MATHEMATICS FOR MANAGEMENT

DETAILED PROGRAM

1. **Dynamical systems, some general definitions.**
2. **Continuous-time dynamical systems.** Linear and nonlinear models. Local bifurcations.
3. **Two-dimensional Dynamical systems in continuous time.** Linear and nonlinear systems. Periodic solutions and limit cycles. Bifurcations.
4. **n-dimensional dynamical systems in continuous time** Linear and nonlinear systems.
5. **Discrete-time dynamical systems.** Definitions. 1D map, linear and nonlinear. Local bifurcations.
6. **Two-dimensional discrete dynamical systems.** Linear systems. Nonlinear maps. Local and Global analysis.
7. **Piecewise-Linear dynamical systems.**
8. **Examples and applications of dynamical systems.** Dynamic of populations, epidemic models, oligopoly theory, financial markets.

INSTRUCTIONS FOR THE EXAM

Students who attend the lessons may obtain the final evaluation by completing the following two tasks:

1) **A GROUP PRESENTATION.** Attending students will be subdivided into groups (3/4 members) that will work on a topic selected by Prof. Tramontana. In the period between the end of the lessons (mid-November) and the Winter break, the groups will discuss a 15min presentation on that topic. For the talk the groups must also use the Software E&FChaos. Members of the groups can only obtain two evaluations: **POSITIVE or NEGATIVE (It is possible that the talks will be given through an online meeting among the components of the group and the teacher)**
2) **COMPLETION EXAM.** Members of the groups who obtained a **POSITIVE** evaluation at the group presentation, **only in** the exam session of Nov/Dec 2020 may complete the exam by answering to 2 written open questions and by solving one exercise.

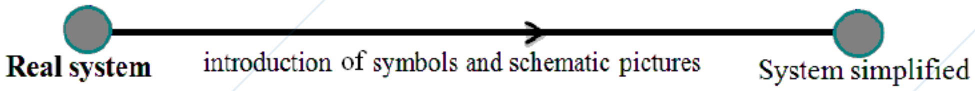
STUDENTS WHO DON'T ATTEND THE LESSONS, RECEIVED A NEGATIVE EVALUATION AT THE GROUP PRESENTATION OR TAKE THE EXAM AFTER DEC.2020 must face the **COMPLETE** exam.

COMPLETE EXAM: Students must answer to 3 written open questions and they must solve an exercise. At least one of the questions concerns to knowledge of the Software E&FChaos and/or an application of the theory.

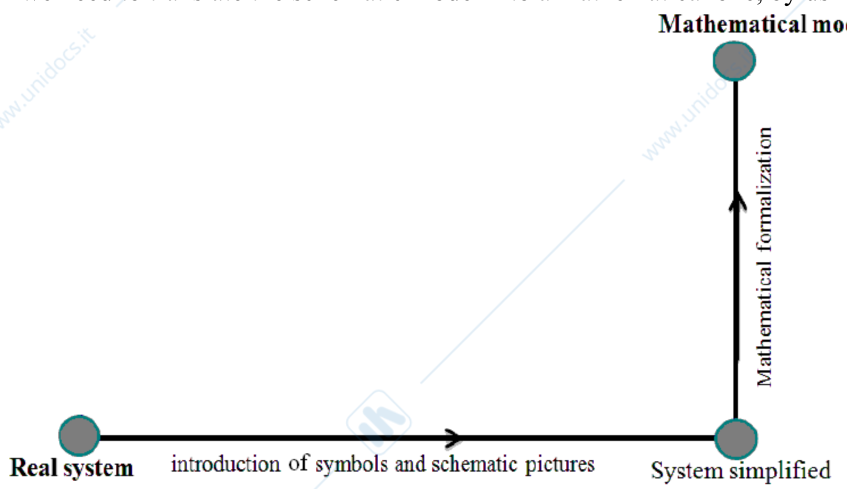
INTRODUCTION – GENERAL DEFINITION

Mathematical modelling

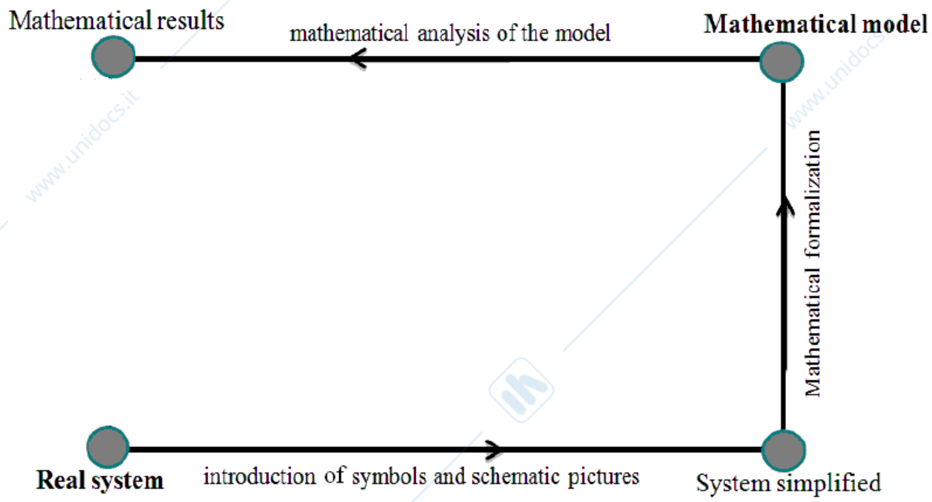
Real systems (physical, biological, social, economic, managerial, etc...) can be represented mathematically starting from a rigorous and critical analysis of the main features and
basic principles of the system we want to describe.
We need to find measurable quantities (i.e. that can be expressed in numbers) characterizing its state.
We must identify its behavior.
We can obtain a schematic and simplified representation of the
system.



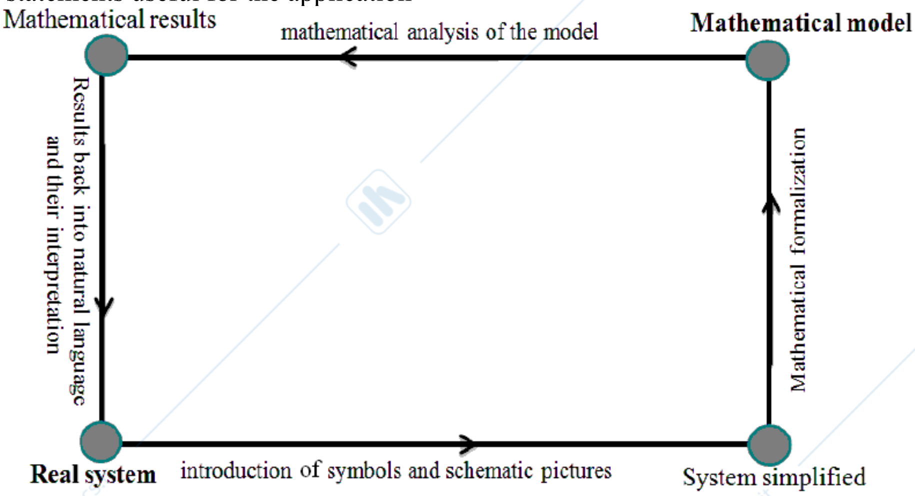
Then we need to translate the schematic model into a mathematical one, by using symbols and operators.



At this stage we can study mathematically the model by using mathematical tools, proofs and/or numerical methods. The output is mathematical/numerical (a proof, an expression, a simulation...)



Finally we must translate the mathematical results into the natural language of the system described to obtain laws or statements useful for the application



Dynamical systems

Dynamical systems are systems that change over time
Dynamical systems are described by using dynamic variables
Dynamic variables are functions of time that define their state as time goes on.
We can say that dynamical systems describe the time evolution of such variables.

Examples
Physics: the motion of particles, planets, fluids...
Ecology: dynamics of interacting populations
Biology: blood circulation, virus diffusion...
Economics & Finance: dynamics of prices, quantities...
Management: supply chain models, “learning and pricing”...

General definitions

We have seen that a mathematical description of a real system that evolves as time goes on is what we call dynamical system. The state of a system is described by a set of n measurable quantities, called state variables
$x_1, \dots, x_n$ where x_i € R, $i = 1, \dots, n$
Examples: the prices of n commodities; macro-indicators (salaries, level of occupation, ination rate, GDP...), the number (or densities) of individuals of species, the positions of particles...
Sometimes it is represented by a vector representing a point in an n-dimensional spece:
$x = (x_1, \dots, x_n)$ € R^n

State space and Initial state

In the applications sometimes only a subset of R^n is suitable to represent the real system (for instance, non negative prices...).

Definition: The **state space** (or phase space) $M \subseteq R^n$ is the set of admissible values of the state variables.

The theory of dynamical systems starts with the knowledge of the state of the system at a certain time t_0
Definition: The **initial condition** (or initial state) is denoted by $x(t_0) = (x_1(t_0), x_2(t_0), \dots, x_n(t_0))$

Continuous and discrete time

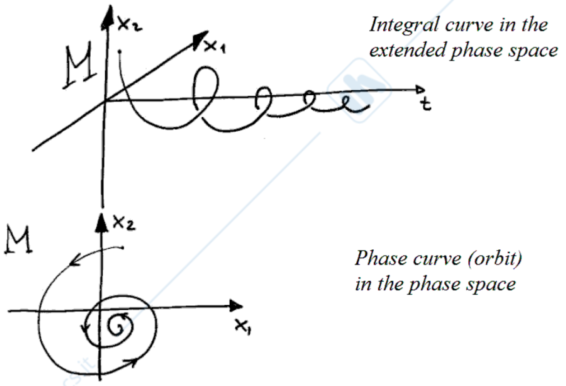
Starting from t_0 we are interested in computing the state of the system at time $t > t0$, i.e. the purpose of dynamical systems is to know the operator:
$x(t) = G(t, x(t_0))$ with $x(t)$ € $M \subseteq R^n$ and $G(.) : M \rightarrow M$.

Generally we are interested in the evolution of the system as $t \rightarrow +\infty$, i.e its asymptotic evolution.
$x(t)$ is the parametric equation of a trajectory, as t varies.

When variables may change continuously with time, then t is a real number and:

$$x_i : R \rightarrow R$$

we are in continuous time and the trajectory is a curve in the space R^n (phase curve, with arrows representing the direction of increasing time) or an integral curve in the n+1 dimensional space (R^n; t):



Discrete time

When variables may only change at discrete time steps (event-driven time), then t is a natural number and:

$$x_i : N \rightarrow R$$

we are in discrete time and the trajectory is a sequence of points. The time evolution of the system jumps from one point to the next one in the sequence.

Equlibrium and trapping sets

An equlibrium is a trapping point (i.e. any trajectory through it remains in it for each successive time:

$$x(t_0) = x^* \rightarrow x(t) = x^* \text{ for } t \geq t_0$$

so any trajectory starting inside a trapping set cannot escape from it.
Definition (trapping set)
A set $A \subseteq M$ is trapping if $x(t_0)$ € A $\rightarrow x(t) = G(t, x(t_0))$ € A for each $t > t_0$

Invariant sets

Definition (invariant set)
A closed set $A \subseteq M$ is invariant if $G(t, A) = A$ i.e. each subset $A' \subset A$ is not trapping.
It means that any trajectory starting inside an invariant set remains there, and all the points of the invariant set can be reached by a trajectory starting inside it.
An equilibrium point is the simplest invariant set. We will see several more...
what happens to a trajectory starting in a neighborhood of (but outside) an invariant set?

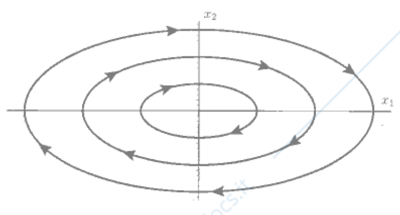
1. The trajectory may enter the invariant set (and never escape from it);
2. It may remain outside (close to it or going elsewhere).

Stability

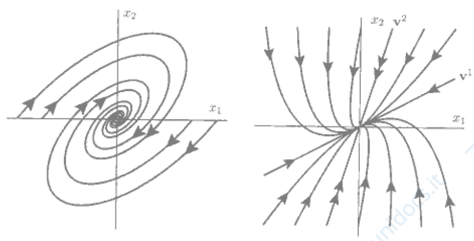
Definition (Asymptotic stability)
An invariant set A is asymptotically stable (often called attractor) if:

i. for each neighborhood U of A there exists another neighborhood V of A, with $V \subseteq U$, such that any trajectory starting from V remains inside U; (Lyapunov stability)
ii. $\lim_{t \rightarrow +\infty} G(t, x)$ € A for each initial condition x € V.

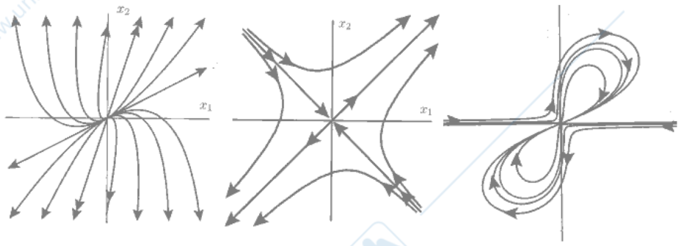
So initial conditions su-ciently close to A not only remain around it (Lyapunov stability), but tend to it in the long run.



Stable



Asymptotically stable

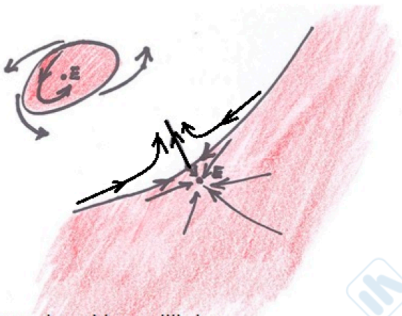


Unstable

Local stability

We are talking about local stability, i.e. considering an arbitrarily small neighborhood of an invariant set. But what happen to initial conditions far from an invariant set?
We need a new definition
Definition (Basin of attraction)
The basin of attraction of an attractor A is the set of all points x € M such that $\lim_{t\to+\infty}$ G (t, x) € A.
If B(A) = M then A is called global attractor.

Basin extension and vulnerability



More vulnerable equilibrium
With larger basin (but closer basin boundary)

Math & Simulations

Unfortunately the theory of dynamical systems usually does not permit to obtain analytical results about the global properties of a system;
There exist analytical results about the local stability of invariant sets;
By using numerical simulations we can gather informations about the global properties of a system.

EXAMPLES 1D MODELS IN CONTINUOUS TIME

Dynamics of a natural population

Let us denote by x(t) the number of individuals in a population (of insects, bacteria, fishes, human beings...);
The natality rate is denoted by n > 0;
The mortality rate is denoted by m > 0.
Malthusian model The growth of the population is then regulated by the following dynamic equation, linear and homogeneous:

x_ = nx - mx = (n - m)x

The Malthusian growth model Results

- The extinction (x = 0) is the only equilibrium point;
- The equilibrium is globally stable (i.e. the population goes to extinction) if the mortality rate exceeds the birth rate (m > n);
- If the birth rate exceeds the mortality rate (n > m) then the population growth exponentially

Introduction of immigration/emigration

Populations do not live in isolation but they interact with other populations.
Let us introduce a constant immigration (resp. emigration) term b > 0 (resp. b < 0)
The growth of the population is then regulated by the following affine dynamic equation:

x_ = ax +b where a = n - m.

Results

- The unique equilibrium point is now: $x^* = - b/a$
- If $a < 0$ and $b > 0$ (i.e. endogenously decreasing population with constant immigration) the equilibrium is positive and globally stable;
- If $a > 0$ and $b < 0$ (i.e. endogenously increasing population with constant emigration) the equilibrium is positive and unstable.

A Walrasian model

If we consider the price formation in a partial market of a single commodity, we can assume that the price adjusts (increasing/decreasing) according to the so-called law of demand:

$$p\_ = k [D (p) - S (p)]$$

where D (.) and S (.) are demand and supply functions, respectively, and $k > 0$ measures the speed of adjustment of the price.

Demand and Supply functions

Demand and supply functions are assumed to be respectively decreasing and increasing with respect to the price. In particular we assume linear functions:

$$D(p) = a - bp; \; S(p) = a_1+b_1p$$

where all the parameters are strictly positive.
Price dynamics The dynamic equation is linear and affine:

$$p\_ = f (p) = -k(b+b_1)p + k(a-a_1)$$

Results

- The unique equilibrium point is given by: $p^* = \frac{a - a_1}{b + b_1}$
- Given the sign of the parameters, provided that $a > a_1$, the price always converge towards its positive equilibrium value, in fact the slope of f (.) is equal to $-k(b+b_1) < 0$.

Logistic growth

The Malthisian growth model is not realistic;
When the population becomes too high, scarcity of food or space causes mortality, that can be assumed proportional to the population density;
By introducing this extra mortality term (sx) we obtain:

$$x\_ = f (x) = nx - (m+sx)x = ax - sx^2$$

that is nonlinear.

- The model admits two equilibria:
  $$x_0^* = 0 \quad \text{and} \quad x_1^* = a/s$$
  where x_0^* is the extinction equilibrium, while x_1^* is called carrying capacity.
- To study the local stability of the equilibria is necessary to compute the derivative:
  $$f'(x) = a - 2sx$$
  from which we easily obtain that $f'(x_0^*) = a$, so if $a < 0$ (that is mortality exceeds new births) we have extinction, otherwise the extinction equilibrium is unstable.

Carrying capacity

- Concerning the carrying capacity equilibrium we have that:
  $$f'(x_1^*) = a$$
  so it is exactly the opposite of what happens with the extinction equilibrium.
  If $a > 0$ it is locally stable, otherwise it is unstable.
- If a varies from a negative to a positive value we can see a change of stability of the two, always existing, equilibria. This is called a Transcritical Bifurcation.
- Given the nonlinearity of the model local stability does not imply global stability (a geometric representation can be useful).

Let us focus on the density of a fish population characterized by a logistic growth equation;
We also assume that in each time period a constant quota h is harvested;
We get the following nonlinear dynamic model:

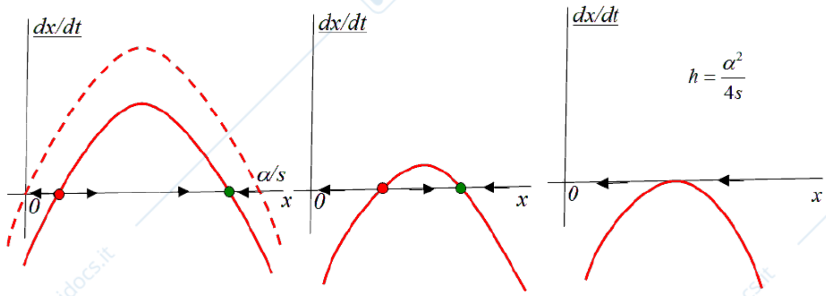
x_ = f (x) = x(a -sx)-h

- The model admits two equilibria:

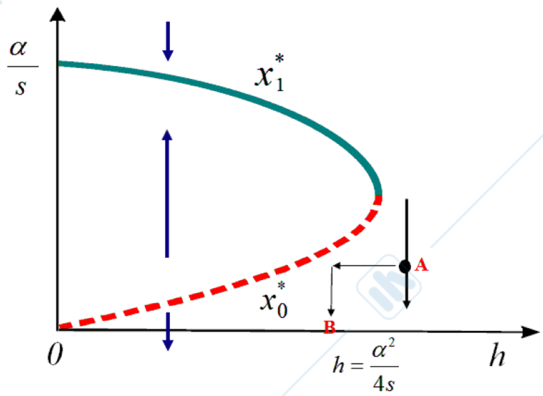
$$x_0^* = \frac{\alpha - \sqrt{\alpha^2 - 4hs}}{2s} \quad \text{and} \quad x_1^* = \frac{\alpha + \sqrt{\alpha^2 - 4hs}}{2s}$$

that exists privided that $h < \frac{\alpha^2}{4s}$.

