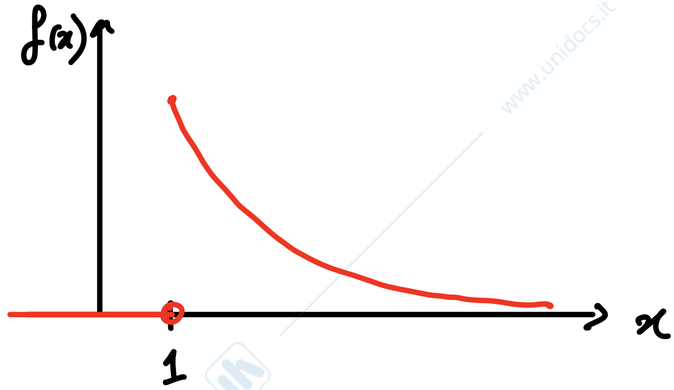
1) $f(x) = c \begin{cases} 0 & x < 1 \\ e^{-3x} & x \geq 1 \end{cases}$

Compute:

a) c such that $f(x)$ is a probability density function.



We impose that $c \int_1^{\infty} e^{-3x} \, dx = 1$

$$c \left. \frac{e^{-3x}}{-3} \right|_1^{\infty} = c \frac{e^{-3}}{3} = 1 \rightarrow c = 3e^3$$

b) the cumulative probability function

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \int_1^{x} 3e^3 e^{-3u} \, du & x \geq 1 \end{cases}$$

$$3e^3 \int_1^x e^{-3u}\,du = 3e^3 \left.\frac{e^{-3u}}{-3}\right|_1^x = -e^3\left(e^{-3x} - e^{-3}\right)$$

$$= 1 - e^{-3(x-1)} .$$

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 1 - e^{-3(x-1)} & x \geq 1 \end{cases}$$

c) the quantile x_p at level $p \in (0,1)$

$$1 - e^{-3(x_p - 1)} = p$$

$$e^{-3(x_p-1)} = 1-p$$

$$-3(x_p - 1) = \ln(1-p)$$

$$x_p = 1 - \frac{1}{3}\ln(1-p).$$

d) $$E[X \mid X<2] = \frac{\int_1^2 x\, 3e^3\, e^{-3x}\, dx}{Prob(X<2)} =$$

$$= \frac{3e^3 \int_1^2 x e^{-3x}\, dx}{F_X(2)} = \frac{3e^3 \int_1^2 x e^{-3x}\, dx}{1-e^{-3(2-1)}}$$

$$= \frac{3e^3}{1-e^{-3}} \left(\frac{e^{-3x}}{-3} x \Big|_1^2 - \int_1^2 \frac{e^{-3x}}{-3} 1\, dx \right) =$$

$$= \frac{3e^3}{1-e^{-3}} \left(\frac{2e^{-6} - e^{-3}}{-3} + \frac{1}{3} \int_1^2 e^{-3x}\, dx \right) =$$

$$= \frac{e^3}{1-e^{-3}} \left(e^{-3} - 2e^{-6} + \int_1^2 e^{-3x}\, dx \right) =$$

$$= \frac{e^3}{1-e^{-3}} \left(e^{-3} - 2e^{-6} + \frac{e^{-3x}}{-3} \Big|_1^2 \right) =$$

$$= \frac{e^3}{1-e^{-3}} \left(e^{-3} - 2e^{-6} + \frac{e^{-6} - e^{-3}}{-3} \right)$$

e) $\text{Prob}(x^2 > 2)$

$x^2 > 2 \qquad (x > \sqrt{2}) \cup (x < -\sqrt{2})$

$\text{Prob}(x^2 > 2) = \text{Prob}\big((x > \sqrt{2}) \cup (x < -\sqrt{2})\big) =$

$= \text{Prob}(x > \sqrt{2}) + \cancelto{0}{\text{Prob}(x < -\sqrt{2})} =$

$\downarrow$ since $(x > \sqrt{2})$ and $(x < -\sqrt{2})$ are disjoint.

$= 1 - F_x(\sqrt{2}) = 1 - \left(1 - e^{-3(\sqrt{2}-1)}\right) =$

$= e^{-3(\sqrt{2}-1)}$.

2) Let X be a random variable with moment generating function

$$m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^2 \quad \text{for} \quad t < \lambda$$

Compute the mean and the variance of X.

$$E[X] = \frac{d}{dt} m_X(t)\Big|_{t=0} = \frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)^2\Big|_{t=0} =$$

$$= 2\,\frac{\lambda}{\lambda-t}\left(-\frac{\lambda}{(\lambda-t)^2}\right)(-1)\Big|_{t=0} = \frac{2\lambda^2}{(\lambda-t)^3}\Big|_{t=0} =$$

