Studio di funzione

1) Calcolo del dominio.
   Criticità:
   a) Presenza di un denominatore → porre il denominatore $\neq 0$.
   b) Presenza di un radicale con indice pari → porre l'argomento ≥ 0
   c) Presenza di un logaritmo → porre l'argomento > 0
   d) Presenza di tgx e ctgx → rientra in a)
   e) Presenza di arcsenx e arccosx → il dominio è $[-1; 1]$
2) Limiti agli estremi del C.E.
   Individuazione di eventuali asintoti verticali, obliqui, orizzontali.
3) Segno della funzione ed ~~int~~ eventuali intersezioni con gli assi.
4) Studio della crescenza della funzione ed eventuali estremi relativi.
5) Studio della convessità ed eventuali punti di flesso.
6) Disegno del grafico.

$$f(x) = \frac{x^2-1}{x^2-2x}$$

$x^2-2x \neq 0 \qquad x \neq 0 \wedge x \neq 2 \qquad D_f = (-\infty;0) \cup (0;2) \cup (2;+\infty)$

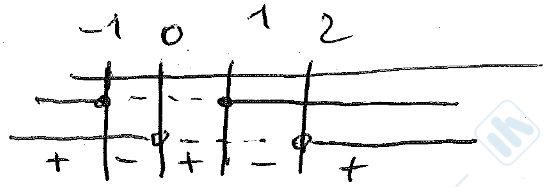
$\{x : x \in \mathbb{R},\ x \neq 0 \wedge x \neq 2\}$

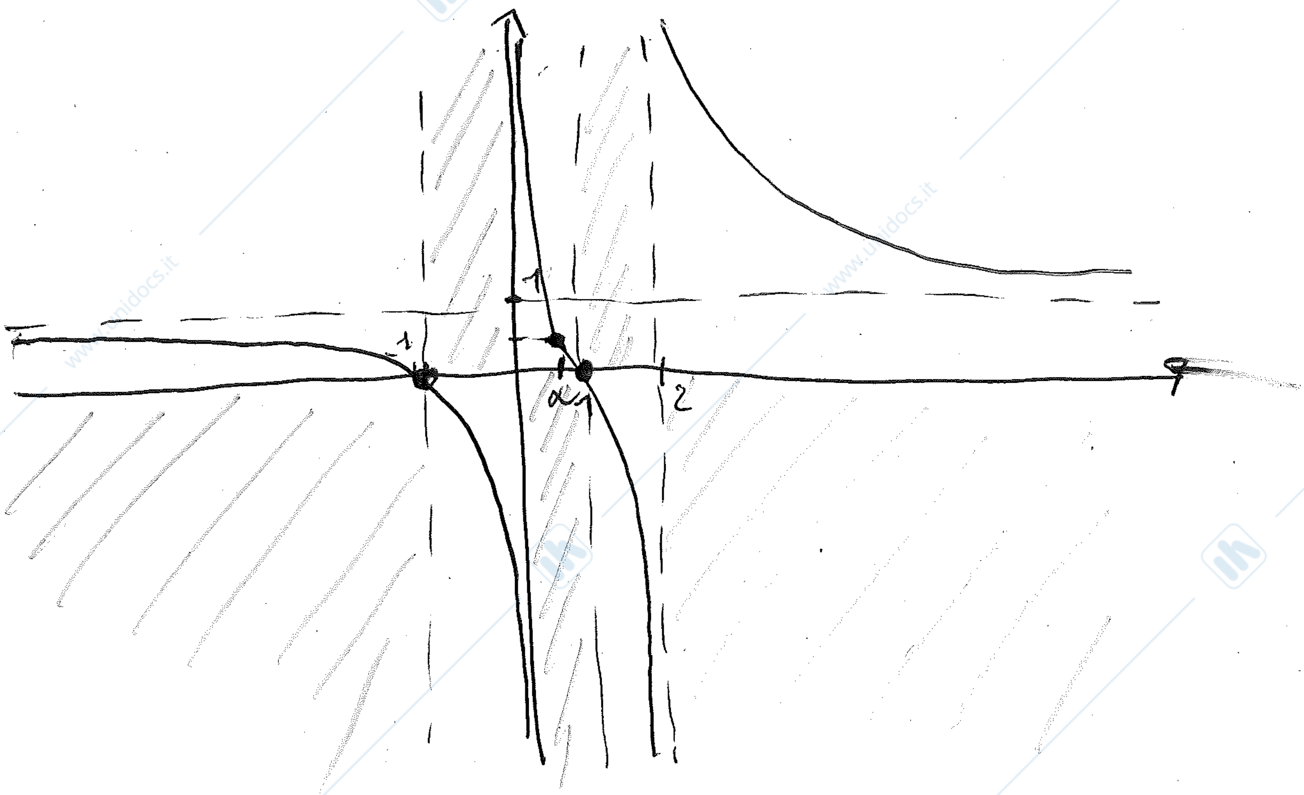
$\lim_{x \to -\infty} \frac{x^2-1}{x^2-2x} = 1 \qquad \lim_{x \to +\infty} \frac{x^2-1}{x^2-2x} = 1 \qquad y = 1$ as. orizzontale

