1st LESSON

$\mathbb{N}$ (natural numbers) : $\rightarrow$ INDUCTION PRINCIPLE $\{0, 1, 2, 3 \dots\}$

Operations

Sum $(m, n) \rightarrow m + n$

PROPERTIES

1) Commutativity

$m + n = n + m \quad \forall\, n, m \in \mathbb{N}$

2) Associativity

$(m + n) + k = m + (n + k) = m + n + k \quad \forall\, m, n, k \in \mathbb{N}$

3) Neutral element

$m + 0 = 0 + m = m \quad \forall m \in \mathbb{N}$

Product $(m, n) \rightarrow m \cdot n$

PROPERTIES

1) Commutativity

$m \cdot n = n \cdot m \quad \forall\, n, m \in \mathbb{N}$

2) Associativity $\quad (m \cdot n) k = m (n \cdot k) \quad \forall\, n, m, k \in \mathbb{N}$

3) Neutral element $\quad n \cdot 1 = 1 \cdot n = n \quad \forall n \in \mathbb{N}$

4) Distributivity

$k (m + n) = mk + nk \quad \forall\, n, m, k \in \mathbb{N}$

Induction Principle

1) let $T \subset \mathbb{N}$

if i) $0 \in T$

ii) $n \in T$ $\quad n+1 \in T$

then, $T = \mathbb{N}$ $\quad \forall n \in \mathbb{N}$

2) let $P(n)$ be a property of $n \in \mathbb{N}$

if $P(0)$ is true

$P(n) \rightarrow P(n+1)$ $\quad \forall n \in \mathbb{N}$

then, $P(n)$ is true $\quad \forall n \in \mathbb{N}$

P is a theorem in $\mathbb{N}$

Theorem of Gauss

Sum all numbers from 1 to 100

	1	2	3	4	5	6	7
$+\downarrow$							
	100	99	98	97	96	95	94
	101	101	101	101	101	101	101

$$\text{sum} = 101 \cdot 50 = 5050$$

Generalisation: sum all numbers from 1 to $\mathbb{N}$

1) N is even

	1	2	3	4	5
$+\downarrow$	N	$N-1$	$N-2$	$N-3$	$N-4$
	$N+1$	$N+1$	$N+1$	$N+1$	$N+1$

$$(N+1)\frac{N}{2}$$

2) N is odd

	0	1	2	3	4	5
$+\downarrow$	N	$N-1$	$N-2$	$N-3$	$N-4$	$N-5$

$$N \cdot \frac{N+1}{2}$$

Heuristic !!!

In order to have a rigorous proof we need induction principles

for the 2nd formulation

$$P(N) = 1 + 2 + 3 \dots + N = \frac{N(N+1)}{2}$$

$$P(N) = \sum_{n=1}^{N} n = \frac{N(N+1)}{2}$$

If a_n is a list of numbers

$a_0 = 4$, $a_1 = 3$, $a_{1000} = 1001 \dots$

$$\sum_{n=0}^{N} a_n = a_0 + a_1 + a_2 + \dots + a_N$$

To prove $P(n)$ $\forall N$ we have to proceed by induction

i) $P(0) = \frac{0(0+1)}{2} = 0$ sum of the first 0 numbers

then, $P(0)$ is true INDUCTIVE BASIS

ii) assume $P(N)$ is true

$$1 + 2 + 3 \dots + N = \sum_{N=1}^{N} n(N) = \frac{N(N+1)}{2}$$

iii) show that $P(n+1)$ is true

$$1 + 2 + 3 \dots N+1 = \sum_{N=1}^{N+1} n(N) = \frac{(N+1)(N+1+1)}{2} = \frac{(N+1)(N+2)}{2}$$

compute $\underbrace{1 + 2 + 3 + \dots + N}_{\frac{N(N+1)}{2}} + N+1 = \frac{N(N+1)}{2} + N+1 = \frac{N(N+1) + 2(N+1)}{2} = \frac{(N+1)(N+2)}{2}$

By induction the formula is true for $\forall N \in \mathbb{N}$

Definitions by induction

Factorial of a number $n \in \mathbb{N}$

$n! = n(n-1)(n-2)\dots 1$

$\hookrightarrow$ RIGOROUS DEFINITION: i) $0! = 1$ (by definition)

ii) $(n+1)! = (n+1) \cdot \underbrace{n \cdot (n-1) \dots 1}_{n!} =$

$= n!\,(n+1)$

1) Re-do the proof with Σ

2) Prove that:

$$1 + 4 + 9 + \dots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

is true by induction

Hyp. $\dfrac{N(N+1)(2N+1)}{6}$

$$N = N+1 \to \frac{(N+1)(N+2)(2N+2+1)}{6} = \frac{(N+1)(N+2)(2N+3)}{6}$$

IND.: $\underbrace{1 + 4 \dots + N^2}_{\frac{N(N+1)(2N+1)}{6}} + (N+1)^2 = \dfrac{N(N+1)(2N+1)}{6} + (N+1)^2 =$

$$= \frac{N(N+1)(2N+1) + 6(N+1)^2}{6} = \frac{(N+1)(N(2N+1) + 6(N+1))}{6}$$

$(N+2)(2N+3) = (N)(2N+1) + 6(N+1) =$

$= 2N^2 + 7N + 6 = 2N^2 + 7N + 6$