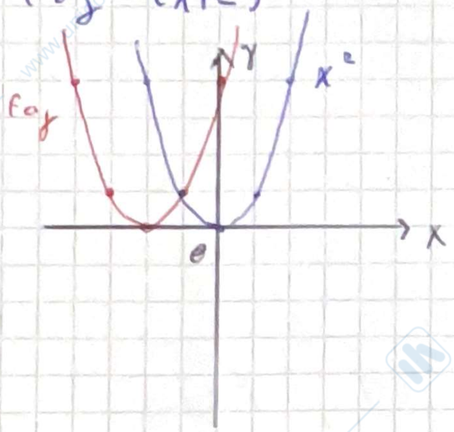
2) $(g \circ f)(x) = f(x) + c$

$G(g \circ f)$ si ottiene da $G(f) \begin{cases} \text{in alto} & c > 0 \\ \text{in basso} & c < 0 \end{cases}$

esempio

$f(x) = x^2 \qquad g(x) = x + 2$

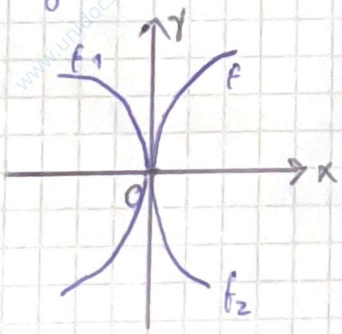
- $f \circ g \quad (x+2)^2$



Simmetria

sia $g(x) = -x$

(i) $(f \circ g)(x) = f(g(x)) = f(-x) = f_1(x)$



(ii) $(g \circ f)(x) = g(f(x)) = -f(x) = f_2(x)$

(iii) $(g \circ f \circ g)(x) = -f(-x) = f_3(x)$

Pari, dispari, né pari né dispari

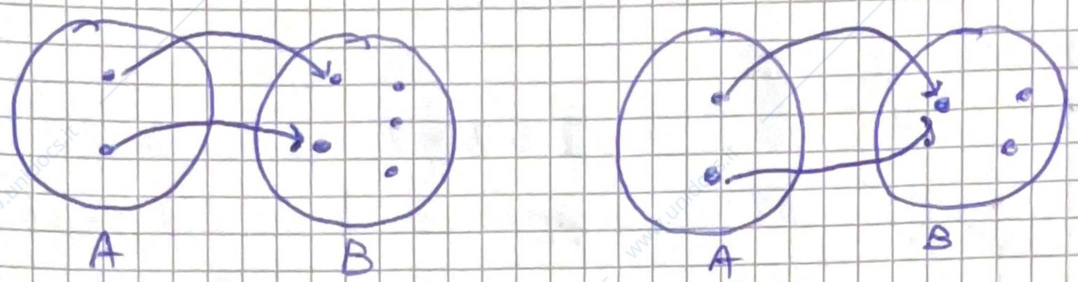
Sia $A \subseteq \mathbb{R}$ simmetrica

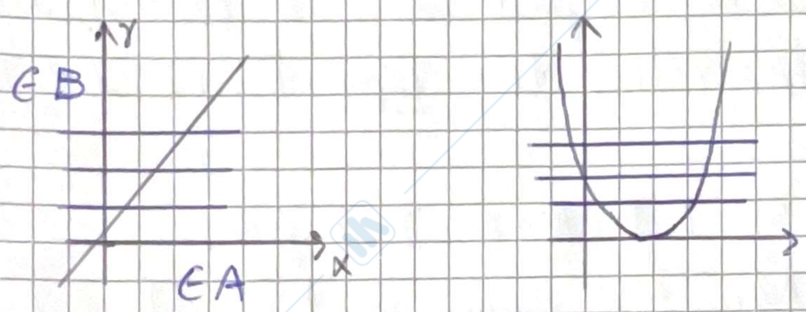
$x \in A \iff -x \in A$

Sia $f: A \to \mathbb{R}$

- è detta pari se $f(-x) = f(x)$
- è detta dispari se $f(-x) = -f(x)$

Funzione iniettiva





Def. Sia $f: A \subseteq \mathbb{R} \to \mathbb{R}$, è detta iniettiva se ad ogni due valori distinti di A associa due valori distinti di $\mathbb{R}$