$\lim_{x \to 0^-} \frac{x^2-1}{x^2-2x} = -\infty \qquad \lim_{x \to 0^+} \frac{x^2-1}{x^2-2x} = +\infty \qquad x = 0$ as. vert.

$\lim_{x \to 2^-} \frac{x^2-1}{x^2-2x} = -\infty \qquad \lim_{x \to 2^+} \frac{x^2-1}{x^2-2x} = +\infty \qquad x = 2$ as. vert.

$\frac{x^2-1}{x^2-2x} \geq 0 \qquad N\ x \leq -1 \vee x \geq 1 \qquad D\ x < 0 \vee x > 2$





$$f'(x)=\frac{2x(x^2-2x)-(2x-2)(x^2-1)}{(x^2-2x)^2}=$$

$$=\frac{\cancel{2x^3}-4x^2-\cancel{2x^3}+2x^2+2x-2}{(x^2-2x)^2}=\frac{-2x^2+2x-2}{(x^2-2x)^2}=$$

$$=\frac{-2(x^2-x+1)}{(x^2-2x)^2}$$

$f'(x)\geq 0 \quad x^2-x+1<0 \quad \Delta<0$

$\nexists\, x\in\mathbb{R}$

$\Downarrow$

$x^2-x+1>0\ \forall x\in\mathbb{R} \Rightarrow f(x)<0\ \forall x\in \mathrm{Dom} f$

$\Rightarrow f(x)$ è sempre decrescente

$$f''(x)=\frac{-2[(2x-1)(x-2)^2x^2-2(x^2-x+1)(x^2-2x)(2x-2)]}{(x^2-2x)^4}=$$

$$=\frac{-2[(2x-1)(x-2)^{\cancel{2}}x^{\cancel{2}}-\cancel{4x}(x^2-x+1)\cancel{(x-2)}(x-1)]}{x^3(x-2)^3}=$$