A geometric representation may help:



Bifurcation diagram



A typical *FOLD BIFURCATION*

The SI epidemic model

Let us consider a fixed population of N individuals;
The N individuals can be subidivded into Suscepitbles (S) and Infected (I), and at each time period

$$I(t)+S(t) = N$$

The number of Infected evolves according to:

I_ = aIS

where a ≥ 0 measures the strenght of the epidemy and the product IS is a proxy for the amount of meetings between Infected and Susceptibles.

The model and its equilibria

By replicing S with N - I we get:

I_ = f (I) = aI (N - I)

that is a nonlinear one-dimensional dynamical system.

- The model admits two equilibria:

$$I^*_0 = 0 \qquad \text{and} \qquad I^*_1 = N$$

that can be considered, without epidemy and all infected equilibrium, respectively.

Local Stability

The derivative of the differential equation is:

$$f'(I) = a(N - 2I)$$

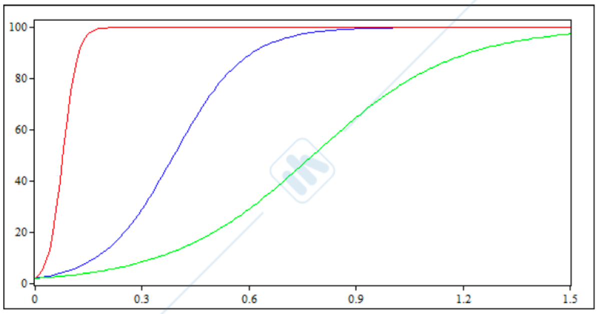
that calculated at the two equilibria becomes:

$$f'(I_0^*) = aN > 0 \qquad \text{and} \qquad f(I_1^*) = -aN < 0$$

so, in any case the whole population will be infected.

The role of 'a'

Here are some timeplots with $N = 100$ and different values for α



$\alpha = 0.5$ $\alpha = 0.1$ $\alpha = 0.05$

CONTINUOUS-TIME DYNAMICAL SYSTEMS - 1D

ODE

In continuous-time the evolution equations are expressed by a set of ordinary differential equations (ODE) involving the time derivative of each state variable:

$$\frac{dx_i(t)}{dt} = \dot{x} = f_i(x_1(t), \ldots, x_n(t); \alpha), \qquad i = 1, \ldots, n$$
$$x_i(t_0) = \bar{x}_i$$

For example:

_ $x_1 = x_2$
_ $x_2 = -bx_1 - ax_2$

with $x_1(0) = x_0$ and $x_2(0) = v_0$

- A solution of a dynamical system is a set of functions:

$$x_i(t) = g(\bar{\mathrm{x}}, \alpha, t) \text{ for } i = 1, .., n$$

example $(n = 1)$:

$$x(t) = x_0 e^{\alpha(t - t_0)}$$

- In general we are not able to get explicit expressions of the solutions (besides some particular cases);
- Nevertheless, we are able to give a qualitative description of invariant sets, their stability properties and the asymptotic properties of the solution (qualitative or topological theory of dynamical systems).

Theorem (existence and uniqueness)

If the functions f_i have continuous partial derivatives $\frac{\partial f_i}{\partial x_k}$ in M and x(t0) ∈ M, then there exists a unique solution x_i (|t), i = 1, …, n, of the system such that $x(t_0)$ = x_, and each x_i (t) is a continuous function.

The linear case

In this Section we study the following dynamic equation:

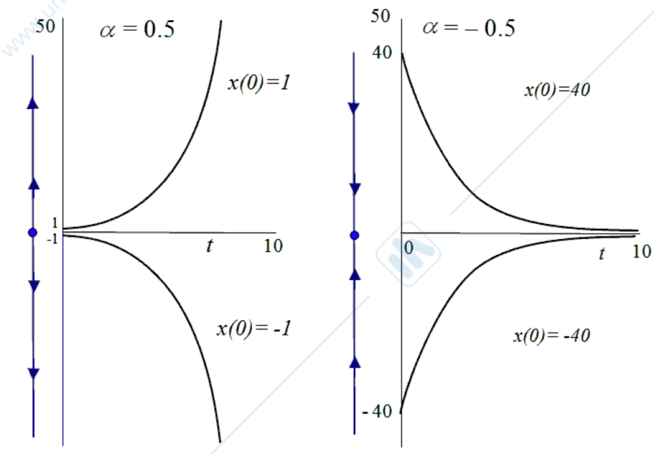
_ x = f (a, x) = ax
$x(t_0) = x_0$

- The rate of growth of the dynamic variable is proportional to itself (a is the constant of proportionality);
- If a > 0 and $x_0 > 0$ then the dynamic variable x increases, leading to an increase in the time derivative, so the increasing becomes faster and faster (exponential growth);

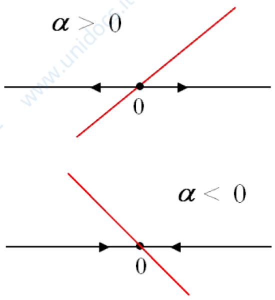
In this Section we study the following dynamic equation:

x = f (a, x) = ax
$x(t_0) = x_0$

- If a > 0 and $x_0 < 0$ then the dynamic variable x decreases faster and faster (unstable behavior);
- If a < 0 than if $x_0 > 0$ (resp. $x_0 < 0$) then the dynamic variable x decreases (resp. increasing) tending towards a value x = 0 (globally stable equilibrium);



Graphical analysis



Algebraic analysis

1. The first step consists in finding the equilibria (or fixed points or steady states). In order to do that we must solve the equation:

Equilibrium condition $f(a, x) = 0$

from which we obtain $ax = 0 \leftrightarrow x^* = 0$

2. The local stability of the equilibrium is obtained by calculating the derivative of the dynamic function f (a, x) with respect to the dynamic variable, calculated in the equilibrium point: $f'(x^*)$

   And then we have: Local stability condition
   if $f'(x^*) > 0 \rightarrow x^*$ is locally unstable
   if $f'(x^*) < 0 \rightarrow x^*$ is locally stable
   if $f'(x^*) = 0 \rightarrow x^*$ is neutrally stable

Extension to the affine case

The linear non homogeneous (or a-ne) case is defined as follows:

$$x = f(x) = ax + b$$
$$x(t_0) = x_0$$

We have now the instruments to study it by using algebra.

Algebraic study

Let's start by searching for equilibria:
Equilibrium condition: $f(x) = 0$ from which we obtain. $ax + b = 0 \leftrightarrow x^* = -b/a$
So the equilibrium is still unique but different from zero.

Stability analysis

For the local stability we need to calculate the derivative of the dynamic function, calculated in correspondence of the equilibrium: $f'(x^*) = a$
And then we have:

Local stability condition
if $a > 0 \rightarrow x^*$ is locally unstable
if $a < 0 \rightarrow x^*$ is locally stable
if $a = 0 \rightarrow x^*$ is neutrally stable

that is exactly the same conditions of the linear case.

Summarizing

- The linear and a-ne case admit only one equilibrium;
- The positivity (negativity) of the slope of the linear function f (x) determines the stability (instability) of the equilibrium;
- For linear and a-ne systems, local stability implies global stability.

NONLINEAR CONTINUOUS 1D & LOCAL BIFURCATIONS

1D nonlinear models

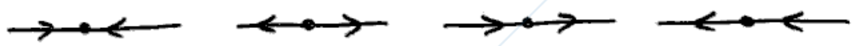
Let us consider a generalized one-dimensional dynamic equation in continuous time:

$$\dot{x} = f(x)$$

where f (x) can be linear or nonlinear. It is still true that from the solutions of the equation
Equilibrium condition: $f(x) = 0$
we can fine the equilibria, which can ba more than one, and even zero.

Behavior around an equilibrium

- If the system starts at an equilibrium it stays there forever;
- Oscillations are not possible (consequence of the Theorem of uniqueness);
- Four different phase portraits may characterize the behavior of a 1D system around an equilibrium:
- If the system starts at an equilibrium it stays there forever;
- Oscillations are not possible (consequence of the Theorem of uniqueness);
- Four dierent phase portraits may characterize the behavior of a 1D system around an equilibrium:



Hyperbolic equilibria

An equilibrium x* is called hyperbolic iff $f'(x^*) \neq 0$
If the equilibrium is hyperbolic then the function f (x) can be approximated by the first order Taylor expansion:

$$f(x)=f(x^*)+f'(x^*)(x-x^*)+o(x-x^*) = f'(x^*)(x-x^*)+o(x-x^*)$$

So the system can be linearly approximated by neglecting the higher order terms. By using the change of variable $X = x - x^{x}$ (displacement between x and the equilibrium point), our dynamic equation becomes:

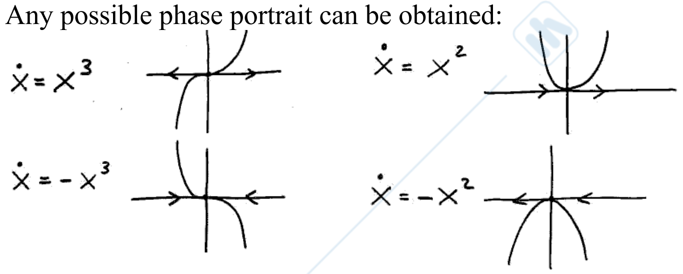
$$\dot{X} = aX \qquad \text{where } a = f'(x^*)$$

So we have the following result:
Local asymptitc stability in 1D continuous time
An equilibrium x* is locally asymptotically stable iff $f'(x^*) < 0$
while it is locally asymptotically unstable iff $f'(x^*) > 0$
This result leaves out two important cases:

1. What happens with initial conditions far from an equilibrium? (not analytical answer)
2. What happens around non hyperbolic equilibria?

Non hyperbolic equilibria

In order to say something about the local stability of non hyperbolic equilibria we should better investigate the function (looking at high order derivatives or the shape of the function).
Any possible phase portrait can be obtained:



Structural unstability

Non-hyperbolic equibria are also said to be structurally unstable, in the sense that:
Any arbitrarily small modification of the shape of the function f (x) generally leads to a modification in the stability properties as well as in the number of equilibrium points.

Such slight modifications (i.e. a small variation of a parameter) that lead to dynamic scenarios qualitatively different are called bifurcations.

Definitions

<u>Qualitatively equivalent systems</u>

Two one-dimensional dynamical systems $\dot{x} = f(x)$ and $\dot{x} = g(x)$ are qualitatively equivalent if they have the same number of equilibrium points that orderly have, along the phase line, the same stability properties.

This equivalence relation defines equivalence classes.

<u>Local Bifurcations</u>

If a slight changing causes a passage from one equivalence class to another one, a local bifurcation is said to occur.

One (or more) non hyperbolic equilibrium points are involved in these bifurcation situations.

Fold Bifurcation

When a Fold Bifurcation occurs:

- Two equilibrium points are created simoultaneously, as a parameter varies;
- At the bifurcation value they are coincident;
- After the bifurcation value, they are separated: one locally stable and one unstable

An example is given by the dynamical system:

$$\dot{x} = f(x) = \mu - x^2$$

as the parameter μ varies through the bifurcation value $\mu_0 = 0$.

- If we apply the equilbrium condition $\dot{x} = 0$ to the system we find two potential equilbria:

$$x_1^* = +\sqrt{\mu} \quad \text{and} \quad x_2^* = -\sqrt{\mu}$$

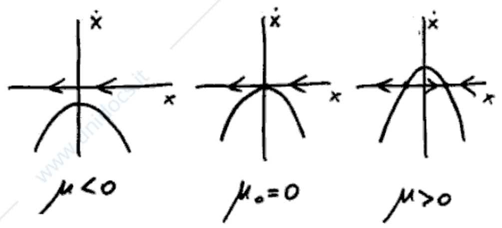
- The equilibrium points exist only if $\mu \geq 0$ and at the bifurcation value $\mu_0 = 0$ they are coincident;

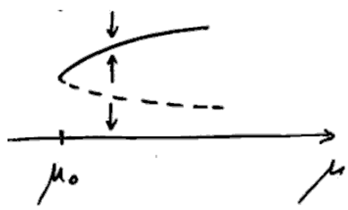
- At the bifurcation value the equilibrium points are non hyperbolic, in fact:

$$f'(x) = -2x \implies f'(x_1^*) = -2\sqrt{\mu} \text{ and } f'(x_2^*) = +2\sqrt{\mu}$$

  that vanish when $\mu = \mu_0 = 0$.
- For $\mu > 0$ we have that $f'(x_1^*) < 0$ while $f'(x_2^*) > 0$, so x_1^* is locally stable and x_2^* is unstable.

Graphically





Bifurcation Diagram

In a bifurcation diagram we have the bifurcation parameter on the horizontal axis, while in the vertical axis are reported the equilibrium values, respresented by a continuous line when stable and by a dashed line when unstable.

Transcritical Bifurcation

When a Transcritical (or stability exchange) Bifurcation occurs:

- Two equilibrium points, one stable and one unstable merge at the bifurcation value;
- After the bifurcation they still exist but with opposite stability properties (i.e. the one previously stable becomes unstable and viceversa);

An example is given by the dynamical system:

$$\dot{x} = f(x) = \mu x - x^2$$

as the parameter μ varies through the bifurcation value $\mu_0 = 0$.

If we apply the equilbrium condition _ x = 0 to the system we find two, always existing, equilbria:

$$x_1^* = 0 \quad \text{and} \quad x_2^* = \mu$$

At the bifurcation value the equilibrium points are coincident and non hyperbolic, in fact:

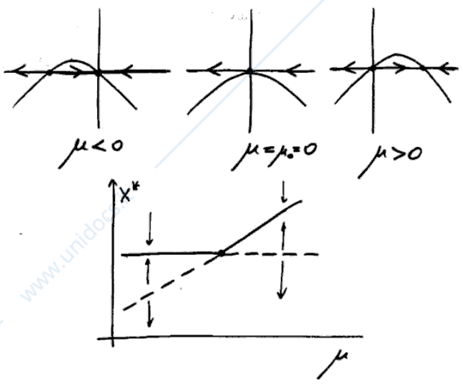
$$f'(x) = \mu - 2x \rightarrow f'(x_1^*) = \mu \quad \text{and} \quad f'(x_2^*) = -\mu$$

and they both vanish when $\mu = \mu_0 = 0$.

For $\mu > 0$ we have that $f'(x_1^*) > 0$ while $f'(x_2^*) < 0$, so x_1^* isunstable and $x_2^*$is locally stable;

At the opposite, for $\mu < 0$ we have that $f'(x_1^*) < 0$ while $f'(x_2^*) > 0$, so x_1^* is locally stable and $x_2^*$is unstable.

Graphically



Pitchfork Bifurcation

When a (Supercritical) Pitchfork Bifurcation occurs:

- There is a transition from a single, locally stable, equilibrium point to three equilibria (two further equilibria are created simoultaneously);
- At the bifurcation value the three equilibria are coincident;
- After the bifurcation the already existing equilibrium becomes unstable with the two new equilibria are both locally stable;
- The unstable equilibrium point is the boundary between the basins of attraction of the two coexisting stable equilibria.

An example is given by the dynamical system:

$$\_\ x = f(x) = \mu x - x^3$$

as the parameter μ varies through the bifurcation value $\mu_0 = 0$.

If we apply the equilbrium condition _ x = 0 to the system we find three potential equilbria:

$$x^*_0 = 0 \quad \text{and} \quad x^*_{1;2} = \pm \mu^{1/2}$$

with x^*_0 always existing while $x^*_{1;2}$ only exist if $\mu > 0$.

At the bifurcation value the equilibrium points are coincident and non hyperbolic, in fact:

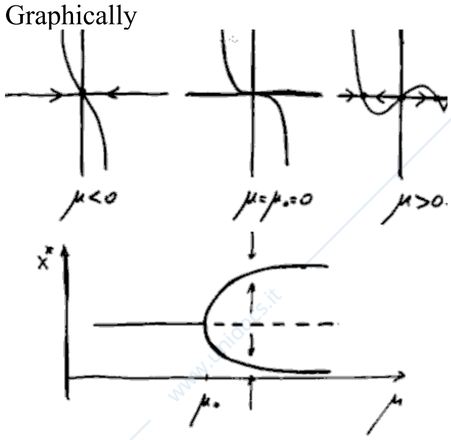
$f'(x) = \mu - 3x^2 \rightarrow f'(x^*_0) = \mu$ and $f'(x^*_{1;2}) = -2\mu$

and they all vanish when $\mu = \mu_0 = 0$.

For $\mu > 0$ we have that $f'(x^*_0) > 0$ while $f'(x^*_{1;2}) < 0$, so x^*_0 is unstable and $x^*_{1;2}$ are both locally stable;

At the opposite, for $\mu < 0$ we have that $f\,0(x^*_0) < 0$, so x^*_0 is locally stable.

Graphically



CONTINUOUS 2D LINEAR

Two-dimensional systems in continuous time:

- 2D systems
- Linear systems
- Real Eigenvalues (5 cases)
- Complex eigenvalues

2D systems

We consider dynamical models of systems whose state is described by two variables: x1(t) e x2(t), which are interdependent:

$$x_1 = f_1(x_1(t), x_2(t))$$
$$x_2 = f_2(x_1(t), x_2(t))$$

To get a qualitative global view of the phase portrait, it is possible to represent the two curves of equations:

$$f_1(x_1, x_2) = 0$$
$$f_2(x_1, x_2) = 0$$

usually called nullclines.

Nullclines

The two nullclines subdivide the phase plane into zones characterized by different signs of the time derivatives (x_1, x_2)

Linear systems

We start by considering a linear homogeneous system of two differential equations of first order with constant coefficients of the following (normal) form:

$$\begin{cases} \dot{x}_1 = a_{11}x_1(t) + a_{12}x_2(t) \\ \dot{x}_2 = a_{21}x_1(t) + a_{22}x_2(t) \end{cases}$$

We can write it in matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}; \quad \dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$$

Equilibrium

The system admits only one equilibrium: E (0,0)

In order to the study the (global) stability of the equilibrium we need to find the roots of the following equation: characteristic equation

$$\lambda^2 - Tr(\mathbf{A})\lambda + Det(\mathbf{A}) = 0$$

where

$$Tr(\mathbf{A}) = a_{11} + a_{22}$$
$$Det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$$

and the roots are called eigenvalues.