- $\forall x \neq z \in A$ si ha $f(x) \neq f(z)$

oppure

- $f(x) = f(z)$, $x, z \in A \Rightarrow x = z$

Funzione monotona

Sia $f: A \subseteq \mathbb{R} \to \mathbb{R}$ è detta

(i) monotona strettamente crescente se $\forall x < z$ si ha $f(x) < f(z)$

(ii) monotona non decrescente se $\forall x \leq z$ $f(x) \leq f(z)$

(iii) monotona strett. decrescente se $\forall x < z$ $f(x) > f(z)$

(iv) monotona non crescente se $\forall x \leq z$ $f(x) \geq f(z)$

Relazione tra monotonia e iniettività

Teo $f: A \subseteq \mathbb{R} \to \mathbb{R}$

Se f è strett. monotona su $A \Rightarrow f$ è iniettiva

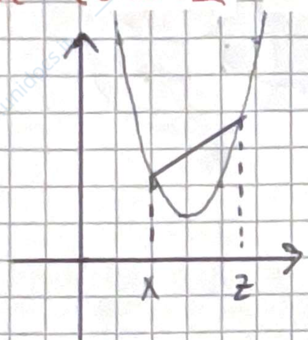
dim.

Per semplicità, supponiamo $f \nearrow$ strettamente

$x \neq z \Rightarrow \begin{cases} x < z \Rightarrow f(x) < f(z) \\ x > z \Rightarrow f(x) > f(z) \end{cases} \Rightarrow f(x) \neq f(z)$

oss. NON VALE $\Leftarrow$ se iniettiva

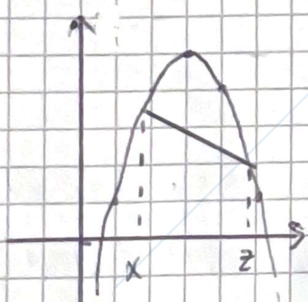
Funzione convessa



Sia $f: I \subseteq \mathbb{R} \to \mathbb{R}$, I intervallo.

Allora f è convessa se $\forall x, z \in I$, $x < z$ si ha il segmento che unisce $(x; f(x))$ e $(z; f(z))$ si trova sopra il grafico di f in $[x, z]$

Funzione concava:



Sia $f: I \subseteq \mathbb{R} \to \mathbb{R}$, I intervallo.

Allora f è concava se $\forall x, z \in I$, $x < z$ si ha il segmento che unisce $(x; f(x))$ e $(z, f(z))$ si trova sotto il grafico della funzione in $[x, z]$

Segno e zeri di una funzione

data $f: A \subseteq \mathbb{R} \to \mathbb{R}$

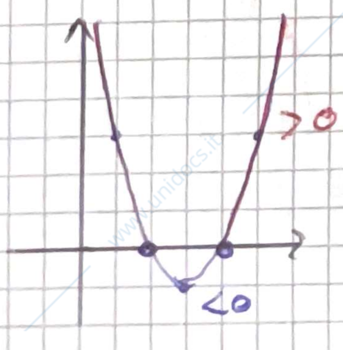
(i) f si dice positiva su A se $f(x) > 0$

f si dice non negativa su A se $f(x) \geq 0$

f si dice negativa su A se $f(x) < 0$

f si dice non positiva su A se $f(x) \leq 0$

(ii) x si chiama zero della funzione se $f(x) = 0$



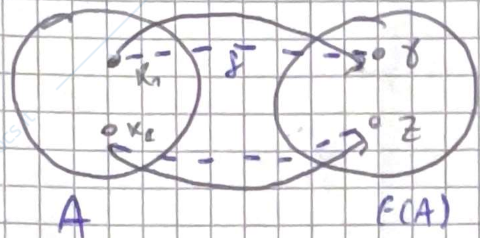
Ripasso prodotto - composizione

Il prodotto in $\mathbb{R}$
- è commutativo
- è associativo
- $\exists$ elemento neutro (1)
- $\exists$ elemento inverso $(x \cdot y) = 1$