$$=\frac{-2[x(2x-1)(x-2)-4(x^2-x+1)(x-1)]}{x^3(x-2)^3}=$$

$$=\frac{-2[(2x^2-x)(x-2)-(x^2-x+1)(4x-4)]}{x^3(x-2)^3}=$$

$$=\frac{-2[2x^3-x^2-4x^2+2x-4x^3+4x^2-4x+4x^2-4x+4]}{x^3(x-2)^3}=$$

$$=\frac{-2[-2x^3+3x^2-6x+4]}{x^3(x-2)^3}$$

$P(2)=-16+24-12+4\neq 0 \qquad P=-2x^3+3x^2-6x+4$

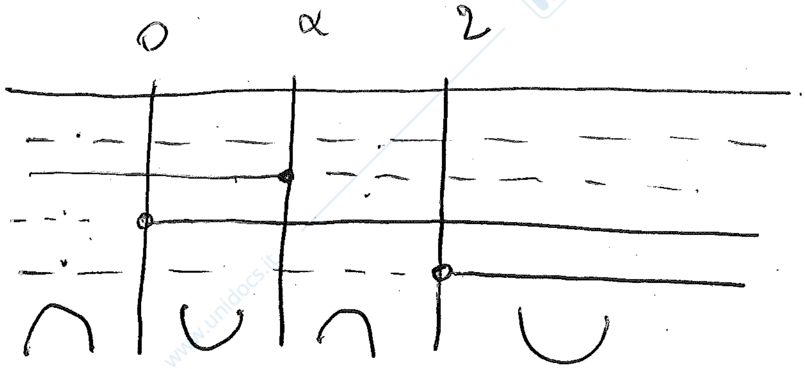
$\alpha \in (0;2)$

$\alpha = 1 \quad P(1) = -2+3-6+4 = -1 < 0$

$P\left(\frac{1}{2}\right) = -\frac{2}{8}+3\cdot\frac{1}{4}-\frac{6}{2}+4 = 4-\frac{5}{2} = \frac{3}{2} > 0$

$\alpha \in \left(\frac{1}{2};1\right)$

$$\begin{cases} -2>0 & \text{mai} \\ P(x)\geq 0 & x<\alpha \\ x^3>0 & x>0 \\ (x-2)^3>0 & x>2 \end{cases}$$



$$f(x) = \begin{cases} x\cdot e^{-\frac{1}{x}} & x\neq 0 \\ 0 & x=0 \end{cases} \qquad D_f \equiv \mathbb{R}$$

$\lim_{x\to-\infty} x e^{-\frac{1}{x}} = -\infty \qquad \lim_{x\to+\infty} x e^{-\frac{1}{x}} = +\infty$

$\lim_{x\to 0^-} x e^{-\frac{1}{x}} = \lim_{x\to 0^-} \frac{e^{-\frac{1}{x}}}{\frac{1}{x}} = -\infty$

$-\frac{1}{x} \to +\infty \to +\infty$

$\lim_{x\to 0^+} x e^{-\frac{1}{x}} = 0 \qquad -\frac{1}{x}\to -\infty$

$x=0$ as. vert.

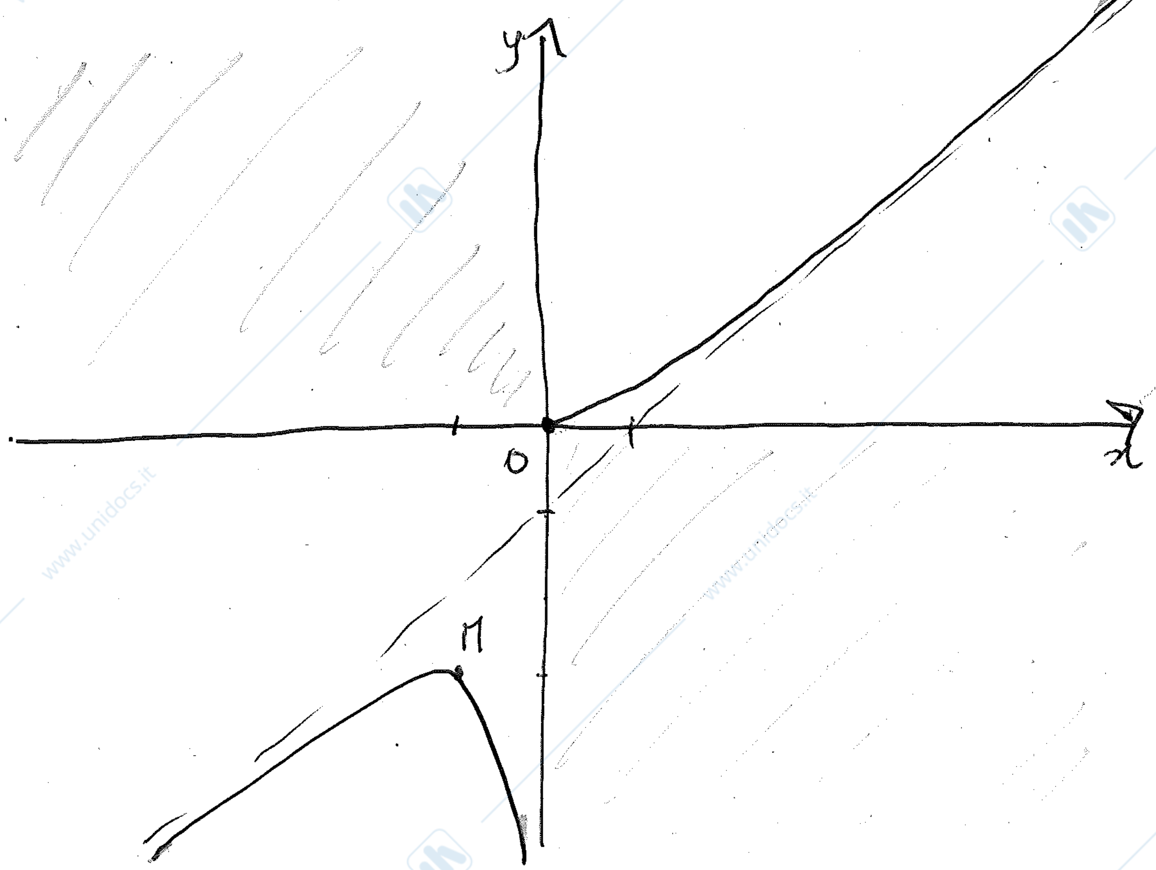
In $x_0=0$ la fz. assume discontinuità di II specie