Case 1: real, distinct and negative eigenvalues ($\lambda_2 < \lambda_1 < 0$)

- Eigenvalues are real provided that:
  $Tr(A)^2 - 4Det(A) > 0$
- In order to have negative eigenvalues it must bu true that:
  $Tr(A) < 0$ and $Det(A) > 0$
- The equilibrium is asymptotically stable and is called stable node (or sink).

Case 1: phase diagram

Case 2: real, distinct and positive eigenvalues ($\lambda_1 > \lambda_2 > 0$)

- Eigenvalues are real provided that:
  $Tr(A)^2 - 4Det(A) > 0$
- In order to have positive eigenvalues it must bu true that:
  $Tr(A) > 0$ and $Det(A) > 0$
- The equilibrium is unstable and is called unstable node (or source).

Case 2: phase diagram

Case 3: real, distinct eigenvalues with opposite sign ($\lambda_2 < 0 < \lambda_1$)

- Eigenvalues are real provided that:
  $Tr(A)^2 - 4Det(A) > 0$
- In order to have eigenvalues with opposite sign, it must bu true that:
  $Det(A) < 0$
- The equilibrium is unstable and is called saddle.
- There exist and invariant line called stable manifold.

Case 3: phase diagram

Case 4: real, coincident and negative eigenvalues ($\lambda_1 = \lambda_2 < 0$)

- Eigenvalues are real and coincident provided that:
  $Tr(A)^2 - 4Det(A) = 0$
- In order to have negative eigenvalues it must bu true that:
  $Tr(A) < 0$
- The equilibrium is called stable improper node (sometimes stable star node).

Case 4: phase diagram

Case 5: real, coincident and positive eigenvalues ($\lambda_1 = \lambda_2 > 0$)

- Eigenvalues are real and coincident provided that:
  $Tr\,(A)^2 - 4Det(A) = 0$
- In order to have positive eigenvalues it must bu true that:
  $Tr\,(A) > 0$
- The equilibrium is called unstable improper node (sometimes unstable star node).

Case 5: phase diagram

Complex eigenvalues

If $Tr\,(A)^2 - 4Det(A) < 0$ than we have two complex conjugate eigenvalues:

$$\lambda_1 = a + ib \quad \text{and} \quad \lambda_2 = a - ib$$

where

$$a = Re(\lambda) = \frac{Tr(\mathbf{A})}{2}$$
$$b = Im(\lambda) = \frac{\sqrt{4Det(\mathbf{A}) - Tr(\mathbf{A})^2}}{2}$$

called real and imaginary part, respectively.

Three kinds of oscillations

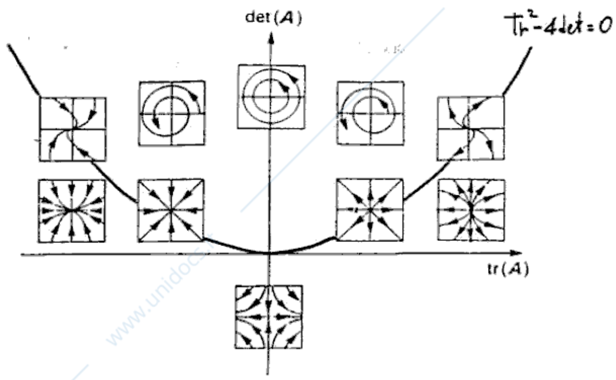
1. if $a < 0$ (i.e. $Tr\,(A) < 0$), orbits are characterized by oscillations of decreasing amplitude and convergence to the equilibrium (called stable focus);
2. if $a > 0$ (i.e. $Tr\,(A) > 0$), orbits are characterized by oscillations of increasing amplitude and divergence from the equilibrium (called unstable focus);
3. if $a = 0$ (i.e. $Tr\,(A) = 0$), orbits are characterized by oscillations of constant amplitude (the equilibrium is called centre).

The imaginary part regulates the time required to complete a whole oscillation $T = 2\pi / b$

Phase diagrams

Summarizing

The Trace and Determinant plane Asymptotical stability only occurs in the quadrant with $Tr\,(A) < 0$ and $Det(A) > 0$



EXAMPLES OF 2D MODELS IN CONTINUOUS TIME

A competitive model

Let us consider the following dynamical system: $\begin{cases} \dot{y}(t) = 2y(t) - 2z(t) \\ \dot{z}(t) = -3y(t) + z(t) \end{cases}$

It can be written in matrix form:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$$

from which we obtain that Tr=3 and Det=-4, so eigenvalues are both real and with opposite sign.

Solution

- The characteristic equation is: $\lambda^2 - 3\lambda - 4 = 0$
- The eigenvalues are: $\lambda_1 = -1$ and $\lambda_2 = 4$
- We can conclude that the equilibrium is a saddle.

A good/bad model

Let us consider now the following dynamical system: $\begin{cases} \dot{y}(t) = -5y(t) + 8z(t) \\ \dot{z}(t) = -y(t) + z(t) \end{cases}$

It can be written in matrix form:

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -5 & 8 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$$

from which we obtain that Tr=-4 and Det=3, so eigenvalues are both real and both negative.

Solution

- The characteristic equation is: $\lambda^2 + 4\lambda + 3 = 0$
- The eigenvalues are: $\lambda_1 = -3$ and $\lambda_2 = -1$
- We can conclude that the equilibrium is a stable node.

A prey/predator model

Let us consider the following dynamical system: $\begin{cases} \dot{y}(t) = y(t) - 5z(t) \\ \dot{z}(t) = y(t) + z(t) \end{cases}$

It can be written in matrix form: $\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$

from which we obtain that Tr=2 and Det=6, and in particular Tr^2-4Det=-20, so eigenvalues are complex conjugate with positive real part.

Solution

- The characteristic equation is: $\lambda^2 - 2\lambda + 6 = 0$
- The eigenvalues are: $\lambda_1 = \frac{2+2i\sqrt{5}}{2} = 1 + i\sqrt{5}$ and $\lambda_2 = \frac{2-2i\sqrt{5}}{2} = 1 - i\sqrt{5}$
- The real part is negative, so we can conclude that the equilibrium is an unstable focus.

Another prey/predator model

Let us consider the following dynamical system: $\begin{cases} \dot{y}(t) = 3y(t) - z(t) \\ \dot{z}(t) = y(t) + z(t) \end{cases}$

It can be written in matrix form: $\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$

from which we obtain that Tr=4 and Det=4, and in particular Tr^2-4Det=0, so eigenvalues are coincident and positive.

Solution

- The characteristic equation is: $\lambda^2 - 4\lambda + 4 = 0$
- The eigenvalues are: $\lambda_1 = \lambda_2 = 1$
- We can conclude that the equilibrium is an unstable improper node.

The Lotka-Volterra model

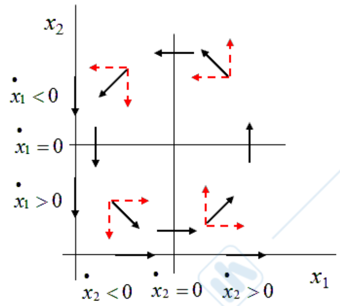
Let us consider the following dynamical system: $\begin{cases} \dot{x}_1 = x_1(\alpha - bx_2) \\ \dot{x}_2 = x_2(cx_1 - d) \end{cases}$

where $x_1 = x_1$ (t) represents the density in a given region of a species, the prey, while $x_2 = x_2$ (t) is the density of another species, predators.

- a > 0 regulates the growth of the prey in absence of the predator.
- d > 0 is the mortality rate of predator in absence of the prey.
- b, c > 0 measure the effects of each species on the other.

Nullclines

- The nullcline $x_1 = 0$ is given by: $x_1 = 0$ and $x_2 = a/b$
- The nullcline $x_2 = 0$ is given by: $x_2 = 0$ and $x_1 = d/c$



Equilibria

Two equilibria exist: $O(0,0)$ and $E\left(\frac{d}{c};\frac{\alpha}{b}\right)$

What about their local stability?

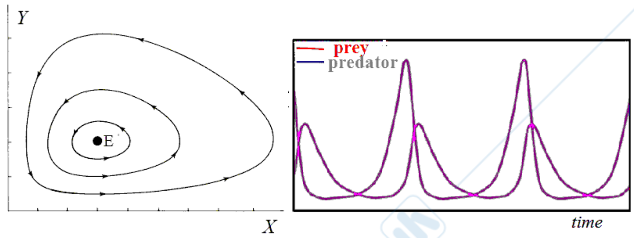
We need to calculate the Jacobian matrix: $J : \begin{bmatrix} \alpha - bx_2 & -bx_1 \\ cx_2 & cx_1 - d \end{bmatrix}$

that in the two equilibria become, respectively:

$$J(O) = \begin{bmatrix} \alpha & 0 \\ 0 & -d \end{bmatrix} \quad \text{and} \quad J(E) = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{c\alpha}{b} & 0 \end{bmatrix}$$

Eigenvalues

- The eigenvalues related to O are clearly (we deal with a tringular matrix so the eigenvalues are the elements of the diagonal): $\lambda_1 = a$ and $\lambda_2 = -d$
  and considering that a, d > 0 we can conclude that O is a saddle.
- Concerning the eigenvalues of E we have that: Tr $(J_E) = 0$ and Det$(J_E) = ad$
  from which we obtain that Tr $(J_E)^2 - 4\text{Det}(J_E) = -4ad < 0$, so eigenvalues are pure imaginary.
- Moreover E is non-hyperbolic, so we have a centre.



The general Lotka-Volterra model

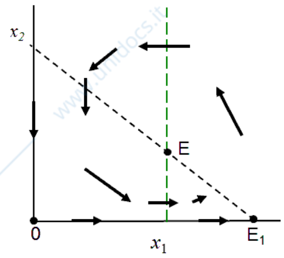
Let us consider the following modified version: $\begin{cases} \dot{x}_1 = x_1(\alpha - sx_1 - bx_2) \\ \dot{x}_2 = x_2(cx_1 - d) \end{cases}$

s > 0 regulates the logistic growth of the prey in absence of the predator.

Nullclines

The nullcline $x_1 = 0$ is now given by: $x_1 = 0$ and $x_2 = \frac{\alpha - sx_1}{b}$

The nullcline $x_2 = 0$ is again given by: $x_2 = 0$ and $x_1 = \frac{d}{c}$



Equilibria

Now we have three equilibria: $O(0,0)$ $A\left(\frac{\alpha}{s},0\right)$ $E\left(\frac{d}{c};\frac{\alpha c - sd}{bc}\right)$

and coexistence is only possible if ac > sd.

The Jacobian matrix: $J : \begin{bmatrix} \alpha - sx_1 - bx_2 & -bx_1 \\ cx_2 & cx_1 - d \end{bmatrix}$

that in the three equilibria become, respectively:

$$J(O) = \begin{bmatrix} \alpha & 0 \\ 0 & -d \end{bmatrix} \quad J(E) = \begin{bmatrix} -\frac{sd}{c} & -\frac{bd}{c} \\ \frac{\alpha c - sd}{b} & 0 \end{bmatrix}$$

$$J(A) = \begin{bmatrix} -\alpha & -b\frac{\alpha}{s} \\ 0 & c\frac{\alpha}{s} - d \end{bmatrix}$$

Eigenvalues

- The eigenvalues related to O are again: $\lambda_1 = a$ and $\lambda_2 = -d$
  so O is still a saddle.
- The eigenvalues related to A are: $\lambda_1 = -\alpha$ and $\lambda_2 = c\frac{\alpha}{s} - d$ so A is a stable node if ac < sd, otherwise it is a saddle, too.
- Concerning the eigenvalues of E we have that: $Tr(J_E) = -\frac{sd}{c} \le 0$ and $Det(J_E) = \frac{d(\alpha c - sd)}{c} > 0$ whenever it is positive.

We also have that $Tr(J_E)^2 - 4Det(J_E) = \frac{d^2 s}{c^2}(s+4) - 4\alpha d < 0 \Longleftrightarrow \alpha > \frac{ds}{c^2}(s+4)$, then it is a stable focus.

The SIR model of Kermack & McKendrick
Moving from the SI model, Kermack & McKendrick (1927) add a new compartment, called Recovered (R) , that is people who have recovered and developed immunity to the infection.
The link goes from Infected (I) to Recovered (R), according to a proportion g called the Recovery rate:

$$\dot{R} = \gamma I$$

So the updated dynamical system becomes: $$\begin{cases} \dot{S} = -\beta S I \\ \dot{I} = \beta S I - \gamma I \\ \dot{R} = \gamma I \end{cases}$$

The 2D model
his is a 2D model (the first two differential equations) governing also the evolution of a third variable (Recovered people). So we are going to study the 2D nonlinear system: $$\begin{cases} \dot{S} = -\beta S I \\ \dot{I} = \beta S I - \gamma I \end{cases}$$

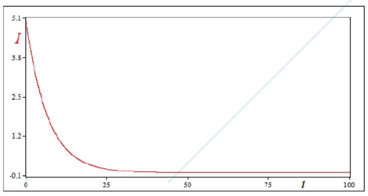
From the equilibrium conditions we find that at the equilibrium I*= 0, but now how we reach this equilibium becomes important.

The dynamics of Infected
The key is to distinguish when Infected immediately decrease until they become zero, or when they start increasing, reach a peak, and then decrease to zero.
The first case occurs when at the beginning I> 0, that is when: $R_0 \equiv \frac{\beta S(0)}{\gamma} > 1$

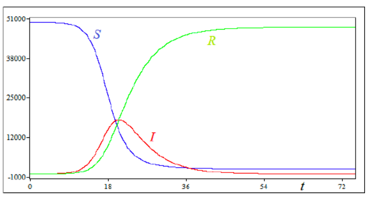
where R_0 is called the basic reproduction number, and it measures how many Susceptibles each Infected will infect on average (to not be confused with the initial number of recovered, R(0)).

The two cases

Let us consider a population of 50.000 people, starting with $S(0) = 49995$, $I(0) = 5$, $R(0) = 0$ and an illness with recovery rate $\gamma = 0.2$. In this case $R_0 = 1 \Longleftrightarrow \beta \simeq 0,000004$.



$\beta = 0.000001$



$\beta = 0.000014$

The Kaldor business cycle model
Let Y (t) be the national income (or output) and K(t) the capital stock at time t.
Kaldor propoes the following model: $$\begin{cases} \dot{Y} = \alpha\,[I(Y,K) - S(Y,K)] \\ \dot{K} = I(Y,K) - \delta K \end{cases}$$

where I denotes investments, S are savings, a > 0 is a speed of reaction and δ> 0 the depreciation rate of capital.
Savings are assumed to be linearly dependent on income:

$$S(Y) = \sigma Y$$

with $0 \leq \sigma \leq 1$.

Investments function is assumed nonlinear, with saturation effects for small and high values of income:

$$I(Y,K) = \sigma\mu + \gamma\left(\frac{\sigma\mu}{\delta} - K\right) + arctan(Y - \mu)$$

The dynamical system becomes: $$\begin{cases} \dot{Y} = \alpha\left[\sigma\mu + \gamma(\frac{\sigma\mu}{\delta} - K) + arctan(Y-\mu) - \sigma Y\right] \\ \dot{K} = \sigma\mu + \gamma(\frac{\sigma\mu}{\delta} - K) + arctan(Y-\mu) - \delta K \end{cases}$$

From the equilibrium conditions _Y = 0 and _K = 0 we get: $$K = \frac{\sigma}{\delta} Y$$ $$\sigma(1 + \frac{\gamma}{\delta})(Y - \mu) = arctan(Y - \mu)$$

The point $P = (\mu, \frac{\sigma\mu}{\delta})$ is always an equilibrium point.
Two further equilibria can be created.
The Jacobian matrix:

$$\mathbf{J}(Y,K) = \begin{bmatrix} \alpha\left(\frac{1}{1+(Y-\mu)^2} - \sigma\right) & -\alpha\gamma \\ \frac{1}{1+(Y-\mu)^2} & -\gamma-\delta \end{bmatrix}$$

at the equilibrium becomes:

$$\mathbf{J}(P) = \begin{bmatrix} \alpha(1-\sigma) & -\alpha\gamma \\ 1 & -\gamma-\delta \end{bmatrix}$$

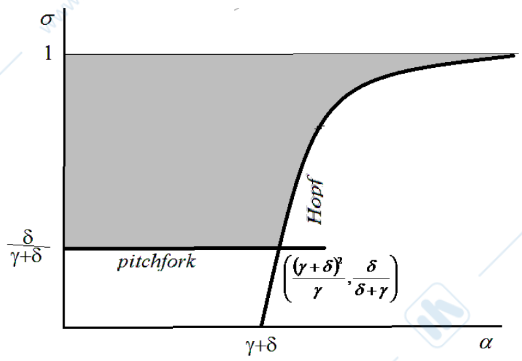
We have:

$$Tr(J(P)) = \alpha(1-\sigma) - \gamma - \delta$$
$$Det(J(P)) = -\alpha(1-\sigma)(\gamma+\delta) + \alpha\gamma$$

The stability conditions:

$$Tr(J(P)) < 0 \Rightarrow \alpha < \frac{\gamma+\delta}{(1-\sigma)} \text{or} \sigma > \frac{\alpha-\gamma-\delta}{\alpha}$$
$$Det(J(P)) > 0 \Rightarrow \sigma > \frac{\delta}{\gamma+\delta}$$

Stability region



CONTINUOUS 2D NONLINEAR & HOPF BIFURCATION

Nonlinear 2D systems
We consider now a two-dimensional dynamical systems of the form:

$$\dot{x}_1 = f_1(x_1(t), x_2(t))$$
$$\dot{x}_2 = f_2(x_1(t), x_2(t))$$

where $f_1(.)$ and $f_2(.)$ are in general nonlinear.
In general we can say that, as for the 1D case, several equilibrium points can coexist. The can be found by solving the system:

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$$

Again, a linear system can be used to understand the local behaviour of a nonlinear system in a neighborhood of the equilibrium point (i.e. locally). In R^n a neighborhood is an open disk of radius r and centre x*.

Taylor Expansion
The local phase portrait in a neighborhood of an equilibrium point $E = (x^*_1, x^*_2)$ by using the linear approximation of the nonlinear system obtained by the Taylor expansion is the following:

$$f_1(x_1, x_2) = f_1(x_1^*, x_2^*) + \left.\frac{\partial f_1}{\partial x_1}\right|_E (x_1 - x_1^*) + \left.\frac{\partial f_1}{\partial x_2}\right|_E (x_2 - x_2^*) + o(.)$$
$$f_2(x_1, x_2) = f_2(x_1^*, x_2^*) + \left.\frac{\partial f_2}{\partial x_1}\right|_E (x_1 - x_1^*) + \left.\frac{\partial f_2}{\partial x_2}\right|_E (x_2 - x_2^*) + o(.)$$

where o (.) represents higher order infinitesimal terms.