Composizione di funzione ($\circ$)
- non è commutativa
- è associativa
- $\exists$ elemento neutro $I(x) = x$
- $f \circ g = I$

Funzione Inversa

$f: A \subseteq \mathbb{R} \to f(A)$ e f. iniettiva



Allora, $\forall y \in f(A)$ $\exists! \, x \in A \mid f(x) = y$

def. $g(A) \to A \mid g(y) = x$

g è funzione detta inversa e usiamo la notazione $f^{-1} = g$

$$\left(\begin{array}{l} f(1) = 5 \\ \Rightarrow g(5) = 1 \end{array} \right)$$

Proprietà

Sia $f: A \to f(A)$ iniettiva, f^{-1}, $f(A) \to$ funz. inversa

$$A \xrightarrow{f} f(A) \xrightarrow{f^{-1}} A$$

$f^{-1} \circ f = I_A$ $(I_A : A \to A$ con $I_A(x) = x)$

$f \circ f^{-1} = I_{f(A)}$

$$f(A) \xrightarrow{f^{-1}} A \xrightarrow{f} f(A)$$

esempi

1) $f(x) = 2x+1$ $\quad f: \mathbb{R} \to \mathbb{R}$

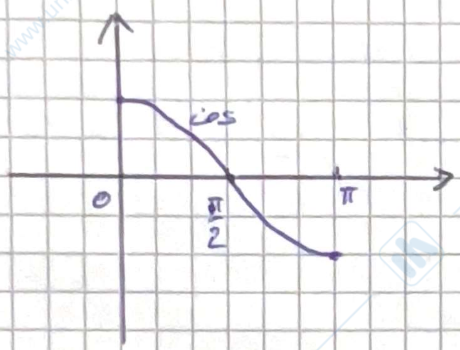
calcolo funz. inversa

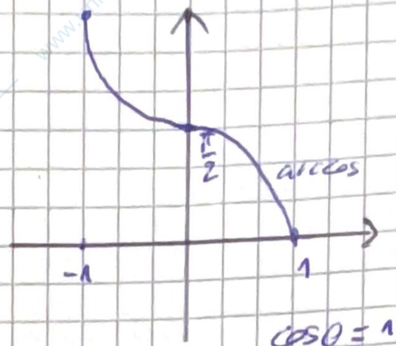
$f(x) = y$ $\quad y = 2x+1$ $\quad \frac{y-1}{2} = x$ $\quad f^{-1}(x) = \frac{x-1}{2}$

2) $f(x) = x^2$ $\quad f: [0, +\infty) \to [0, +\infty)$

Funzioni goniometriche inverse

1) $f(x) = \cos x \qquad f: [0, \pi] \to [-1, 1]$

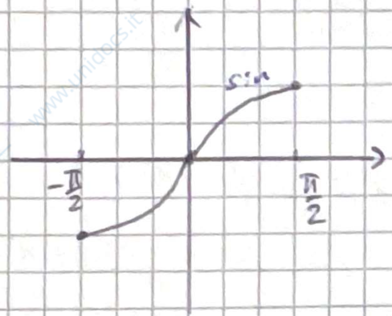


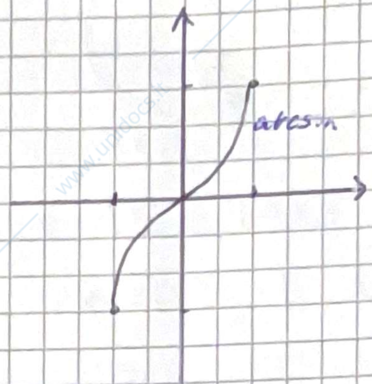


$\cos 0 = 1 \Rightarrow \arccos 1 = 0$

$f([0,\pi]) = [-1,1]$ • f. iniettiva $\Rightarrow \exists$ inversa : $\arccos : [-1,1] \to [0,\pi]$