$\lim_{x\to\pm\infty} \frac{x e^{-\frac{1}{x}}}{x} = 1 = m$

$q = \lim_{x\to\pm\infty} x e^{-\frac{1}{x}} - x = \lim_{x\to\pm\infty} x\left(e^{-\frac{1}{x}}-1\right) = \lim_{x\to\pm\infty} \frac{e^{-\frac{1}{x}}-1}{-\frac{1}{x}}(-1) = -1$ $\to 1$

$y = x - 1$ asintoto obliquo

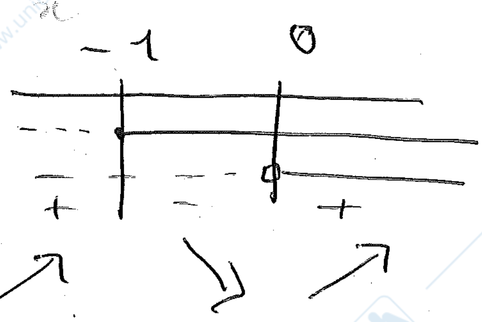
$f(x) \geq 0$ per $x \geq 0$



$$f'(x) = \begin{cases} e^{-\frac{1}{x}} + x \cdot e^{-\frac{1}{x}} \cdot \frac{1}{x^2} = e^{-\frac{1}{x}}\left(1 + \frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$e^{-\frac{1}{x}} + \frac{e^{-\frac{1}{x}}}{x}$

$f'(x) \geq 0 \quad \frac{x+1}{x} \geq 0 \quad N) \; x \geq -1 \quad D) \; x > 0$



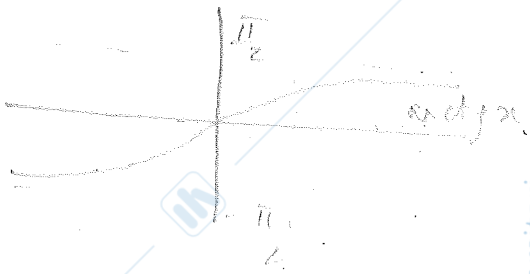
$M(-1; -e)$

$$f''(x) = \begin{cases} \cancel{\frac{e^{-\frac{1}{x}}}{x^2}} + \frac{1}{x} \cdot \frac{e^{-\frac{1}{x}}}{x^2} + \cancel{e^{-\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f''(x) = \begin{cases} \frac{e^{-\frac{1}{x}}}{x^3} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f''(x)>0 \quad x>0$