We define Jacobian matrix as the matrix collecting the four partial derivatives:

Jacobian matrix

$$\mathbf{J}(x_1,x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1,x_2) & \frac{\partial f_1}{\partial x_2}(x_1,x_2) \\ \frac{\partial f_2}{\partial x_1}(x_1,x_2) & \frac{\partial f_2}{\partial x_2}(x_1,x_2) \end{bmatrix}$$

By substituting the Taylor expansion we can rewrite the 2D system in a neighborhood of an equilibrium point as:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \mathbf{J}(x_1^*,x_2^*)\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + o(.)$$

where $X_1 = x_1 - x^*_1$ and $X_2 = x_2 - x^*_2$ are coordinates measuring the displacement from the equilibrium.

Linearization Theorem

If a nonlinear system have an equilibrium x* such that all the eigenvalues of J(x*) have nonvanishing real part (i.e. are hyperbolic), then in a neighborhood of x* the local phase portrait of the nonlinear system is qualitatively equivalent to that of the linear approximation.

As a consequence any hyperbolic equilibrium of a nonlinear dynamical system can be classified as a stable (unstable) node, or saddle, or a stable (unstable) focus for the corresponding linear approximation.

Local asymptotic stability

Let x* be an equilibrium point of _x = f (x), x ∈ R^n. If all the eigenvalues of J(x*) have negative real part, then x* is a locally asymptotically stable equilibrium. If at least one eigenvalue of the Jacobian matrix J(x*) has positive real part, then x* is unstable.

It follows that given a nonlinear system, the following are the first steps to perform:

1. To find equilibrium points by solving the system f_i (.) = 0;
2. To evaluate the local stability of each equilibrium point through the study of the Jacobian matrix calculated at the equilibrium point and the corresponding eigenvalues.

A new invariant set

- While in 1D continuous time systems the only possible invariant set is an equilibrium point, a new kind of invariant set is possible with 2D continuous time systems.
- There also exist invariant closed orbits or limit cycles (Γ) on which periodic trajectories exist.
- Also for invariant closed orbits the question about stability arises.

Theorems

Jordan curve lemma

Any closed orbit (Γ) divides the plane into two connected and disjoint regions, one inside and one outside the closed curve. Both regions are trapping.
As a consequence, a trajectory starting inside Γ cannot exit, while one starting outside cannot enter.

Poincaré-Bendixson Theorem

Let _x = f(x) be a set of two ordinary dierential equations defined in an open set G ⊆ R^2, and let D ⊂ G be a compact (i.e. closed and bounded) trapping set that does not contain any equilibrium point. Then D must contain at least one closed invariant orbit of the dynamical system.

A corollary of the Poincaré-Bendixson Theorem is:
if Γ is a closed orbit such that its interior region is entirely included into G, then Γ must include at least one equilibrium point.

Local bifurcations

- We have seen in 1D systems that transitions of an equilibrium point from stable to unstable do not just imply a transition from stability to instability, but are associated with the creation/distruction of other equilibrium points;
- Fold, Transcritical or Pitchfork bifurcations can also occur in 2D systems whenever eigenvalues are real. (i.e. nodes and saddles that become non-hyperbolic at the bifurcation)
- In 2D systems also the creation/distruction of closed invariant orbits can be the consequence of a bifurcation.
- The bifurcation with no 1D analogue is called ANDRONOV-HOPF bifurcation and it is related to the presence of complex conjugate eigenvalues.

The Andronov-Hopf theorem (I)

- Let us consider the 2D dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu), \mathbf{x} \in \mathbb{R}^2, \mu \in \mathbb{R}$$

with $\mathbf{f}$ formed by two smooth functions, and let $\mathbf{x}^*(\mu)$ be an isolated equilibrium point, i.e. $\mathbf{f}(\mathbf{x}^*, \mu) = 0$.
- Let us assume that *the eigenvalues are complex* $\lambda_{1,2}(\mu) = \alpha(\mu) \pm i\omega(\mu)$ for μ in a neighborhood of μ_0 and that for $\mu = \mu_0$ they are *pure imaginary* (i.e. the real part venishes: $\alpha(\mu_0) = 0$, $\omega(\mu_0) = \omega_0 > 0$.
- If $\left.\frac{\partial Re\lambda_{1,2}}{\partial \mu}\right|_{\mu=\mu_0} > 0$ holds, then

- x^* is a stable focus for $\mu < \mu_0$;
- x^* is an unstable focus for $\mu > \mu_0$.

The Andronov-Hopf theorem (II)

Innitely many closed invariant curves exist for $\mu = \mu_0$, which are neutrally stable (centre);

supercritical case (soft loss of stability)
When the equilibrium from stable focus is transformed into an unstable focus, a small stable limit cycle is created around it, attracting the trajectories starting inside the cycle (except for the equilibrium), as well as those starting outiside it.

subcritical case (hard loss of stability)
An unstable closed orbit surrounds the stable equilibrium and constitutes the boundary of its basin of attraction. As the bifurcation parameter approaches the bifurcation value, the basin shrinks because the unstable orbit collapses to the equilibrium point, and then disappears. (hard loss of stability)

Normal form
Let us consider the following dynamical system:

$$\dot{x}_1 = \mu x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + \mu x_2 - x_2(x_1^2 + x_2^2)$$

The unique equilibrium is: $x^* = (0;0)$
The Jacobian matrix calculated in the equilibrium is: $J(x^*) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$

whose eigenvalues are $\lambda_{1,2} = \mu \pm i$. For $\mu = 0$ a Andronov-Hopf supercritical bifurcation occurs.

CONTINUOUS ND SYSTEMS & CHAOS

Linear systems
Let us consider a linear system: _x = Ax, $x \in R^n$
where A is an n x n matrix of constant coe-cients.
The unique equilibrium is an n-dimensional vector with all the components equal to zero: O (0, …, 0)
It is possible to write the characteristic equation, det(A- λI) = 0:

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n = 0$$

whose solutions are the (real or complex) n eigenvalues of the system.

Stability condition
Let us put in decreasing order the n-eigenvalues of the linear system according to their real part: $Re(\lambda_1) > Re(\lambda_2) > \ldots > Re(\lambda_n)$

λ_1 is called the dominant eigenvalue.
The equilibrium O is globally asymptotically stable iff: $Re(\lambda_1) < 0$
that is, all the eigenvalues have a negative real part.

Considerations

- There are not any general algebraic condition of the global stability of an equilibrium of an n-dimensional system;

- For some particular configurations of the matrix A, it is possible to find necessary and su-cient conditions for the stability (Routh-Hurwitz criterion; Gerschgorin Circle Theorem...).

Nonlinear systems
Let us consider an n-dimensional dynamical system in the form: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x};\mu),\ \mathbf{x} \in \mathbb{R}^n,\ \mu \in \mathbb{R}.$

and let where x*(μ) be an equilibrium point, implicitly defined as a solution of the nonlinear system f (x;μ) = 0 of n equations with n unknows.
To study the local stability of the equilibrium point we can look at the eigenvalues of the Jacobian matrix computed at the equilibrium point $\mathbf{J}(\mathbf{x}^*(\mu)) = \left[\left. \frac{\partial f_i}{\partial x_i} \right|_{\mathbf{x}^*} \right]$

so, the linearization is still available.

Nonlinear systems

If the dominant eigenvalue of J(x*) has a negative real part then the equilibrium point is locally stable.
If, by varying the value of a parameter μ, the real part of the dominant eigenvalue passes from negative to positive, then one of the bifurcations we know occurs.

First novelty
The Jordan curve lemma no longer holds. In fact, trajectories con jump from inside to outside a closed invariant curve by moving outside their plane.

A tale with three characters:

- Pierre Simon Laplace

We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe ad those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes. (Laplace, 1776)
This statement is now known as the Laplacian determinism, and the intellect which is assumed to know the equations of motion of the Universe and its state and any given state is sometimes called Laplace's demon.

- Henri Poincaré

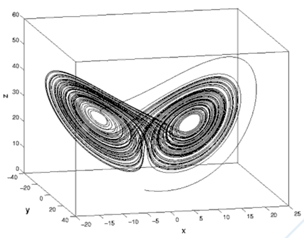
If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately.
If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that is governed by laws. But is is not always so; it may happen that small differences in the initial conditions produce very great ones in the nal phenomena. A small error in the former will produce an enourmous error in the latter. Prediction becomes impossible, and we have an apparently fortuitous phenomenon. (Poincaré, 1903)

- Edward Lorenz

In the Sixties, the American mathematician Edward Lorenz was stdying a system of differential equations used in weather forecasting: $\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1) \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = x_1 x_2 - \beta x_3 \end{cases}$

He found an invariant set that was aperiodic.
He computed two distinct timeplot by varying one initial condition, subtracting 10^{-6}. They were completely different.

The discovery of deterministic chaos
This is the strange attractor found by Lorenz:



Deterministic chaos
When the present determines the future, but the approximate present does not approximately determine the future. (Lorenz,1972)
In 1972 Lorenz presented a paper entitled: “Predictability: Does the Flap of a Butterfly's Wings in Brazil set of a Tornado in Texas?”
From the moment, the sensitivity to initial conditions characterizing chaotic motion is popularly known as butterfly effect.
This what we have learnt about deterministic chaos:

- Chaotic motion is aperiodic.
- Chaotic motion is sensitive to initial conditions (butterfly effect).
- Despite its deterministic nature, chaotic motion is basicallyfunpredictable.

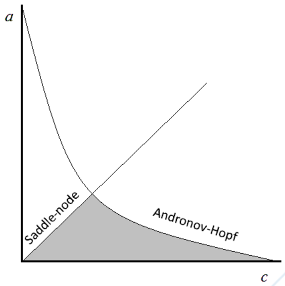
EXAMPLES OF 3D MODELS IN CONTINUOUS TIME

The Rössler model
Let us consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + a x_2 \\ \dot{x}_3 = b x_1 - c x_3 + x_1 x_3 \end{cases}$$

The only equilibrium of the model is O(0;0;0).
It is possible to demonstrate that the stability conditions are the following:
c > a c > ab (c -a)(1+b-ac) > c - ab
This implies that for b = 1 the stability conditions become: $a < min\left[c, \frac{1}{c}\right]$

Stability region
In the (c,a) paramters' plane this is the stability region:



A financial model with Lorenz equations
In 1995, Malliaris and Stein derive a financial model expressed by the Lorenz equations:

$$\begin{cases} \dot{x} = s(-x+y) \\ \dot{y} = x(r-z) - y \\ \dot{z} = -bz + xy \end{cases}$$

where x(t) is the excess volatility of the price of a financial asset; y(t) the volatility of the average Bayesian errors by traders; z(t) measures the excess speculation.
Parameters are all positive.

Equilibria
From the equilibrium condition x_ = y_ = z_ = 0 we get three equilibria:

$$E_0 = (0,0,0);$$
$$E_1 = \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$$
$$E_2 = \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$$

E_1 and E_2 only exist if r > 1.

Stability
It can be proven that E_0 is locally stable provided that: r < 1 suggesting the occurrence of a Pitchfork Bifurcation.
The other two equilbria are stable provided that: $1 < r < r^* \quad \text{with} \quad r^* = \frac{(3+b+s)s}{(s-1-b)} > 1$

DISCRETE-TIME DYNAMICAL SYSTEMS - 1D

Discrete time and Difference Equations
Dynamical systems with discrete time ($t \in N$) naturally arise in social modelling. This is the case where changes of a system occur as a consequence of decisions that cannot be continuously revised;

Given a characteristic time interval Δt, taken as a unit of time advancement ($\Delta t = 1$), if x(t) 2 R^n is the state of the system at a given time t, then the state at the next time t +1 is obtained by the application of a map:
x (t +1) = T(x(t)) is said to be a system of difference equations.

$$x(t+1) = T(x(t))$$

A map

$x(t) \longrightarrow$ [T] $\longrightarrow x(t+1)$

A repeated application of the map T (or iteration) represents a unit-time advancement of the state of the dynamical system:

$$x(1) = T(x(0)); \quad x(2) = T(x(1)) = T(T(x(0)) = T^2(x(0)); \quad \ldots; \quad x(n) = T^n(x(0))$$

The simplest case
Let us start by considering the linear homogeneous iterated map:

$$x(t+1) = f(x(t)) = ax(t)$$

with initial condition $x(0) = x_0$.
The first problem is: how to identify equilibrium points?
Equilibrium points in 1D discrete time maps
An equilibrium point satis_es the following condition: $x^* = x(t+1) = x(t)$
so equilibrium points are the roots of the equation: $x^* = f(x^*)$

The equilibrium
In the case of the linear homogeneous map, equilibria solve the equation: $x^* = ax^*$
which is only solved by $x^* = 0$.
Now the question about stability arises;
In this case it is not so di-cult to find a general solution of the system. It can be obtained inductively:
$x(1) = ax_0$ $\quad x(2) = ax(1) = a^2x_0$ $\quad x(3) = ax(2) = a^3x_0 \ldots$ from which:
General solution $\quad x(t) = x_0 a^t \quad t \in \mathbb{N}$

Global Stability
Once the general solution is known, to study the stability it is enough to find the conditions such that: $\lim_{x \to +\infty} x(t) = x^*$
So we have that x^* is globally stable provided that:
Global stability condition $\quad |a| < 1 \quad$ that is $-1 < a < 1$.

0 < a < 1
Let us deepen this case by using a = 1/2
For any initial condition $x(0) = x_0 > 0$ we have:

$$x(1) = \tfrac{1}{2}x_0 > 0$$
$$x(2) = \tfrac{1}{2}x(1) = \tfrac{1}{4}x_0 > 0$$
$$\ldots$$
$$x(n) = \tfrac{1}{2^n}x_0 > 0$$

and clearly, for $n \to +\infty$ this series converges to $x^* = 0$.
Moreover, if $x_0 > 0$ than $x(n) > 0$, $\forall n \in N$, and similarly if $x_0 < 0$ than $x(n) < 0$, $\forall n \in N$. So trajectories are monotone.

A graphical interpretation

-1 < a < 0
Let us deepen this case by using a = -1/2 .
For any initial condition $x(0) = x_0 > 0$ we have:

$$x(1) = -\tfrac{1}{2}x_0 < 0$$
$$x(2) = -\tfrac{1}{2}x(1) = \tfrac{1}{4}x_0 > 0$$
$$\ldots$$
$$x(n) = \tfrac{1}{2^n}x_0 > 0 \text{ if } n \text{ is even}, \; -\tfrac{1}{2^n}x_0 < 0 \text{ if } n \text{ is odd}$$

and clearly, for $n \to +\infty$ this series converges to $x^* = 0$.
In this case trajectories are oscillatory.
Oscillations are possible even in the simplest 1D case

A graphical interpretation

a > 1

Let us deepen this case by using a = 2.
For any initial condition $x(0) = x_0 > 0$ we have:

$$x(1) = 2x_0 > 0$$
$$x(2) = 2x(1) = 4x_0 > 0$$
$$\dots$$
$$x(n) = 2^n x_0 > 0$$

and clearly, for n→ + ∞ this series diverges to infinite.
Moreover, if $x_0 > 0$ than $x(n) > 0$, $\forall n \in N$, and similarly if $x_0 < 0$ than $x(n) < 0$, $\forall n \in N$. So trajectories are monotone.

A graphical interpretation

a < -1

Let us deepen this case by using a = -2.
For any initial condition $x(0) = x_0 > 0$ we have:

$$x(1) = -2x_0 < 0$$
$$x(2) = -2x(1) = 4x_0 > 0$$
$$\dots$$
$$x(n) = 2^n x_0 > 0 \text{ if } n \text{ is even}, \ -2^n x_0 < 0 \text{ if } n \text{ is odd}$$

and clearly, for n→ + ∞ this series oscillates, diverging in absolute value.

a = 1 and a = -1

- If a = 1 than $x(t) = x_0 \ \forall t$

so any initial condition is an equilibrium point.

- If a = -1 than $x(t) = \begin{cases} x_0 \text{ if } t \text{ is odd} \\ -x_0 \text{ if } t \text{ is even} \end{cases}$

so the trajectory alternate always two values: x_0 and $-x_0$.

Summarizing

- If |a| < 1 trajectories converge to the equilibrium point. The map is called a contraction.
- If |a| > 1 trajectories diverge (at least in absolute value). The map is called an expansion.
- If a > 0 trajectories are monotone.
- If a < 0 trajectories are oscillatory.

The affine case
Let us consider the more general linear a-ne map: $x(t+1) = ax(t)+b$
The main difference is due to the new position of the unique equilibrium, obtained as follows:

for a a ≠ 1.

$$x^* = ax^* + b$$
$$\Downarrow$$
$$x^* = \frac{b}{1-a}$$

Are there other differences with respect to the linear homogeneous case?

Graphical explanation

Analytical proof
It is possible to make a change of coordinates: $X(t) = x(t) - x^* = x(t) - \frac{b}{1-a}$ so:

$$x(t) = X(t) + \frac{b}{1-a}$$

then we get:

$$x(t+1) = ax(t) + b$$
$$\Downarrow$$
$$X(t+1) + \frac{b}{1-a} = a\left[X(t) + \frac{b}{1-a}\right] + b$$
$$\Downarrow$$
$$X(t+1) = aX(t)$$

Solution
We know that: $X(t) = X(0)a^t$
and then:

$$x(t) = \left(x_0 - \frac{b}{1-a}\right)a^t + \frac{b}{1-a}$$

so we have that:
for $|a| < 1$ solutions converge to x^*;
for $|a| > 1$ solutions go away from x^*;
for $a = 1$ solutions follow the rule $x(t+1) = x(t) + b$, increasing or decresing according to the sign of b;
for $a = -1$ solutions oscillate between x_0 and $-x_0$.

EXAMPLES OF 1D MODELS IN DISCRETE TIME

Compound Interest
Consider in time t = 0 you invested the amount C at a compound interest with effective interest rate equal to r > 0.
If a generic time you reached the amount of M(t) > 0, the next period you obtain:

$$M(t+1) = M(t)+rM(t) = (1+r)M(t)$$

that is a linear homogeneous difference equation.
This trajectory is divergent because 1+r > 1.
Even more interesting is the fact that we can find the general solution:

$$M(n) = C(1+r)^n$$

The Cobweb model
We consider the price formation in a partial market of a single commodity.
The quantity demanded by consumers is a decreasing function of the price (demand function):

$$Q^d(t) = D(p(t))$$

Producers decide the amount of output on the basis of an increasing function of price (supply function):

$$Q^s(t) = S(p^e(t))$$

where pe(t) represents the expected price for time time when they must decide how much to produce.
In equilibrium we have:

$$Q^d(t) = Q^s(t)$$

Price dynamics
We assume static expectations (i.e. $p^e(t) = p(t-1)$), so we have:

$$D(p(t)) = S(p(t-1))$$

By assuming simple linear functions such as:

$$D(p) = a-bp \text{ and } S(p) = -c+dp$$

where parameters are all positive.
We obtain the following dynamics of price (after a simple time translation):
The Cobweb model

$$p(t+1) = T(p(t)) = -\frac{d}{b}p(t) + \frac{a+c}{b}$$

that is linear non homogeneous 1D map.