$$= \frac{2}{\lambda}.$$

$$VAR[X] = E[X^2] - E[X]^2$$

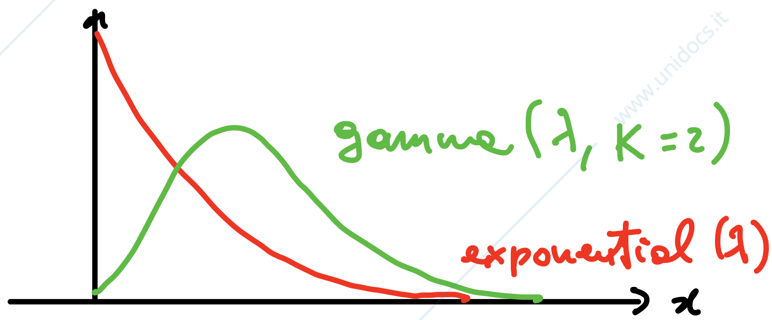
$$E[X^2] = \frac{d^2}{dt^2}\left(\frac{\lambda}{\lambda-t}\right)^2\Big|_{t=0} = \frac{d}{dt}\left(\frac{2\lambda^2}{(\lambda-t)^3}\right)\Big|_{t=0} =$$

$$= 2\lambda^2 \frac{d}{dt}(\lambda - t)^{-3}\Big|_{t=0} = 2\lambda^2(-3)(\lambda-t)^{-3-1}(-1)\Big|_{t=0} =$$

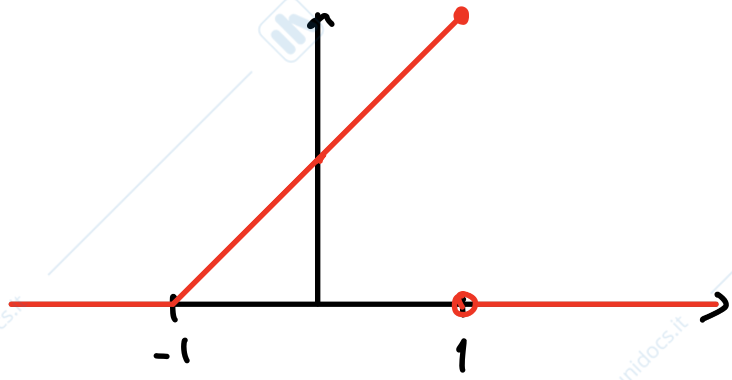
$$= 6\,\frac{\lambda^2}{(\lambda-t)^4}\Big|_{t=0} = \frac{6}{\lambda^2}$$

$$VAR[X] = \frac{6}{\lambda^2} - \left(\frac{2}{\lambda}\right)^2 = \frac{2}{\lambda^2}.$$

__Remark__: $m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^2$ is the m.g.f. of the gamma distribution with parameter λ and $K=2$.



3) $f(x) = c \begin{cases} 0 & x < -1 \\ x+1 & -1 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$



Compute:

a) c such that $f(x)$ is a probability density function

We impose that $c \int_{-1}^{1} (x+1)\, dx = 1$

that is $c\left(\frac{x^2}{2} + x\right)\Big|_{-1}^{1} = c\left(\frac{1}{2} + 1 - \frac{1}{2} + 1\right) =$

$= 2c = 1 \longrightarrow c = \frac{1}{2}$.

b) the cumulative probability

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}\int_{-1}^{x}(1+u)\,du & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$$\frac{1}{2}\int_{-1}^{x}(1+u)\,du = \frac{1}{2}\left(u + \frac{u^2}{2}\right)\Big|_{-1}^{x} = \frac{1}{2}\left(x + \frac{x^2}{2} + 1 - \frac{1}{2}\right) =$$

$$= \frac{1}{2}\left(\frac{x^2}{2} + x + \frac{1}{2}\right) = \frac{x^2+2x+1}{4}.$$

Hence $$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{x^2+2x+1}{4} & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

c) the quantile at level $p \in (0, 1)$

$$F_X(x_p) = p \qquad \frac{x_p^2 + 2x_p + 1}{4} = p$$

$$x_p^2 + 2x_p + 1 = 4p \qquad (x_p+1)^2 = 4p$$

$$(x_p+1) = \pm 2\sqrt{p} \rightarrow x_p = -1 \pm 2\sqrt{p}.$$

FOR $p \to 1$ $\quad x_p \to -1-2=-3$ is outside of the interval $[-1, 1]$

then the negative solution is not acceptable.
We take only the positive determination

$$x_p = -1 + 2\sqrt{p}.$$

d) $E[X \mid X < x_p]$ for $p \in (0,1)$

$$E[X \mid X < x_p] = \frac{\frac{1}{2}\int_{-1}^{x_p} x(1+x)\,dx}{p} =$$

$$= \frac{1}{2p}\left(\frac{x^2}{2} + \frac{x^3}{3}\right)\Bigg|_{-1}^{x_p} = \frac{1}{2p}\left(\frac{x_p^2}{2} + \frac{x_p^3}{3} - \frac{1}{2} + \frac{1}{3}\right)$$

e) $\text{Prob}\left(e^{X} < \frac{1}{2}\right)$

$$\text{Prob}\left(e^{X} < \frac{1}{2}\right) = \text{Prob}\left(X < \ln\frac{1}{2}\right) =$$

$$= \text{Prob}(X < -\ln 2) = F_X(-\ln 2) =$$

$-\ln 2 > -1$

$$= \frac{(-\ln 2)^2 - 2\ln 2 + 1}{4}.$$