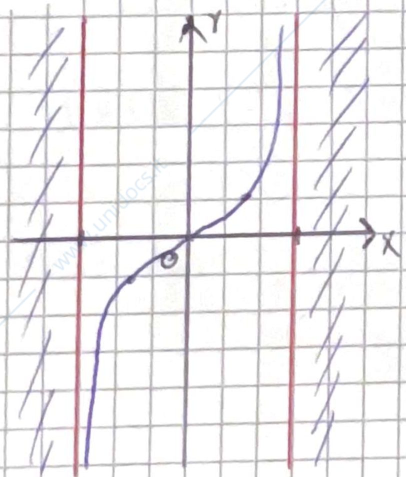
2) $f(x) = \sin x \qquad f: [-\frac{\pi}{2}; \frac{\pi}{2}] \to [-1,1]$

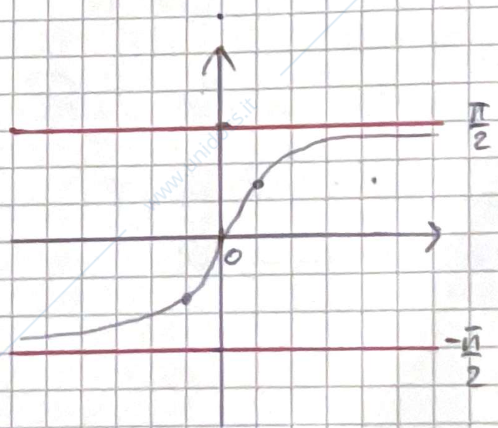




$f([-\frac{\pi}{2}, \frac{\pi}{2}]) = [-1, 1]$ • f. iniettiva $\Rightarrow \exists$ inversa : $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

3) $f(x) = tg\, x \qquad f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$





$f([-\frac{\pi}{2}, \frac{\pi}{2}]) = \mathbb{R}$ • iniettiva $\Rightarrow \exists$ inversa : $arctg : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

Funzione limitata

Sia $f: A \subseteq \mathbb{R} \to \mathbb{R}$

1) f. è limitata superiormente se $f(A)$ è un insieme superiormente limitato oppure se $\exists M \in \mathbb{R} \mid f(x) \leq M \ \forall x \in A$

2) f. è limitata inferiormente se $f(A)$ è un insieme inferiormente limitato oppure se $\exists m \in \mathbb{R} \mid f(x) \geq m \ \forall x \in A$

es.
$f(x) = \operatorname{arctg} x \qquad f(\mathbb{R}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\inf f = -\frac{\pi}{2} \qquad \sup f = \frac{\pi}{2}$

Max Globale

$f: A \subseteq \mathbb{R} \to \mathbb{R}$, $x^* \in A$ è detto punto di massimo globale se $f(x^*) \geq f(x) \ \forall x \in A$

Min Globale

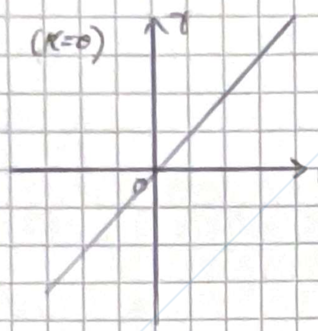
$f: A \subseteq \mathbb{R} \to \mathbb{R}$, $x_* \in A$ è detto punto di minimo globale se $f(x_*) \leq f(x) \ \forall x \in A$

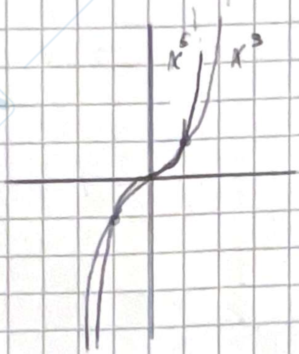
Grafici di funzioni elementari

1) Funzioni potenze a esp. naturale

$f(x) = x^n \quad x \in \mathbb{N}$

(i) n dispari $\qquad f(x) = x^{2k+1} \quad k \in \mathbb{N}_0 \qquad f: \mathbb{R} \to \mathbb{R}$