Equilibrium
The first step consists in funding the unique equilibrium point of the map.
By using the equilibrium condition $p(t+1) = p(t) = p^*$ we obtain: $p^* = \frac{a+c}{b+d}$

Considering that the slope is negative: $-\frac{d}{b} < 0$

we already know that dynamics are oscillatory.

Global stability
The Equilibrium Point is globally stable provided that:
Stability condition

$$\left|-\frac{d}{b}\right| < 1$$

that is:

$$-\frac{d}{b} > -1$$
$$\Downarrow$$
$$b > d$$

Financial market
Let us consider the market of a single asset.
The makert is populated by three types of agent:

- Fundamentalists traders;
- Chartists (or Technical) traders;
- A market maker, who regulates the price according to the total excess demand.

The fundamental value of the asset is exogenously given and known by all the agents: P = F

Fundamentalists
Fundamentalists believe in an immediate correction of mispricing, that is a price different from its fundamental value.
As a consequence, their excess demand is given by:
Fundamentalists trading rule

$D^F_t = f(F - P(t))$ where $f > 0$ is their speed of reaction.
So Fundamentalists buy the asset when it is undervalued ($P(t) < F$) while they sell it when it is overvalued ($P(t) > F$).

Chartists
Chartists, at the opposite, believe in the short run persistence of optimism/pessimism, identied as an high (resp. low) price. that is a price different from.
As a consequence, their excess demand is given by:
Chartists trading rule
$D^C_t = c(P(t)-F)$
where $c > 0$ is their speed of reaction.
So Chartists buy the asset when it is overvalued ($P(t) > F$) while they sell it when it is undervalued ($P(t) < F$).

The market maker
The market maker adjusts the asset price according to the rule: $P(t+1) = P(t)+D(t)$
where $D(t) = D^F(t)+D^C(t)$ is the total demand excess.
By substituting the trading rules of the two types of investors we get:

$$P(t+1) = P(t) + f(F - P(t)) + c(P(t) - F)$$
$$\Downarrow$$
$$P(t+1) = (1+c-f)P(t) + F(f-c)$$

This is a linear a-ne difference equation.

Equilibrium and stability
From the stability condition $P(t+1) = P(t) = P^*$ we get:

$$P^* = F$$

so *the fundamental values is the only equilibrium value.*
The equilibrium is globally stable provided that:

$$-1 < 1+c-f < +1$$

that can be written as:

$$c < f \cup f < c+2$$

So in order to have convergence to the fundamental value, *chartists should not be more aggressive than fundamentalists*, but also *fundamentalists should not be too aggressive* (*overshooting phenomenon*)

NONLINEAR DISCRETE 1D & LOCAL BIFURCATIONS

1D nonlinear models
Let us consider a generalized one-dimensional dynamic equation in discrete time: $x(t+1) = f(x(t))$
where $f(x)$ can be linear or nonlinear and initial condition $x(0) = x_0$.
It is still true that from the solutions of the equation
Equilibrium condition

$$f(x) = x$$

we can find the equilibria, which can be more than one, and even zero.

Hyperbolic equilibria
An equilibrium x^* is called hyperbolic iff: $|f'(x^*)| \neq 1$
If the equilibrium is hyperbolic then the function $f(x)$ can be approximated by the first order Taylor expansion:

$$f(x)=f(x^*)+f'(x^*)(x-x^*)+o(x-x^*)=f'(x^*)(x-x^*)+o(x-x^*)$$

So the system can be linearly approximated by neglecting the higher order terms. By using the change of variable $X = x - x^*$ (displacement between x and the equilibrium point), our dinamic
equation becomes: $X(t+1) = aX(t)$
where $a = f'(x^*)$.

So we have the following results:
Local asymptotic stability in 1D discrete time
An equilibrium x^* is locally asymptotically stable iff: $|f'(x^*)| < 1$
while it is locally asymptotically unstable iff: $|f'(x^*)| > 1$

Non hyperbolic equilibria
In discrete time we have two dierent typologies of non hyperbolic equilibria:
1. A nonhyperbolic equilbrium such that: $f'(x^*) = +1$;
2. A nonhyperbolic equilbrium such that: $f'(x^*) = -1$.

The first case is analogous to the condition $f'(x^*) = 0$ for 1D continuous time models.
Nonhyperbolic equilibria such that $f'(x^*) = -1$ have no analogue in continuous time.
In fact, it is characterized by oscillatory behavior.

Decreasing 1D maps
- Decreasing 1D maps are at the origin of the differences with respect to 1D continuous time systems
- As an example, consider the map:

$$x(t+1) = \frac{1}{x(t)}$$

- From any initial $x(0) = x_0$ we have:

$$x(1) = 1/x_0$$
$$x(2) = 1/x(1) = x_0$$
$$x(3) = 1/x(2) = 1/x_0$$
$$x(4) = 1/x(3) = x_0$$
$$\dots$$

that is called a *periodic cycle* of period two: $C_2(x_0, \frac{1}{x_0})$.

Periodic cycles

Let us consider now the map: $x' = f(x) = x^2-1$

where the notation with the unit-time advancement operator "'" is used. $C_2(0;-1)$ is a cycle of period two. By using a generic initial condition, for instance $x_0 = 3/2$ we get:

$x_0 = 1.5$; $x_1 = 1.25$;
$x_2 = 0.5625$; $x_3 = -0.6836$;
$x_4 = -0.5327$; $x_5 = -0.7162$;
$x_6 = -0.4870$; $x_7 = -0.7628$;
$x_8 = -0.4181$; $x_9 = -0.8252$;
$x_{10} = -0.3191$; $x_{11} = -0.8982$;
... ...
$x_{100} = -0.0001$ $x_{101} = -0.99999$

A periodic cycle, as any other invariant set, can be attractive (stable)

Periodic cycles

A periodic cycle of period k is a set of points $C_k = \{c_1, c_2, ..., c_k\}$ such that
- $c_i \neq c_1$, $i = 2, ..., k$,
- $f(c_i) = c_{i+1}$, $i = 1, ..., k-1$,
- $f(c_k) = c_1$.

In other words:

$f^k(c_i) = c_i$ for each $i = 1; ..., k$

Any periodic point c_i of a k-cycle C_k is a fixed point of the map $f^k(x)$ but not a fixed point of $f^j(x)$ with $j < k$.

Stability of a periodic cycle

The stability of a a fixed point c_i of $f^k(x)$ is given by the condition: $\left|\frac{\partial f^k}{\partial x}(c_i)\right| < 1$

It can be proved that the same result can be obtained by using the following rule:

Stability of a k-cycle

$$\left|\frac{\partial f^k}{\partial x}(c_i)\right| = |f'(c_1) \cdot f'(c_2) \cdot ... \cdot f'(c_k)| = \left|\prod_{i=1}^{k} f'(c_i)\right| < 1$$

That for instance for a 2-cycle is the condition: $|f'(c_1) \times f'(c_2)| < 1$

A note on composite functions

In order to build composite functions $f^k(x)$ it is necessary to apply k times function $f(x)$.

For instance, if:
$$f(x) = x^2 - 1$$
we have that:
$$f^2(x) = f \circ f(x) = (x^2-1)^2 - 1 = x^4 - 2x^2$$
and
$$f^3(x) = f \circ f \circ f(x) = f \circ f^2(x) = (x^4-2x^2)^2 - 1 = x^8 - 4x^6 + 4x^4 - 1$$
and so on.

Local bifurcations

We consider a one-dimensional discrete time dynamical system whose structure depends on a parameter $\alpha \in R$: $x(t+1) = f(x(t); \alpha)$

Let $x^*(\alpha)$ be a fixed point implicitly defined by the equilibrium equation $f(x; \alpha) = x$.

We analyze now the local bifurcations occurring when by varying the parameter a, the fixed point loses stability at a bifurcation value such that: $f'(x^*) = +1$

Fold Bifurcation

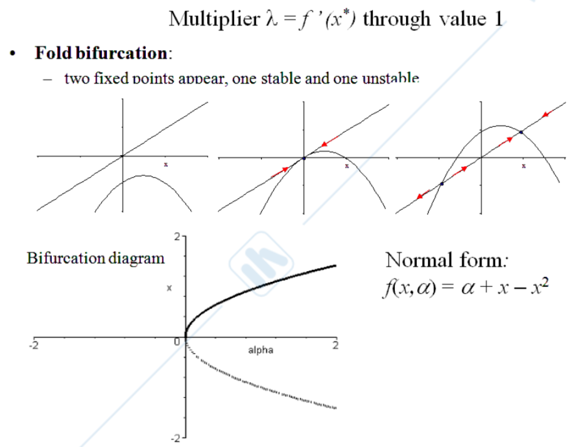
When a Fold Bifurcation occurs:
- Two equilibrium points are created simoultaneously, as a parameter varies;
- At the bifurcation value they are coincident;
- After the bifurcation value, they are separated: one locally stable and one locally unstable

An example is given by the dynamical system:
$$x' = f(x) = a + x - x^2$$
as the parameter α varies through the bifurcation value $\alpha_0 = 0$.

Fold Bifurcation

- If we apply the equilbrium condition $x' = x$ to the system we find two potential equilbria: $x_1^* = +\sqrt{\alpha}$ and $x_2^* = -\sqrt{\alpha}$
- The equilibrium points exist only if $\alpha \geq 0$ and at the bifurcation value $\alpha_0 = 0$ they are coincident;
- At the bifurcation value the equilibrium points are non hyperbolic, in fact:
  $$f'(x) = 1 - 2x \implies f'(x_1^*) = 1 - 2\sqrt{\alpha} \text{ and } f'(x_2^*) = 1 + 2\sqrt{\alpha}$$
  that are both equal to $+1$ when $\alpha = \alpha_0 = 0$.
- For $\alpha > 0$ we have that $0 < f'(x^*_1) < 1$ while $f'(x^*_2) > 1$, so x^*_1 is locally stable and x^*_2 is locally unstable.

Graphically



Transcritical Bifurcation

When a Transcritical (or stability exchange) Bifurcation occurs:

- Two equilibrium points, one stable and one unstable merge at the bifurcation value;
- After the bifurcation they still exist but with opposite stability properties (i.e. the one previously stable becomes unstable and viceversa);

An example is given by the dynamical system:

$$x' = f(x) = \alpha x + x - x^2$$

as the parameter α varies through the bifurcation value $\alpha_0 = 0$.

Transcritical Bifurcation

- If we apply the equilbrium condition $x' = x$ to the system we find two, *always existing*, equilibria:

$$x_1^* = 0 \quad \text{and} \quad x_2^* = \alpha$$

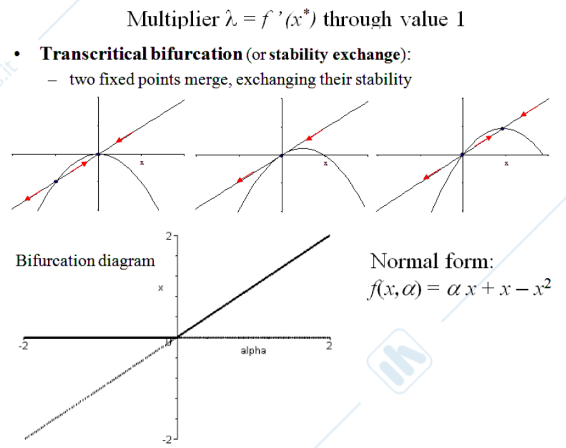
- At the bifurcation value the equilibrium points are *coincident and non hyperbolic*, in fact:

$$f'(x) = 1 + \alpha - 2x \implies f'(x_1^*) = 1 + \alpha \text{ and } f'(x_2^*) = 1 - \alpha$$

  and they both are equal to $+1$ when $\alpha = \alpha_0 = 0$.

- For $\alpha > 0$ we have that $f'(x_1^*) > 1$ while $0 < f'(x_2^*) < 1$, so x_1^* is locally unstable and x_2^* is locally stable;
- At the opposite, for $\alpha < 0$ we have that $0 < f'(x_1^*) < 1$ while $f'(x_2^*) > 1$, so x_1^* is locally stable and x_2^* is locally unstable.

Graphically



Pitchfork Bifurcation

When a (Supercritical) Pitchfork Bifurcation occurs:

- There is a transition from a single, locally stable, equilibrium point to three equilibria (two further equilibria are created simoultaneously);
- At the bifurcation value the three equilibria are coincident;
- After the bifurcation the already existing equilibrium becomes unstable with the two new equilibria are both locally stable;
- The unstable equilibrium point is the boundary between the basins of attraction of the two coexisting stable equilibria.

An example is given by the dynamical system:

$$x' = f(x) = \alpha x + x - x^3$$

as the parameter α varies through the bifurcation value $\alpha_0 = 0$.

- If we apply the equilbrium condition $x' = x$ to the system we find three potential equilbria:

$$x_0^* = 0 \quad \text{and} \quad x_{1,2}^* = \pm\sqrt{\alpha}$$

with x_0^* always existing while $x_{1,2}^*$ only exist if $\alpha > 0$.

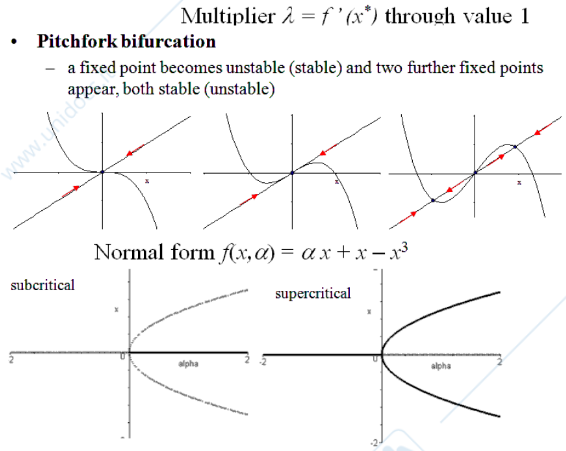
- At the bifurcation value the equilibrium points are *coincident and non hyperbolic*, in fact:

$$f'(x) = 1 + \alpha - 3x^2 \implies f'(x_0^*) = 1 + \alpha \text{ and } f'(x_{1,2}^*) = 1 - 2\alpha$$

and they are all equal to $+1$ when $\alpha = \alpha_0 = 0$.

- For $\alpha > 0$ we have that $f'(x_0^*) > 1$ while $0 < f'(x_{1,2}^*) < 1$, so x_0^* is locally unstable and $x_{1,2}^*$ are both locally stable;
- At the opposite, for $\alpha < 0$ we have that $0 < f'(x_0^*) < 1$, so x_0^* is locally stable.

Graphically



EXAMPLES OF NONLINEAR 1D MODELS IN DISCRETE TIME - Local Bifurcations, Periodic Cycles and Chaos

Financial market
Let us consider the market of a single asset.
The makert is populated by three types of agent:

- Fundamentalists traders;
- Chartists (or Technical) traders;
- A market maker, who regulates the price according to the total excess demand.

The fundamental value of the asset is exogenously given and known by all the agents: P = F

Fundamentalists
Fundamentalists believe in an immediate correction of misprincing, that is a price different from its fundamental value. They become increasingly aggressive as the mispracing become larger.
As a consequence, their excess demand is given by:
Fundamentalists Nonlinear trading rule

$$D^F_t = f(F - P(t))^3$$

where f > 0 is their speed of reaction.

Chartists
Chartists, at the opposite, believe in the short run persistence of optimism/pessimism, identifed as an high (resp. low) price. that is a price different from.
As a consequence, their excess demand is given by:
Chartists trading rule

$$D^C_t = c(P(t) - F)$$

where c > 0 is their speed of reaction

The market maker
The market maker adjusts the asset price according to the rule:

$$P(t+1) = P(t) + D(t)$$

where $D(t) = D^F(t) + D^C(t)$ is the total demand excess.
By substituting the trading rules of the two types of investors we get:

$$P(t+1) = P(t) + f(F - P(t))^3 + c(P(t) - F)$$

This is a nonlinear (cubic) difference equation.

A more convenient notation
It is more convenient to intruduce the variable “mispricing” x(t), defined as:

$$x(t) = P(t) - F$$

We can rewrite the system by substituting F on both sides:

$$P(t+1) - F = P(t) - F + f(F - P(t))^3 + c(P(t) - F)$$

and in terms of misprincing:

$$x(t+1) = x(t)+cx(t)-fx(t)^3$$

or

$$x' = x(1+c)-fx^3$$

Equilibrium and stability
From the stability condition x' = x = x* we get: $x_0^* = 0$ and $x_{1,2}^* = \pm\sqrt{\frac{c}{f}}$

so the fundamental values is not necessarily the only equilibrium value but also other two, symmetric, equilibria may exist.
Three equilibria exist provided that: $\frac{c}{f} > 0$

So for feasible values of the parameters three equilibria coexist.
Local stability
For studying the local stability of the equilibria we must study the derivative:

$$f'(x) = 1+c-3fx^2$$

The Fundamental equilibrium x^*_0 is always locally unstable, in fact:

$$f'(x^*_0) = 1+c > 1$$

Concerning the other two equilibria: $|f'(x^*_{1,2})| < 1$ $0 < c < 1$
c = 0 is a pitchfork bifurcation value.

Caoutious chartists
A nonlinearity can be added to the model also by caoutious Chartists, that is they become less aggressive when the mispricing is large.
We can write their excess demand as follows:
Chartists trading rule

$$D^C_t = c \arctan(P(t)-F)$$

where that arctan function permits to modelize their prudent behavior.

The market maker
The market maker adjusts the asset price according to the rule:

$$P(t+1) = P(t)+D(t)$$

where $D(t) = D^F(t)+D^C(t)$ is the total demand excess.
By substituting the trading rules of the two types of investors we get:

$$P(t+1) = P(t)+f(F-P(t)+c \arctan(P(t)-F)$$

This is a nonlinear difference equation.

A more convenient notation
It is more convenient to intruduce the variable mispricing x(t), defined as:

$$x(t) = P(t)-F$$

We can rewrite the system by substituting F on both sides:

$$P(t+1)-F = P(t)-F+f(F-P(t)+c \arctan(P(t)-F)$$

and in terms of misprincing:

$$x(t+1) = x(t)+c \arctan x(t)-fx(t)$$

or

$$x' = x(1-f)+c \arctan x$$

Equilibrium and stability
From the stability condition x' = x = x* we get: $x^*_0 = 0$
so the fundamental value is still an equilibrium value but we are not able to obtain algebrically the presence of further equilibria.
It stability is given by:

$$|f'(x^*_0)| < 1$$

that is:
$f-2 < c < f$
so, if $f > 2$ then there exists a range of feasible values of the parameter c such that the fundamental equilibrium is locally stable.
And then?