$f(x)=\frac{c}{x}+\operatorname{arctg}x \quad c\in\mathbb{R}$

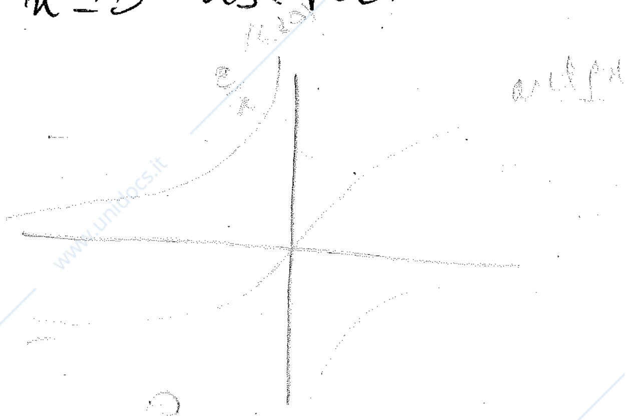


$x\neq 0 \quad Dom f = (-\infty;0)\cup(0;+\infty)$

$\boxed{c<0}$ (per $c=0$ si cade nel caso banale di $f(x)=\operatorname{arctg}x$)

$$\left.\begin{array}{l}\lim\limits_{x\to-\infty}\frac{c}{x}+\operatorname{arctg}x=-\frac{\pi}{2}\\ \lim\limits_{x\to+\infty}\frac{c}{x}+\operatorname{arctg}x=\frac{\pi}{2}\end{array}\right\} \begin{array}{l}y=-\frac{\pi}{2}\ \text{as. orizz.}\\ y=\frac{\pi}{2}\ \text{(per ogni } c)\end{array}$$

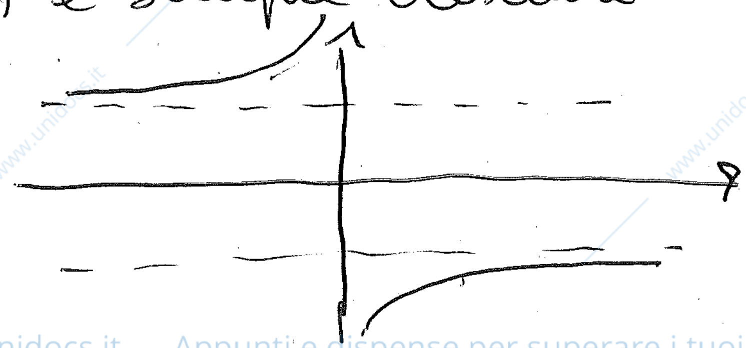
$$\left.\begin{array}{l}\lim\limits_{x\to0^-}\frac{c}{x}+\operatorname{arctg}x=+\infty\\ \lim\limits_{x\to0^+}\frac{c}{x}+\operatorname{arctg}x=-\infty\end{array}\right\} x=0\ \text{as. vert.}$$



$$f'(x)=-\frac{c}{x^2}+\frac{1}{x^2+1}=$$

$$=\frac{-c(x^2+1)+x^2}{x^2(x^2+1)}=\frac{(1-c)x^2-c}{x^2(x^2+1)}>0 \quad \forall x\in Dom f$$

$\Rightarrow f(x)$ è sempre crescente



$c > 0$

$$\left.\begin{array}{l} \lim\limits_{x\to 0^-} \dfrac{c}{x} + \operatorname{arctg} x = -\infty \\ \lim\limits_{x\to 0^+} \dfrac{c}{x} + \operatorname{arctg} x = +\infty \end{array}\right\} x = 0 \text{ as. vert.}$$