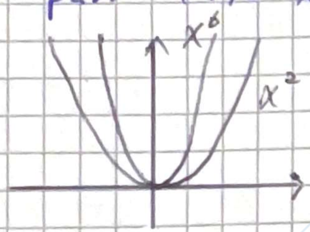
$x^3 \leq x^5$

- $x \geq 0 \iff x \geq 1$
- $x < 0 \iff x > -1$

- funz. dispari $f(-x) = -f(x)$
- funz. strettamente crescente
- f. iniettiva su $\mathbb{R}$

$\exists$ f. inversa $x = \sqrt[3]{y}$

(ii) n pari $\quad f(x) = x^{2k} \quad k \in \mathbb{N} \qquad f: \mathbb{R} \to \mathbb{R}$

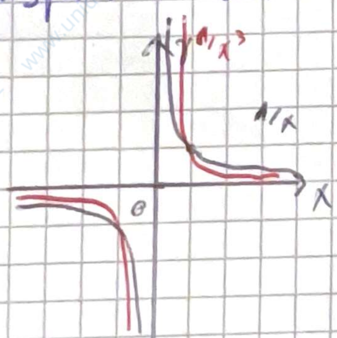


- f. pari
- f. strett. ↗ $[0, +\infty)$
- f strett. ↘ $(-\infty, 0]$

2 Funz. potenza a intero negativo

$f(x) = \frac{1}{x^n}$ o $x^{-n} \quad n \in \mathbb{N} \qquad f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$

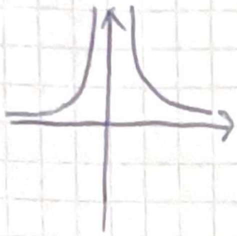
(i) dispari $\quad f(x) = \frac{1}{x^{2k+1}} \quad k \in \mathbb{N}_0$



$\frac{1}{x} \leq \frac{1}{x^3}$: $x > 0$

$x \leq 1$

(ii) n. pari: $f(x) = x^{-2k}$ $\quad f(x) = \frac{1}{x^{2k}}$ $\quad k \in \mathbb{N}$ $\quad f: \mathbb{R} \setminus \{0\} \to (0, +\infty)$

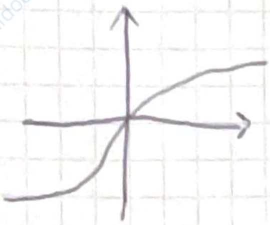


pari, non invertiva

3) Funzione potenza a esponente $1/n$

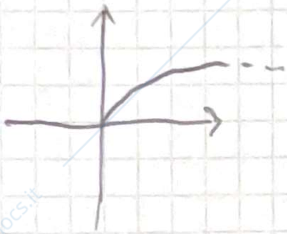
$f(x) = x^{\frac{1}{n}}$ $\quad n \in \mathbb{N}$ $\quad$ (funz. inversa di x^n $f^{-1}(x)$)

(i) n dispari $\quad x^{\frac{1}{2k+1}}$ $\quad k \in \mathbb{N}_0$ $\quad f: \mathbb{R} \to \mathbb{R}$ $\quad x^{\frac{1}{n}} = \begin{cases} \sqrt[n]{x} & x>0 \\ -\sqrt[n]{-x} & x<0 \end{cases}$



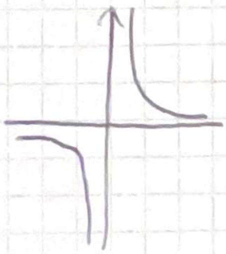
(ii) n pari $\quad x^{\frac{1}{2k}}$ $\quad k \in \mathbb{N}$