Stability of symmetric equilibria
Let us move directly from the scenario in which: $0 < c < 1$
so the fundamental fixed point is locally unstable and the other two symmetric equilibria $x^*_{1,2}$ are both locally stable.
At c = 1 we have that: $f'(x^*_{1;2}) = -1$
so the two equilibria become nonhyperbolic.
By furtherly increasing c is two flip bifurcations simoultaneously occur, created the coexistence of two locally stable 2-cycles.
A further increase in the value of c causes a cascade of period-doubling bifurcations, leading to chaos.

The Cobweb model
We consider the price formation in a partial market of a single commodity.
The quantity demanded by consumers is a decreasing function of the price (demand function):

$$Q^d(t) = D(p(t))$$

Producers decide the amount of output on the basis of an increasing function of price (supply function):

$$Q^s(t) = S(p^e(t))$$

where pe(t) represents the expected price for time time when they must decide how much to produce.
In equilibrium we have:

$$Q^d(t) = Q^s(t)$$

A Production saturation effect dynamics
We assume static expectations (i.e. $p^e(t) = p(t-1)$), so we have:

$$D(p(t)) = S(p(t-1))$$

We still assume a simple linear demand function, such as:

$$D(p) = a-bp$$

where parameters are all positive.
We introduce a nonlinearity in the supply function, representing a production saturation effect:
Nonlinear supply function

$$S(p) = \arctan(\lambda p(t-1))$$

where $\lambda > 0$ represents the slope of the supply at the reference point $p = 1$.

The map is now the following:

$$p' = f(p) = \frac{1}{b}\left[a - \arctan(\lambda(p-1))\right]$$

By using λ as a bifurcation parameter the equilibrium undergoes a Flip bifurcation.

FLIP BIFURCATION, LOGISTIC MAP & BASINS OF ATTRACTION

The new case of Nonhyperbolic equilibria
Let us consider a generalized one-dimensional dynamic equation in discrete time:

$$x(t+1) = f(x(t))$$

where $f(x)$ can be linear or nonlinear and initial condition $x(0) = x_0$.
We know that:
Local asymptotic stability in 1D discrete time
An equilibrium x^* is locally asymptotically stable iff $|f'(x^*)| < 1$
while it is locally asymptotically unstable iff $|f'(x^*)| > 1$

A new kind of bifurcation
We consider a one-dimensional discrete time dynamical system whose structure depends on a parameter $\alpha \in R$: $x(t+1) = f(x(t); \alpha)$
Let $x^*(\alpha)$ be a fixed point implicitly defined by the equilibrium equation $f(x; \alpha) = x$.
We analyze now the local bifurcations occurring when by varying the parameter α, the fixed point loses stability at a bifurcation value such that: $f'(x^*) = -1$

Period-doubling (Flip) Bifurcation
When a (supercritical) Period-doubling bifurcation occurs:
- A cycle of period two is created, as a parameter varies;
- At the bifurcation value the points of the cycle are coincident with the equilibrium value ;
- After the bifurcation value, they are separated: the equilibrium point is unstable while the cycle is locally stable

An example is given by the dynamical system:

$$x' = f(x) = -(1+\alpha)x + x^3$$

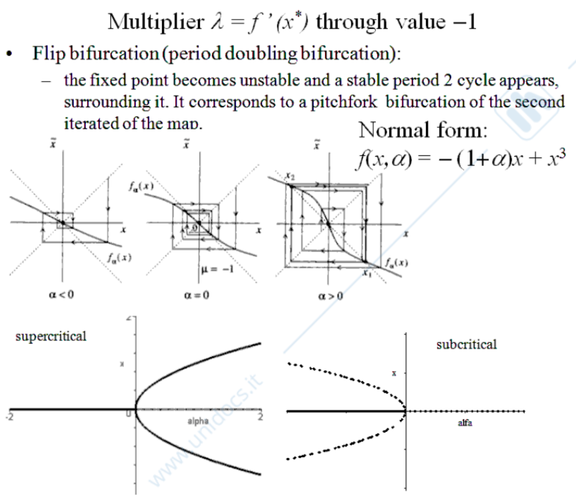
as the parameter α varies through the bifurcation value $\alpha_0 = 0$.

Local stability of an equilibrium
- If we apply the equilbrium condition $x' = x$ we find that one of the equilbrium points is the following: $x^*_0 = 0$
- At the bifurcation value the equilibrium point is non hyperbolic, in fact:
  $$f'(x) = -1 - \alpha + 3x^2 =) \rightarrow f'(x^*_0) = -1 - \alpha$$
  that is equal to -1 when $\alpha = \alpha_0 = 0$.
- For $\alpha > 0$ we have that $f'(x^*_0) < -1$ so the equilibrium is unstable, while for $\alpha < 0$ we have that $-1 < f'(x^*_0) < 1$ so the equilibrium is locally stable.

The map $f^2(x)$
In order to study the presence of a 2-cycle we should find equilibria of the map $f^2(x)$ different from the fixed points of $f(x)$.
It is possible to demonstrate that at $\alpha = 0$, two new equilibria of $f^2(x)$ born, coincident with x^*_0 at the bifurcation value.
Moreover we know that:

$$\frac{\partial f^2}{\partial x}(x_0^*) = f'(x_0^*) f'(x_0^*) = +1$$

For the map $f^2(x)$ a Pitchfork Bifurcation is occurring.
Graphically



The Logistic map

Let us consider the following quadratic map:

$$x(t+1) = \mu x(t)\,(1-x(t))$$

with $\mu > 0$.

Graphically this is a concave parabola.

By applying the equilibrium condition we find two equilibria: $x_0^* = 0$ and $x_1^* = 1 - \frac{1}{\mu}$

Stability of equilibria

To study the local stability of the equilibria we need the derivative:

$$f'(x) = \mu\,(1-2x)$$

At the equilibrium values: $f'(x^*_0) = \mu$ and $f'(x^*_1) = 2-\mu$

So, we have:

x_0^* is locally stable iff $-1 < \mu < 1$

while

x_1^* is locally stable iff $1 < \mu < 3$

At $\mu = 1$ a Transcritical Bifurcation occurs.

Flip bifurcation

- At $\mu = 2$ we have that:

  $$f'(x_1^*) = 0$$

  the point is called *superstable*.

- At $\mu > 2$ we have that:

  $$f'(x_1^*) < 0$$

  so trajectories are *oscillatory*.

- At $\mu = 3$ we have that:

  $$f'(x_1^*) = -1$$

  so *a Flip Bifurcation occurs* and a 2-cycle $C_2 = \{\alpha, \beta\}$ is created. At the bifurcation value $\alpha = \beta = x_1^*$.

Periodic cycles

Period two cycles can be found by looking at:

$$F(x) = f^2(x) = \mu\,[\mu x(1-x)(1-\mu x(1-x))]$$

whose fixed points are the solutions of the equation:

$$x\left[\mu^2(1-x)(1-\mu x(1-x))-1\right] = 0$$

We already know that x_0^* and x_1^* are solutions but other two fixed points are given by:

$$\frac{\mu+1 \pm \sqrt{(\mu-3)(\mu+1)}}{2\mu}$$

that are our α and β, only existing for $\mu \geq 3$ and a period-2 cycle for $f(x)$.

Sequence of cycles

At $\mu = 1+\sqrt{6} \simeq 3.449$ we have that:

$$F'(\alpha) = F'(\beta) = f'(\alpha)f'(\beta) = -1$$

a second Flip Bifurcation occurs.

C_2 loses stability and a stable cycle of period 4 of f (x) is created. If μ is further increased also the period 4 loses stability though Flip bifurcation, creating a period 8 cycle, and so on...

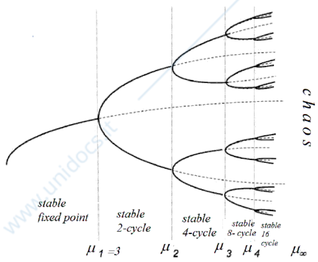
Cascade of Period-doubling Bifurcations

Infinitely many stable cycles of period 2^n are created, which become unstable as μ is increased.

Periodic cycles

Flip bifurcations become more and more frequent as μ is increased (in order to find a new bifurcation μ must be increased less and less);

Bifurcations accumulated at the limit point μ_∞ = 3:56994571869… (Feigenbaum constant)



Chaotic region
After the Figenbaum point dynamics are chaotic.
We can characterized chaos as follows:

1. Infinitely many unstable periodic points exist, dense in the invariant set;
2. An aperiodic trajectory, dense in the set, also exist;
3. Trajectories are sensitive to initial conditions.
4. If c is the maximum value of the parabola, trajectories are bounded in the interval $[c_1;c]$ with $c_1 = f(c)$.
5. For certain ranges of the bifurcation parameters, after the

Feigenbaum point, chaos seems to disappear for a while and dynamics are captured by an attracting periodic cycle (periodic windows, that are infinitely many).

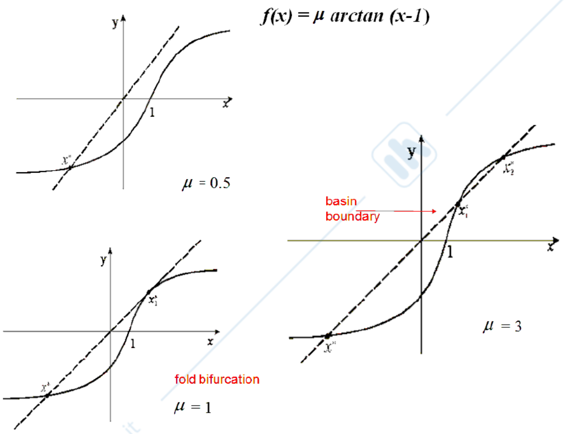
Basin of attraction (I)
Consider the discrete dynamical system: $x(t+1) = T(x(t))$; $x \in R$ characterized by an invariant set $A \subset R$.
Basin of attaction
The Basin of attraction of A is the set of all the points that generate trajectories converging to A
$B(A) = \{x \mid T^n(x) \to A \text{ as } n \to +\infty\}$

Basin of attraction (II)
If U(A) is a neighborhood of A whose points converge to A, the total basin of A can be defined as: $B(A) = \bigcup_{n=0}^{\infty} T^{-n}(U(A))$

where $T^{-1}(x)$ represents the preimages of x and $T^{-n}(x)$ the set of points mapped into x after n iterations of the map T.

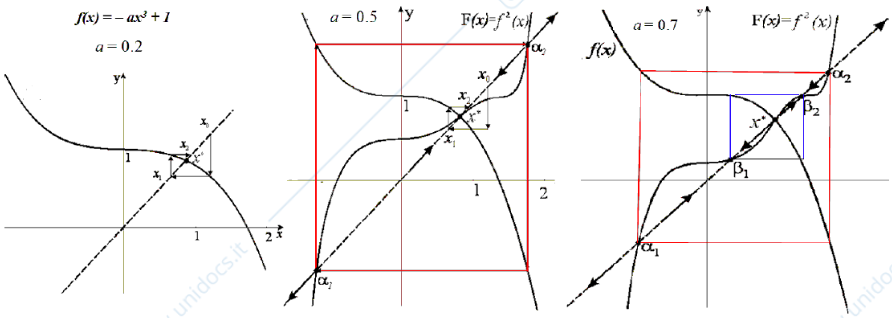
Increasing maps
Let us consider one-dimensional, continuous, increasing and invertible maps.
The only possible invariant sets are the fixed points.
Let us consider the case with several fixed points: $x^*_1 < x^*_2 < \ldots < x^*_k$
Fixed points are alternatingly stable and unstable, and the unstable fixed points are the boundaries that separate the basins of the stable ones.

Increasing maps



Decreasing maps
Let us consider one-dimensional, continuous, decreasing and invertible maps.
The only possible invariant sets are one fixed point and cycles of period two.
In fact, if f (x) is decreasing, then $f^2(x)$ is increasing, and may only have fixed points, one of which is the unique fixed point x* of f (x).
Fixed points of $f^2(x)$ appear in pairs, at opposite sides with respect to x*.
Periodic points of the cycles of period 2 are alternatingly stable and unstable, and the unstable ones are the boundaries that separate the basins of the stable ones.

Decreasing maps



DISCRETE 2D MAPS

2D maps

We consider dynamical models of systems whose state is described by two variables: $x_1(t)$ and $x_2(t)$, with $t \in N$, which are interdependent:

$$x'_1 = T_1(x_1;x_2)$$
$$x'_2 = T_2(x_1;x_2)$$

where we used the more convenient unit-time advancement operator.

To get a qualitative global view of the phase portrait, it is possible to represent the two curves of equations:

$$T_1(x_1;x_2) = x_1$$
$$T_2(x_1;x_2) = x_2$$

usually called nullclines.

Equilibria

Equilibria can be found by solving the system: $\begin{cases} T_1(x_1,x_2) = x_1 \\ T_2(x_1,x_2) = x_2 \end{cases}$

Linear systems

We start by considering a linear homogeneous system of two difference equations of first order with constant coefficients of the following (normal) form: $\begin{cases} x'_1 = a_{11}x_1 + a_{12}x_2 \\ x'_2 = a_{21}x_1 + a_{22}x_2 \end{cases}$

We can write it in matrix form:

$$x' = \mathbf{A}x$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad x' = \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}$$

Equilibrium

The system admits only one equilibrium:

$$E(0;0)$$

In order to the study the (global) stability of the equilibrium we need to find the roots of the following equation:

Characteristic equation

$$P(\lambda) = \lambda^2 - Tr(\mathbf{A})\lambda + Det(\mathbf{A}) = 0$$

where

$$Tr(\mathbf{A}) = a_{11} + a_{22}$$
$$Det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$$

and the roots are called *eigenvalues*, as in the corresponding case in continuous time.

Real and Complex Eigenvalues

We remind that given $\triangle = Tr^2(\mathbf{A}) - 4Det(\mathbf{A})$, we may find three different scenarios:

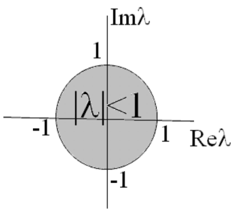
1. If $\triangle > 0$ we have *two real and distinct eigenvalues*, given by:
$$\lambda_{1,2} = \frac{Tr(A) \pm \sqrt{\triangle}}{2}$$
2. if $\triangle = 0$ we have *two real and coincident eigenvalues*, given by:
$$\lambda_1 = \lambda_2 = \frac{Tr(\mathbf{A})}{2}$$
3. if $\triangle < 0$ we have *two complex and conjugated eigenvalues*, given by:
$$\lambda_{1,2} = \frac{Tr(A)}{2} \pm i\frac{\sqrt{-\triangle}}{2}$$
with
$$Re(\lambda) = \frac{Tr(\mathbf{A})}{2} \quad \text{and} \quad Im(\lambda) = \frac{\sqrt{-\triangle}}{2}$$

Stability Condition for linear 2D maps

Global Stability Condition

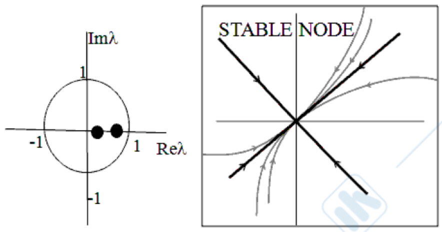
The general solution converges to the equilibrium iff the two eigenvalues are inside the unit circle of the complex plane:

$$|\lambda_i| < 1 \; i = 1, \implies Re^2(\lambda) + Im^2(\lambda) < 1$$



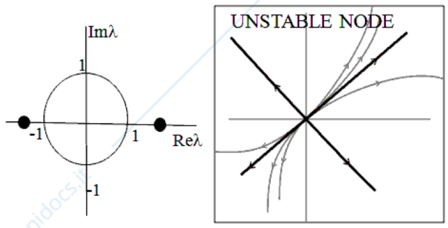
Case 1: real and distinct eigenvalues such that $0 < \lambda_1 < \lambda_2 < 1$

- The equilibrium is asymptotically stable and is called stable node.



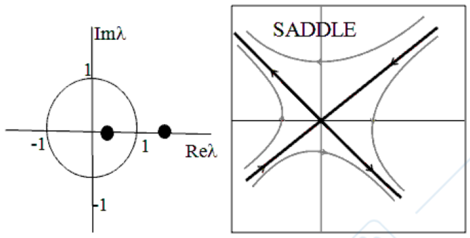
Case 2: real and distinct eigenvalues such that $|\lambda_1| > 1$ and $|\lambda_2| > 1$

- The equilibrium is asymptotically unstable and is called unstable node.



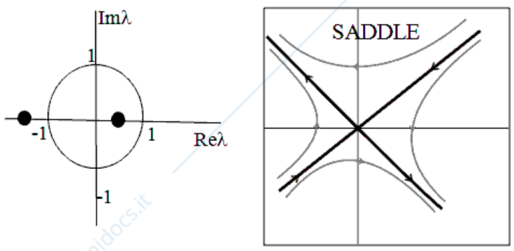
Case 3a: real and distinct eigenvalues such that $|\lambda_1| < 1$ and $|\lambda_2| > 1$

- The equilibrium is asymptotically unstable and is called saddle.



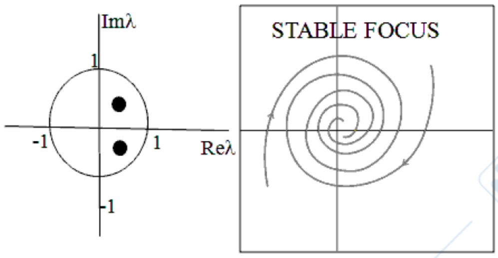
Case 3b: real and distinct eigenvalues such that $|\lambda_1| < 1$ and $|\lambda_2| > 1$

- The equilibrium is asymptotically unstable and is called saddle.



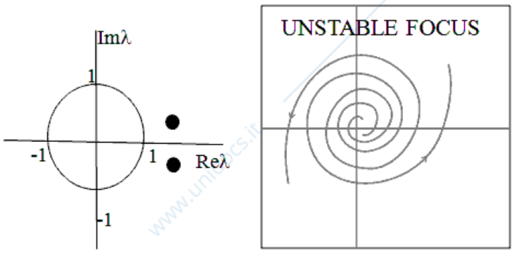
Complex eigenvalues such that $|\lambda_{1,2}| < 1$

- The equilibrium is asymptotically stable and is called stable focus.



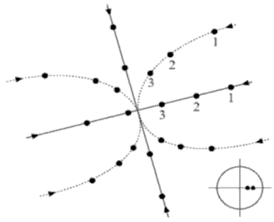
Complex eigenvalues such that $|\lambda_{1,2}| > 1$

- The equilibrium is asymptotically unstable and is called unstable focus.