$$f'(x) = \frac{(1-c)x^2 - c}{x^2(x^2+1)}$$

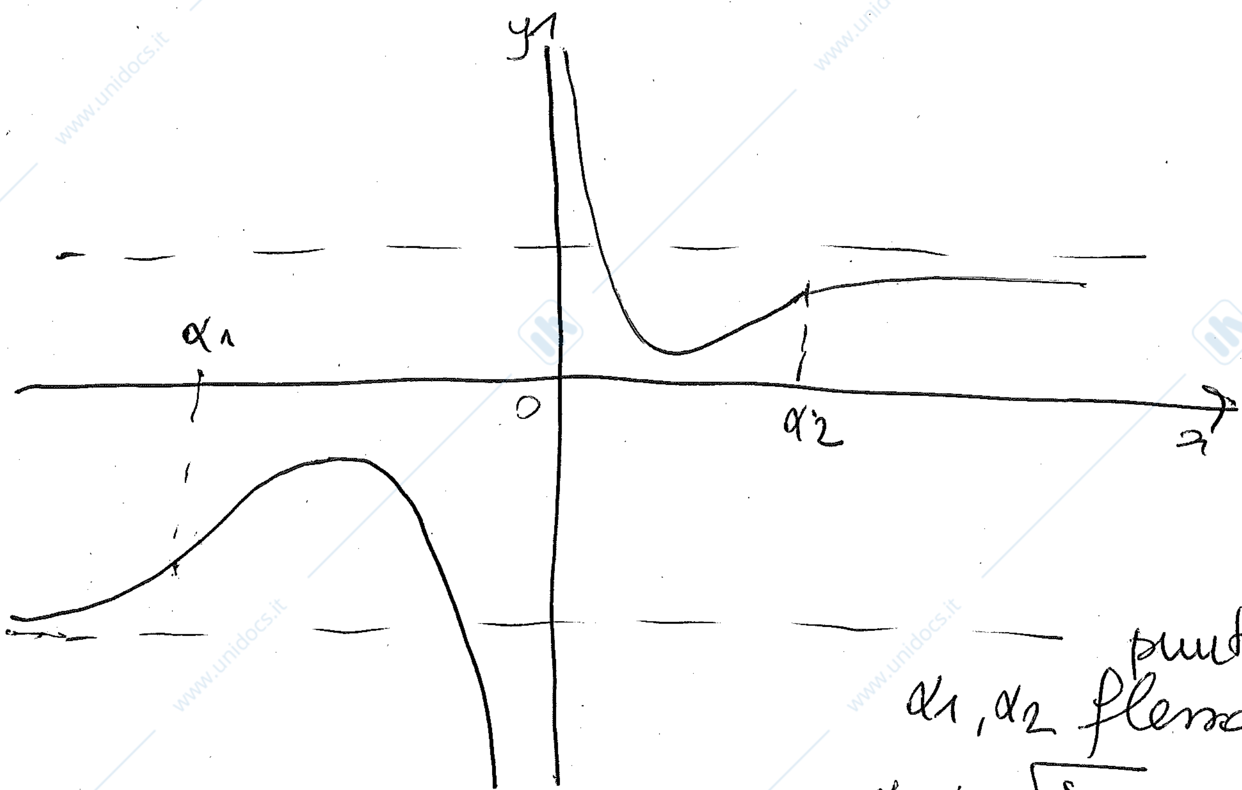
Sottocaso: $0 < c < 1$

$f'(x) \geq 0 \qquad (1-c)x^2 - c \geq 0$

$$x_1 \leq -\sqrt{\frac{c}{1-c}} \quad \vee \quad x_2 \geq \sqrt{\frac{c}{1-c}}$$

x_1 x_2

↗ MAX ↘ MIN ↗



α_1, α_2 punti di flesso

$\alpha_1 < -\sqrt{\frac{c}{1-c}} \qquad \alpha_2 > \sqrt{\frac{c}{1-c}}$

Sottocaso $\underline{c \geq 1}$

$$f'(x) = \frac{(1-c)x^2 - c}{x^2(x^2+1)}$$

$$\underbrace{(1-c)x^2}_{\leq 0} \underbrace{-c}_{< 0} < 0 \quad \forall x \in \mathrm{Dom} f$$

$\Rightarrow f'(x) < 0 \quad \forall x \in \mathrm{Dom} f \Rightarrow f$ sempre decrescente