$f: [0, +\infty) \to [0, +\infty)$



4 Funz. potenze a esponente $-1/n$

(i) n dispari $\quad f(x)\ x^{-\frac{1}{2k+1}}$ $\quad k \in \mathbb{N}_0$



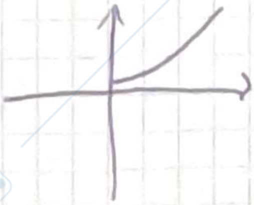
5) Funz. potenze a esp m/n $\quad x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$

1) $\frac{m}{n} > 1$

• m dispari, n pari

$f: [0,+\infty) \to [0,+\infty)$

es. $x^{\frac{3}{2}}$

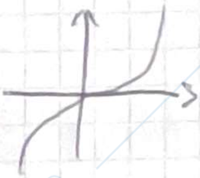


• m pari, n dispari $\quad f: \mathbb{R} \to [0,+\infty)$

$x^{4/3}$

• m, n dispari $\quad f: \mathbb{R} \to \mathbb{R}$

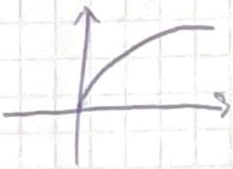
$x^{5/3}$



2) $0 < \frac{n}{m} < 1$

• m dispari, n pari $\quad f: [0,+\infty) \to [0,+\infty)$

$x^{3/4}$



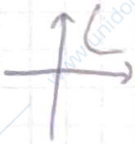
• n pari, n dispari $\quad f: \mathbb{R} \to [0,+\infty)$

$x^{2/3}$

• m, n dispari

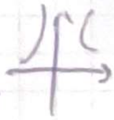
$x^{3/5}$

3) $\frac{n}{m} < 0 \quad \frac{1}{x^{-\frac{m}{n}}}$ • $f: (0,+\infty) \to (0,+\infty)$