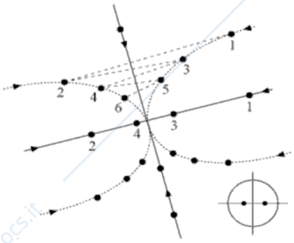


Trajectories

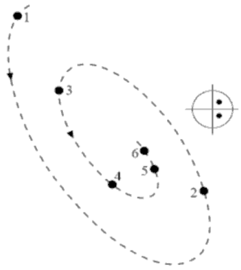
Differently from the corresponding trajectories in continuous time, in discrete time trajectories moves at discrete time pulses



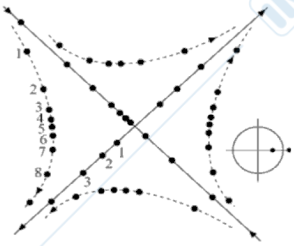
Differently from the corresponding trajectories in continuous time, in discrete time trajectories moves at discrete time pulses



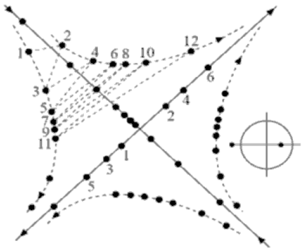
Differently from the corresponding trajectories in continuous time, in discrete time trajectories moves at discrete time pulses



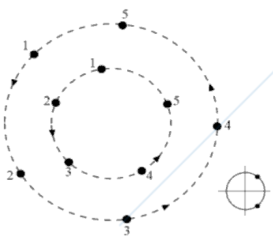
Differently from the corresponding trajectories in continuous time, in discrete time trajectories moves at discrete time pulses



Differently from the corresponding trajectories in continuous time, in discrete time trajectories moves at discrete time pulses



Differently from the corresponding trajectories in continuous time, in discrete time trajectories moves at discrete time pulses



Stability conditions

It is possible to prove that the equilibrium point of a linear 2D map is globally asymptotically stable provided that:

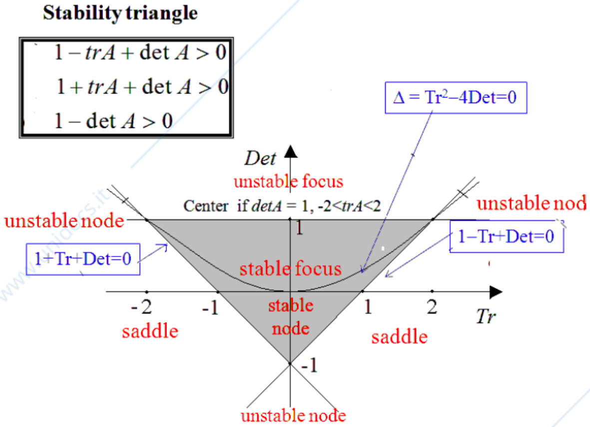
Stability Conditions:

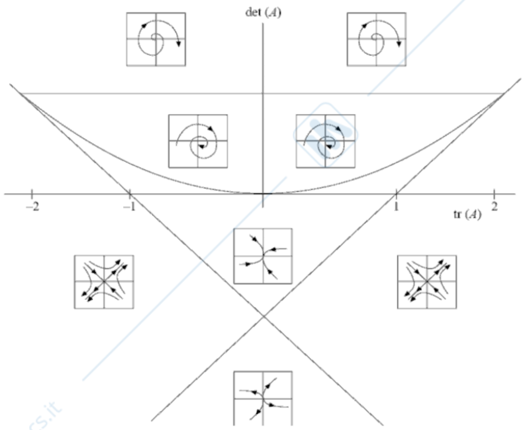
(i) $P(1) = 1 - Tr(\mathbf{A}) + Det(\mathbf{A}) > 0$
(ii) $P(-1) = 1 + Tr(\mathbf{A}) + Det(\mathbf{A}) > 0$
(iii) $Det(\mathbf{A}) < 1$

- If condition (i) is violated, then an eigenvalue exits the unit circle along the real axis in the point $\lambda = 1$;
- If condition (ii) is violated, then an eigenvalue exits the unit circle along the real axis in the point $\lambda = -1$;
- If condition (iii) is violated, then complex conjugate eigenvalues exit the unit circle of the complex plane.

Summarizing

The stability conditions define a triangle of stability in the plane (Tr (A), Det(A))





Nonlinear 2D maps

We consider dynamical models of systems whose state is described by two variables: $x_1(t)$ and $x_2(t)$, with $t \in N$, which are interdependent:

$$x'_1 = T_1(x_1;x_2)$$
$$x'_2 = T_2(x_1;x_2)$$

where we used the more convenient unit-time advancement operator. $T_1(.)$ and $T_2(.)$ can be nonlinear functions.

Equilibria are still obtained by solving:

$$\begin{cases} T_1(x_1^*, x_2^*) = x_1^* \\ T_2(x_1^*, x_2^*) = x_2^* \end{cases}$$

but now they can be more than one and even zero.

Linearization

The map can be linearly approximated around an equlibrium as follows: $\mathbf{x}' - \mathbf{x}^* = \mathbf{J}_T(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*)$

where J_T is the Jacobian matrix: $$\mathbf{J}_T = \begin{bmatrix} \partial T_1/\partial x_1 & \partial T_1/\partial x_2 \\ \partial T_2/\partial x_1 & \partial T_2/\partial x_2 \end{bmatrix}$$

The Global stability conditions in linear 2D maps, become now

Local stability conditions: $|\lambda_1| < 1$ and $|\lambda_2| < 1$

Local Bifurcations

Let us consider a 2D nonlinear dynamical system depending on a parameter, say $\mu \in R$.

Consequently, any fixed point and the Jacobian matrix may depend on μ as well: $\mathbf{x}^* = \mathbf{x}^*(\mu)$ and $\mathbf{J}_T(\mathbf{x}^*) = \mathbf{J}_T(\mathbf{x}^*(\mu))$

When one real eigenvalue exit the unit circle, a local bifurcation occurs.

Neimark-Sacker bifurcation

When two complex conjugated eigenvalues, cross the unit circle, a bifurcation called Neimark-Sacker bifurcation occurs:

Neimark-Sacker bifurcation Theorem

Let $T(x,\mu):\mathbb{R}^2 \to \mathbb{R}^2$ be a one-parameter family of 2-dimensional maps which has a family of fixed points $x^*(\mu)$ at which the eigenvalues are complex conjugate. Assume that for $\mu = \mu_0$:

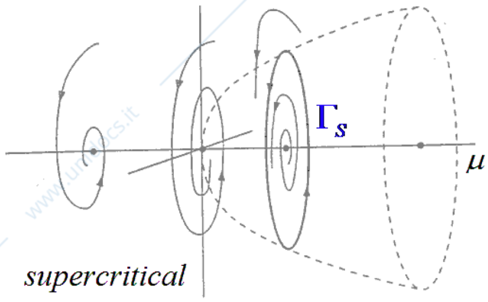
1) $|\lambda(\mu_0)| = 1$

2) $\frac{\partial|\lambda(\mu)|}{\partial\mu}(\mu_0) = d \neq 0$.

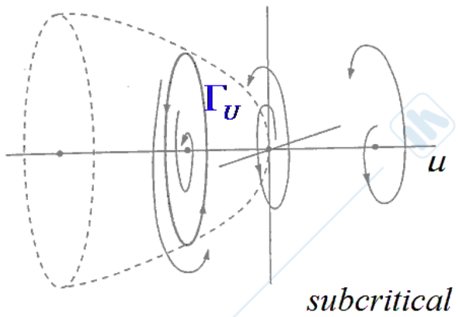
then *there is a simple closed invariant curve in a neighborhood of $x^*(\mu_0)$.*

This Bifurcation may be Supercritical or Subcritical with consequences similar to those of the corresponding Andronov-Hopf Bifurcation in continuous time.

Supercritical NS



Subcritical NS



Example

An example of Supercritical NS comes from the following:

$$x(t+1) = y(t)$$
$$y(t+1) = y(t) - \alpha x(t) + x^2(t)$$

The map admits two fixed points:

$$O(0,0) \quad \text{and} \quad P(\alpha,\alpha)$$

The Jacobian matrix is:

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ 2x-\alpha & 1 \end{bmatrix}$$

Stability of O

We can easily find that:

Tr (J(O)) = 1 and Det(J(O)) = α

and the stability conditions:

$$1 + Tr(J(O)) + Det(J(O)) = 2 + \alpha$$
$$1 - Tr(J(O)) + Det(J(O)) = \alpha$$
$$Det(J(O)) = \alpha$$

from which the condition: $0 < \alpha < 1$

Complex eigenvalues and NS bif.

Considering that $\triangle = 1 - 4\alpha$ we have that:

- For $\frac{1}{4} < \alpha < 1$ eigenvalues are complex and *O is a stable focus*;
- At $\alpha = 1$, *O loses stability via NS bifurcation*;
- At $\alpha > 1$, *O is an unstable focus and a stable invariant curve around it is born.*

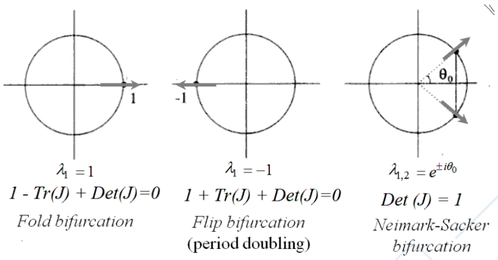
Quasi-periodic vs Chaotic motion
Dynamic motion in discrete time along an invariant curve is called Quasi-periodic:
Quasi-periodic motion

- Dynamics are not periodic, so no point is visited twice;
- Nevertheless, the points of the invariant set are dense in the curve and immediately after a Sup. NS bifucation, the closed invariant curve is filled by points of the trajectory;
- Trajectories are not sensitive to initial conditions. This means that with a slightly different initial condition dynamica will be only slightly different (difference with respect to a chaotic motion);
- The invariant set is of zero measure in the phase space (there is no area covered by it), differently from a chaotic invariant set.

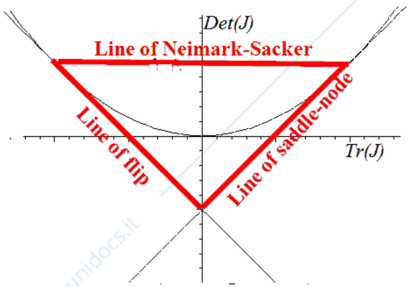
Quasi-periodic and Chaotic motion

- Similarly to what happens with chaotic motion, by varying a parameter of the map, for some parameter's ranges, trajectories may become periodic (frequency locking);
- By continuing moving the bifurcation parameters dynamics may pass from quasi-periodic to chaotic. Like Period-doubling bifurcation, also NS bifurcation is a typical route to chaos.

Summarizing



Stability Triangle



EXAMPLES OF 2D MODELS IN DISCRETE TIME

A financial market with trend followers
Let us consider the market of a single asset. The makert is populated by three types of agent:

- Fundamentalists traders;
- Chartists (or Technical) traders;
- A market maker, who regulates the price according to the total excess demand.

The fundamental value of the asset is exogenously given and known by all the agents: P = F

Fundamentalists
Fundamentalists believe in an immediate correction of misprincing, that is a price different from its fundamental value.
As a consequence, their excess demand is given by:
Fundamentalists trading rule

$$D^F_t = f\,(F - P(t))$$

where $f > 0$ is their speed of reaction.
So Fundamentalists buy the asset when it is undervalued ($P(t) < F$) while they sell it when it is overvalued ($P(t) > F$)

Chartists
Chartists behave as trend followers, that is they extrapolate the most recent price trend into the immediate future.
As a consequence, their excess demand is given by:
Chartists trading rule

$$D^C_t = c(P(t) - P(t-1))$$

where $c > 0$ is their speed of reaction.
So Chartists buy the asset when its price is increasing
($P(t) > P(t-1)$) while they sell it when its price is decreasing

(P(t) < P(t −1)).
The either do not know or do not use the information given by the Fundamental value.

The market maker
The market maker adjusts the asset price according to the rule:

$$P(t+1) = P(t) + D(t)$$

where $D(t) = D^F(t) + D^C(t)$ is the total demand excess.
By substituting the trading rules of the two types of investors we get:

$$P(t+1) = P(t) + f(F - P(t)) + c(P(t) - P(t-1))$$
$$\Downarrow$$
$$P(t+1) = (1+c-f)P(t) - cP(t-1) + Ff$$

This is a difference equation of the second order.

A 2D linear model
It is possible to rewrite a dierence equation of the second order as a 2D map.
In order to do that it is necessary to introduce an auxiliary variable: Z(t) = P(t −1)
that is called a delay variable, that becomes the second dynamic variable of the system.
By substituting into the second order difference equation we finally get:
that is a linear (non homogeneous) 2D map.

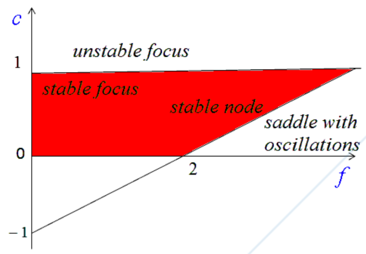
$$\begin{cases} Z(t+1) = P(t) \\ P(t+1) = -cZ(t) + (1+c-f)P(t) + fF \end{cases}$$

Equilibrium and stability
From the stability condition P(t +1) = P(t) = P* and Z(t +1) = Z(t) = Z* we get: P* = Z* = F
so the fundamental values is the only equilibrium value.
To study the Global Stability we need to analyze the matrix of the partial derivatives:

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -c & 1+c-f \end{bmatrix}$$

and $Tr(J) = 1 + c - f$ and $Det(J) = c$.

Stability conditions
The stability conditions are the following:
(i) 1−Tr(J) +Det(J) = f > 0
(ii) 1+Tr(J) +Det(J) = 2+2c −f > 0
(iii) Det(J) = c < 1
While condition (i) always holds, the others lead to the condition: $\frac{f}{2} - 1 < c < 1$



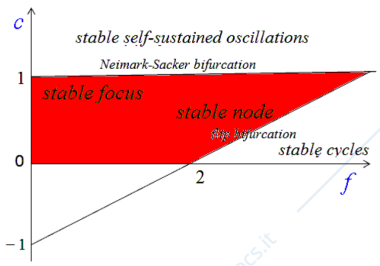
Introducing prudent chartists
Let us consider the same financial market model but now with prudent chartists.
As a consequence, their excess demand is given by:
Chartists trading rule
$D^C t = c \arctan(P(t) - P(t-1))$
where c > 0 is their speed of reaction.

The map
The 2D maps becomes as follows:

$$\begin{cases} Z(t+1) = P(t) \\ P(t+1) = (1-f)P(t) + c\arctan(P(t) - Z(t)) + fF \end{cases}$$

that is a nonlinear 2D map.
The fixed point is the same: (F,F)
It can be proved that the stability conditions are also the same.

Stability region



but the consequences of the loss of stability are totally different...
A discrete L-V model
Let us consider the map:

$$T: \begin{cases} x_1' = x_1(e_1 + a_{11}x_1 + a_{12}x_2) \\ x_2' = x_2(e_2 + a_{21}x_1 + a_{22}x_2) \end{cases}$$

Parameters e_i and a_{ii} for i = 1,2 denote autonomous logistic growth coeffcients.

$a_{ii} < 0$

Parameters a_{12} and a_{21} determine the kind of interaction:

- If a_{12} and a_{21} are both positive, then the two species live in symbiosis;
- If a_{12} and a_{21} are both negative, then the two species live in competition;
- If $a_{12}a_{21} < 0$, then the one species is the predator of the other (prey).

Fixed points

The map admits four fixed points:

- $P_0 = (0,0)$: the *extinction equilibrium*;
- $P_1 = \left(\frac{1-e_1}{a_{11}};0\right)$ and $P_2 = \left(0;\frac{1-e_2}{a_{22}}\right)$: the *one-species equilibrium*;
- $Q = (x_1^*;x_2^*) = \left(\frac{a_{12}(e_2-1)+a_{22}(1-e_1)}{a_{11}a_{22}-a_{12}a_{21}};\frac{a_{21}(e_1-1)+a_{11}(1-e_2)}{a_{11}a_{22}-a_{12}a_{21}}\right)$: the *coexistence equilibrium*.

The Jacobian matrix

To study the local stability of the equilibria we need to calculate the Jacobian matrix:

$$J(x_1,x_2) = \begin{bmatrix} e_1 + 2a_{11}x_1 + a_{12}x_2 & a_{12}x_1 \\ a_{21}x_2 & e_2 + 2a_{22}x_2 + a_{21}x_1 \end{bmatrix}$$

Stability of P_0

The Jacobian matrix calculated in P_0 becomes: $J(P_0) = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$

that is diagonal.

In this case the eigenvalues are real and correspond to the two elements on the diagonal.

So the stability condition is: Stability condition for P_0

$$-1 < e_1 < 1$$
$$-1 < e_2 < 1$$

Stability of P_i

The Jacobian matrix calculated in P_i becomes: $J(P_i) = \begin{bmatrix} 2-e_i & a_{ij}\frac{1-e_i}{a_{ii}} \\ 0 & e_j + \frac{1-e_i}{a_{ii}}a_{ji} \end{bmatrix}$

that is triangular.

Also in this case the eigenvalues are real and correspond to the two elements on the diagonal.

So the stability condition is: Stability condition for P_i

$$-1 < 2-e_i < 1$$
$$-1 < e_j + \frac{1-e_i}{a_{ii}}a_{ji} < 1$$

Stability of P_i

We can rewrite the two conditions as follows:

Stability condition for P_i

$1 < e_i < 3$

$a_{ii}(e_j-1) + a_{ji}(1-e_i) > 0$ bif related to $\lambda = +1$

$a_{ii}(e_j+1) + a_{ji}(1-e_i) < 0$ bif related to $\lambda = -1$

So we have that:

$e_i = 1$ is a Transcritical bifurcation value involving P_0 and P_i

Stability of Q

The Jacobian matrix calculated in Q becomes: $J(Q) = \begin{bmatrix} 1 + a_{11}x_1^* & a_{12}x_1^* \\ a_{21}x_2^* & 1 + a_{22}x_2^* \end{bmatrix}$

In this case the eigenvalues cannot be easily computed.

If we look at the condition:

$$P(1) = 1 - Tr + Det > 0$$
$$\Downarrow$$
$$(a_{11}a_{22} - a_{12}a_{21})x_1^*x_2^* > 0$$

we have that this is realized with positive equilibria and prey-predator competition.

We also know that eigenvalues are complex iff:

$$\triangle = Tr^2 - 4Det < 0$$
$$\Downarrow$$
$$(a_{11}x_1^* - a_{22}x_2^*)^2 + 4a_{12}a_{21}x_1^*x_2^* < 0$$

that requires as a necessary condition that $a_{12}a_{21} < 0$ that is a prey-predator competition.