$x^{-3/4}$

• $f: \mathbb{R} \to (0,+\infty)$



• $f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$

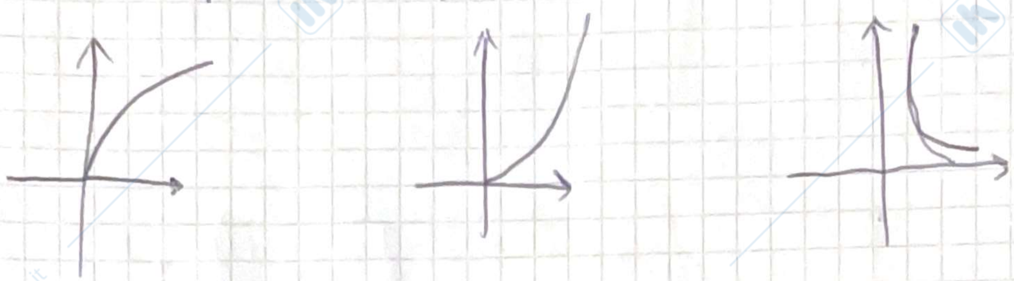
6) Funz. esp. reale

$f(x) = x^{\alpha}$

$f: [0, +\infty) \to [0, +\infty)$ $\alpha > 0$

$f: (0, +\infty) \to (0, +\infty)$ $\alpha < 0$

$0 < \alpha < 1$ $\alpha > 1$ $\alpha < 0$



$x^a \cdot x^b$

$\frac{x^a}{x^b}$ $(x^a)^b$

es. $x^{\frac{2}{3}} \leq 2 - x^2$ graficamente

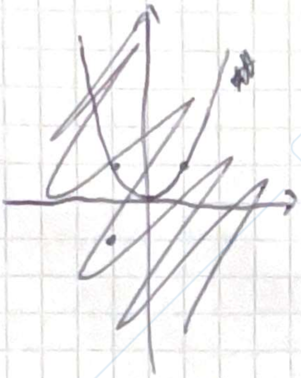
7/10/24

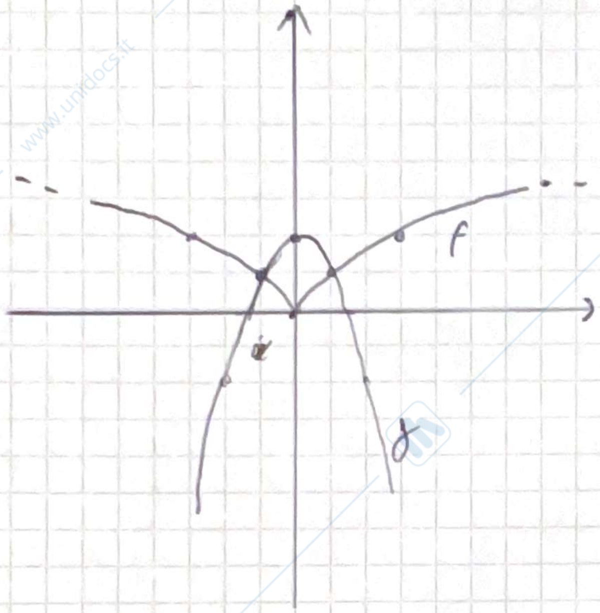
Alcune disequazioni da risolvere con grafici

1) $x^{2/3} \le 2 - x^2$

$f(x)\ x^{\frac{2}{3}} = \sqrt[3]{x^2}$ $\quad f: \mathbb{R} \to \mathbb{R}$

$g(x) = 2 - x^2$ $\quad f: \mathbb{R} \to \mathbb{R}$





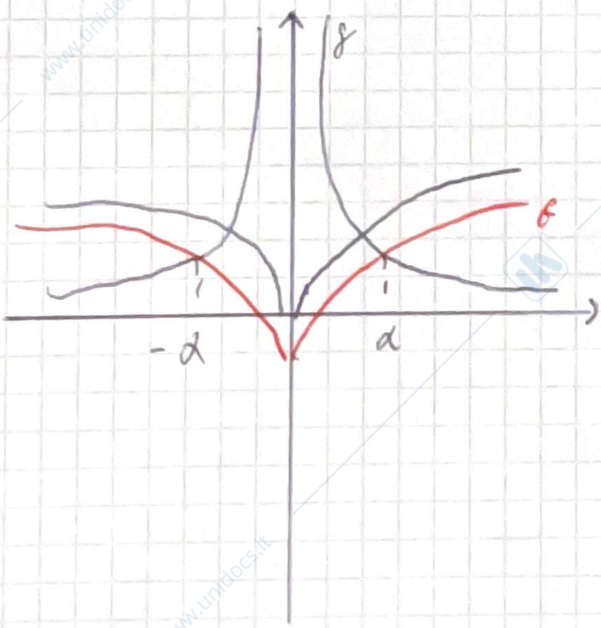
$S = [-1, 1]$

2) $\sqrt{|x|} - 1 \ge x^{-\frac{2}{3}}$

$f(x) = \sqrt{|x|} \to \sqrt{x}$ se $x \ge 0$

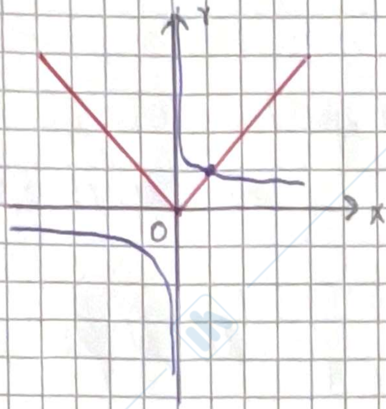
$g(x) = \dfrac{1}{\sqrt[3]{x^2}}$

$D_f\ \mathbb{R} \setminus \{0\}$



$S = (-\infty, -\alpha] \cup [\alpha, +\infty)$

3) $|x| > x^{-\frac{1}{5}}$



$S: (-\infty, 0) \cup (1, +\infty)$

RICORDA

$a \leq b \qquad a, b \in \mathbb{R}$

- $a^{2k+1} \leq b^{2k+1} \qquad \forall\, k \in \mathbb{N}$
- $0 \leq a \leq b \qquad a^{2k} \leq b^{2k}$
- $a \leq b < 0 \qquad a^{2k} \geq b^{2k}$