OLIGOPOLY - GAME THEORY AND BOUNDED RATIONALITY

Oligopoly
An Oligopoly is a market form in which a market is dominated by a few firms. In this sense it differs from:
- Monopoly, where a single firm dominates the market;
- Perfect Competition, in which a lot of small firms take the market price as given.

Strategic Interdependence
The main feature of an oligopoly is the Strategic Interdependence,vthat is the fact that the consequences of a firm's decision depend also upon the decisions of the other concurrent firms.

A consequence of the Strategic Interdependence is the following: In order to maximize their objective functions (usually prot functions), oligopolists must try to forecast the decisions of the others.
This kind of decisions are the main object of a branch of Mathematics called GAME THEORY, developed from the pionnering work of Von Neumann and Morgerstern: "Theory of Games and Economic Behavior" in 1944.

Games
A game is made up by the following elements:
- A set of players;
- A set of strategies available for each player;
- A set of outcomes (called payoffs) associated to each possible combination of strategies.

Different Games:
- Games can be subdivided into:
- Complete/Incomplete Information
- Simultaneous/Sequential
- Static (or one-shot)/ Dynamic
- Zero sum/Non Zero sum
- Coordination/Competitive

How to represent a game
Usully games are represented in one of the two following forms:
1. Normal form (a matrix);
2. Extensive form (a tree).

Normal form is useful for static and simultaneous games, while the Extensive form is more useful for Sequential and Dynamic games.

Most famous games
- Prisoner Dilemma;
- Matching Pennies;
- Rock-Paper-Scissors;
- Chicken game;
- Centipede game

Strategies
To solve a game means to identify combinations of strategies that is reasonable the players will adopt.
Games are simplified when some strategies are of the following form:
- Dominant Strategy: a strategy that is the best alternative for a player for any possible strategy selected by the other players;
- Dominated Strategy: a strategy that is never the optimal strategy to adopt for a player.

Once removed dominated strategies, if there are no dominant strategies then it must be found a way for solving the game.

Max-min and max-max strategies
Von Neumann and Morgerstern proposed a simple rule to solve a game:
Max-min
Each player adopts the strategy to which corresponds the maximum of the minimum payos they may obtain.
This is considerate a quite conservative way of finding solutions.
There exist a lot of drawbacks of this strategy.

The contribution of John Nash
John Forbes Nash (1928-2015) provided the most important definition of Equilibrium used in Game Theory.
Nash Equilibrium
A combination of strategies is a Nash Equilibrium if for each player the strategy in the combination is optimal given the strategies of the others.

Features of the NE
- NE is a robust solution when it is unique (unfortunately sometimes there are many);
- Considering also mixed strategies, at least one NE always exists (Nash Theorem);
- When it is unique, neoclassical economists assume that the player are able to identify it and play it immediately, also in repeated games;
- It can be a good normative concept of equilibrium, but is it descriptive?

Cournot oligopoly

- In 1838 the French Mathematician and Economist Augustin Cournot for the rst time provided a mathematical study of a particular class of oligopoly;
- He focused on the case where two rms (duopoly) compete by deciding the amount of product to bring to the market;
- He considered homogeneous goods, so the cumulative amount of goods produced determine the market price;
- In the terminology of Game Theory, he studied this market form as a static game with complete information.

The Cournot model

Let us consider a market where the (inverse) demand function is the following:

$$P(Q) = a - bQ$$

where a and b are positive demand parameters, while

$$Q = q_1 + q_2.$$

The two firms have the same costs' structure, given by a constant marginal cost:

$$C_i(q_i) = cq_i \text{ for } i = 1;2$$

The firms select the amount of good the produce in order to maximize their profits: $q_i = \arg\max \Pi_i = Pq_i - cq_i$ for $i = 1,2$

The problem of firm 1

If we consider firm 1, she wants to maximize: $\Pi_1 = P(Q)q_1 + cq_1 = [a - b(q_1 + q_2)]\, q_1 - cq_1$

and to do that: $\frac{\partial \Pi_1}{\partial q_1} = a - c - 2bq_1 - bq_2 = 0$

from which:

Reaction Function $q_1 = \frac{a-c}{2b} - \frac{q_2}{2}$

that specifies the optimal choice for any possible decision of the competitor.

Cournot-Nash Equilibrium

Considering the two reaction functions: $\begin{cases} q_1 = \frac{a-c}{2b} - \frac{q_2}{2} \\ q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \end{cases}$

it is possible to find a point belonging to both:

Cournot-Nash Equilibrium

$$E(q_1^*, q_2^*) = E\left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$$

that is a Nash Equilibrium of the duopoly game.

Problem solved?

The model studied by Cournot is based on a lot of important simplifying assumptions:

- Linearity of both demand and cost functions;
- Firms are identical;
- Firms know everything about their rivals;
- Firms perfectly predict the choice of the competitor (rational expections hypothesis)

If we remove one (or more) of these simplifying assumptions, is it still reasonable to think that firms sooner or later will learn to play the Cournot-Nash solution?

Removing the linearity hypothesis

The first more realistic changing to the benchmark model consists in considering nonlinear functions:

- A Nonlinear demand function, for instance an isoelatic one, deriving from Cobb-Douglas utility functions of the consumers: $P(Q) = \frac{1}{Q}$
- A Nonlinear cost function, characterizing decreasing or increasing scale returns: $c(q_i) = c_0 + c_1\sqrt{q_i}$ or $c(q_i) = c_0 + c_1 q_1^2$
- Nonlinearities may arise from other sources.

Removing homogeneity

Moreover we may assume that:

- Firms may be characterized by different cost functions (for instance, different marginal costs...);
- Firms may produce non homogeneous goods (substitute or complementary);
- More heterogeneity may arise from different sources.

Removing perfect rationality

The most strong assumption concerns the assumption of perfect rationality of the firms, that have rational expectations: $E_i(q_j) = q_j$

Decision makers used heuristics (also known as shortcuts or rules of thumb) to solve complicated problems

They also adopt a trial and error procedure to find the best strategy

This makes the model necessarily dynamic, with the fundamental question given by: under which circumstances the choices of economic agents converge to those of perfectly rational decision makers?

Boundedly rational firms

In a context of oligopoly, the assumption of bounded rationality usually involves two aspects:

1. The Expectation formation mechanism;
2. The decisional rules

Concerning expectations a typical alternative to rational expectations are static expectations:

$E_i(q_j(t+1)) = q_j(t)$

A little bit more complicated are adaptive expectations: $E_i(q_j(t+1)) = E_i(q_j(t)) + \lambda\,[q_j(t) - E_i(q_j(t))]$

with $\lambda > 0$.

Decisional rules

Firms may not be able to know the shape of the demand function, and as a consequence of their own profit function. In this case they can use a gradient-like decisional mechanism: $q_i(t+1) = q_i(t) + \gamma(q_i)\dfrac{\partial \Pi_i(t)}{\partial q_i(t)}$

where g(qi) is the (potentially endogenous) speed of adjustment.

Sometimes firms are prudent and adjust their production only partially towards the optimal choice:

$$q_i(t+1) = \lambda q_i(t) + (1-\lambda)\left[q_i^{opt}(t+1) \middle| E_i(q_j(t+1))\right]$$

where $0 \leq \lambda \leq 1$ measures how prudent the firms is.

EXAMPLES OF DUOPOLY MODELS 2D DISCRETE TIME MODELS

From a static to a dynamical game

Let us consider the reaction functions of the original Cournot duopoly: $q_i = \dfrac{a-c}{2b} - \dfrac{q_j}{2}$

Removing the rational expectations hypothesis, this becomes a repeated game: $q_i(t+1) = \dfrac{a-c}{2b} - \dfrac{E_i(q_j(t+1)|I(t))}{2}$

where I (t) denotes Information available at time t.

By introducing static expectations we have that: $E_i(q_j(t+1)|I(t)) = q_j(t)$

The map

By inserting this kind of expectations for both firms we obtain the following dynamical system: $T: \begin{cases} q_1(t+1) = \frac{a-c}{2b} - \frac{q_2(t)}{2} \\ q_2(t+1) = \frac{a-c}{2b} - \frac{q_1(t)}{2} \end{cases}$

Map T is:

- Two-dimensional;
- Linear.

So we expect only one equilibrium, that when it is stable is globally stable.

The Equilibrium

By using the equilibrium condition $q_i(t+1) = q_i(t) = q^*_i$ for i = 1; 2, we get the following equilibrium:

Cournot-Nash Equilibrium $E(q_1^*, q_2^*) = E\left(\dfrac{a-c}{3b}, \dfrac{a-c}{3b}\right)$

So the question concerns its stability.

The matrix of coe-cients is the following: $A: \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$

with Tr = 0 and Det = -1/4.

Stability of E

- Considering that 4= 1, we have real eigenvalues;
- If Tr = 0 we have that: $\lambda_{1,2} = \pm\dfrac{\sqrt{\triangle}}{2}$

  so eigenvalues are either both inside the unit circle or both outside (no saddle);
- In particular we get: $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$

  so the CN Equilibrium is globally stable.

New assumptions

In this model we change two assumptions to the previous version of the game:

1. We use an isoelastic (nonlinear) demand function $P(Q) = \dfrac{1}{Q}$
2. We consider potentially different marginal costs: $C_1(q_1) = c_1 q_1$ and $C_2(q_2) = c_2 q_2$

Let us check if something new occurs.

The problem of firm 1→

Firm 1, as usual, aims at maximizing its profit function:

$$\Pi_1 = P(Q)q_1 - C_1(q_1)$$

Considering our assumptions, we have:

$$\Pi_1 = \frac{q_1}{q_1+q_2} - c_1 q_1$$

The maximizing choice satisfies the so-called *first-order condition*:

$$\frac{\partial \Pi_1}{\partial q_1} = \frac{q_2}{(q_1+q_2)^2} - c_1 = 0$$

that solved for q_1 provides the *reaction function*:

$$q_1 = r(q_2) = \sqrt{\frac{q_2}{c_1}} - q_2$$

The static model
By using perfect rationality the static model provide the rational solution:

$$\begin{cases} q_1 = r(q_2) = \sqrt{\frac{q_2}{c_1}} - q_2 \\ q_2 = r(q_1) = \sqrt{\frac{q_1}{c_2}} - q_1 \end{cases}$$

The system is solved by:

$$O(0,0) \quad \text{and} \quad E\left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2}\right)$$

Obviously we are more interested in E.

The map
By adopting static expectations for both firms we obtain the following dynamical system:

$$T: \begin{cases} q_1(t+1) = \sqrt{\frac{q_2(t)}{c_1}} - q_2(t) \\ q_2(t+1) = \sqrt{\frac{q_1(t)}{c_2}} - q_1(t) \end{cases}$$

Map T is:
- Two-dimensional;
- Nonlinear.

So we expect not necessarily only one equilibrium, that when are stable are locally stable

The Equilibria
By using the equilibrium condition $q_i(t+1) = q_i(t) = q^*_i$ i for i = 1; 2, we get the following equilibria:
Cournot-Nash Equilibria

$$O(0,0) \quad \text{and} \quad E\left(\frac{c_2}{(c_1+c_2)^2}, \frac{c_1}{(c_1+c_2)^2}\right)$$

that are the same of the static model.
So the question concerns their stability.
The Jacobian matrix is the following:

$$J: \begin{bmatrix} 0 & \frac{1}{2\sqrt{c_1 q_2}} - 1 \\ \frac{1}{2\sqrt{c_2 q_1}} - 1 & 0 \end{bmatrix}$$

with $Tr = 0$ and $Det = -\left(\frac{1}{2\sqrt{c_1 q_2}} - 1\right)\left(\frac{1}{2\sqrt{c_2 q_1}} - 1\right)$

Stability of E
Also in this case Tr = 0 so: $\lambda_{1,2} = \pm\frac{\sqrt{\Delta}}{2}$

so eigenvalues are either both inside the unit circle or both outside (no saddle);
But now is more di-cult to prove that eigenvalues are real and to compute them.

Stabiliy conditions
We can use the stability conditions are the following:

$$\begin{array}{ll} (i) & 1 - Tr(J) + Det(J) > 0 \Longrightarrow Det(J) > -1 \\ (ii) & 1 + Tr(J) + Det(J) > 0 \Longrightarrow Det(J) > -1 \\ (iii) & Det(J) < 1 \end{array}$$

So in our case we have that: $-1 < Det(J) < 1$
After some algebraic computation can be proved that:
Local Stability Condition

$$3 - 2\sqrt{2} < \frac{c_1}{c_2} < 3 + 2\sqrt{2}$$

so if firms were identical (c1 = c2) the Equilibrium E would be locally stable, otherwise...

New assumptions
In this model we change the following assumptions to the previous version of the game:
1. We come back to consider a linear demand function: P(Q) = a-bQ
2. Firms are boundedly rational and only partially informed, they adjust their production by following a gradient-like mechanism:

$$q_i(t+1) = q_i(t) + v_i q_i \frac{\partial \Pi_i}{\partial q_i}(q_1, q_2)$$

with:

$$\frac{\partial \Pi_i}{\partial q_i}(q_1, q_2) = a - c_i - 2bq_i - bq_j$$

3. We still consider potentially different marginal costs: $C_1(q_1) = c_1 q_1$ and $C_2(q_2) = c_2 q_2$

The model
By using static expectations we get the following dynamical system:

$$T: \begin{cases} q_1' = q_1(1 + v_1(a - c_1)) - 2bv_1 q_1^2 - bv_1 q_1 q_2 \\ q_2' = q_2(1 + v_2(a - c_2)) - 2bv_2 q_2^2 - bv_2 q_1 q_2 \end{cases}$$

Map T is:
- Two-dimensional;
- Nonlinear.

So we expect not necessarily only one equilibrium, that when are stable are locally stable.

The Equilibria
By using the equilibrium condition $q'_i = q_i = q^*_i$ for i = 1; 2, we get:

$$\begin{cases} q_1(a - c_1 - 2bq_1 - bq_2) = 0 \\ q_2(a - c_2 - 2bq_2 - bq_1) = 0 \end{cases}$$

from which:

Cournot-Nash Equilibria

- $O(0,0)$
- $E_1\left(\frac{a-c_1}{2b},0\right)$ and $E_2\left(0,\frac{a-c_2}{2b}\right)$
- $E^*(q_1^*,q_2^*)$ with:

$$q_1^* = \frac{a+c_2-2c_1}{3b} \quad \text{and} \quad q_2^* = \frac{a+c_1-2c_2}{3b}$$

Local Stability

The Jacobian matrix is the following:

$$J : \begin{bmatrix} 1+v_1(a-c_1-4bq_1-bq_2) & -v_1 bq_1 \\ -v_2 bq_2 & 1+v_2(a-c_2-4bq_2-bq_1) \end{bmatrix}$$

Calculated in O it becomes:

$$J(0,0) : \begin{bmatrix} 1+v_1(a-c_1) & 0 \\ 0 & 1+v_2(a-c_2) \end{bmatrix}$$

where the eigenvalues are on the diagonal.
So if:

$$a > c_1 \quad or \quad a > c_2$$

the Equilibrium O is an unstable node.

Stability of E_i

The Jacobian matrix calculated in E_1 becomes:

$$J\left(\frac{a-c_1}{2b},0\right) : \begin{bmatrix} 1-v_1(a-c_1) & -\frac{v_1}{2}(a-c_1) \\ 0 & 1+\frac{v_2}{2}(a+c_1-2c_2) \end{bmatrix}$$

The Jacobian matrix calculated in E_2 becomes:

$$J\left(0,\frac{a-c_1}{2b}\right) : \begin{bmatrix} 1+\frac{v_1}{2}(a+c_2-2c_1) & 0 \\ -\frac{v_2}{2}(a-c_2) & 1-v_2(a-c_2) \end{bmatrix}$$

If E^* has positive coordinates, both E_1 and E_2 are unstable (nodes or saddle).

Stability of E*

The Jacobian matrix calculated in E* becomes:

$$J(q_1^*,q_2^*) : \begin{bmatrix} 1-2v_1 bq_1^* & -v_1 bq_1^* \\ -v_2 bq_2^* & 1-2v_2 bq_2^* \end{bmatrix}$$

that is difficult to analyze:
We can say that:

$$\triangle = 4b^2\left[(v_1 q_1^* - v_2 q_2^*)^2 + v_1 v_2 q_1^* q_2^*\right] > 0$$

so eigenvalus are real.
Moreover:

$$1 - Tr + Det > 0$$

if the coordinates are positive.
E^* may lose stability only via Flip bifurcation.

NONINVERTIBLE MAPS

Images and preimages
A map T: defined by x' = T(x), transform a point x into a unique point x':

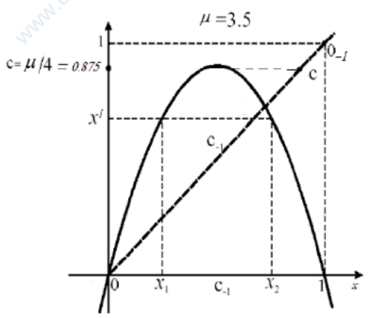
- x' is called the rank-1 image of x;
- x is called the rank-1 preimage of x'

Invertible maps
if $\mathbf{x} \neq \mathbf{y}$ implies $T(\mathbf{x}) \neq T(\mathbf{y})$ for each x, y then T is an *invertible map*.
The inverse mapping $\mathbf{x} = T^{-1}(\mathbf{x}')$ is uniquely defined.

Noninvertible maps
if points x exist, such that they have several rank-1 preimages, then T is a noninvertible map.
The inverse mapping $x = T^{-1}(x')$ is multivalued.
An example in 1D is given by the logistic map: $x' = f(x) = \mu x(1-x)$

where some points x are characterized by two pre-images: $x_1 = f_1^{-1}(x') = \frac{1}{2} - \frac{\sqrt{\mu(\mu-4x')}}{2\mu}$; $x_2 = f_2^{-1}(x') = \frac{1}{2} + \frac{\sqrt{\mu(\mu-4x')}}{2\mu}$

The logistic map

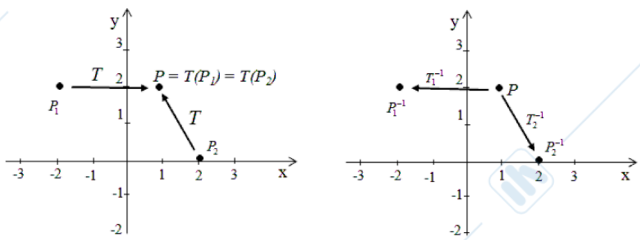


If x'< c = m/4 then x' has two preimages, otherwise no preimages at all.
Only x' = c is characterized by only one preimage.
The maximum value c plays a crucial role

A 2D example
Let us consider the following map: $T: \begin{cases} x' = ax + y \\ y' = x^2 + b \end{cases}$
By using parameters a = 1/2 and b = -2, it is easy to check that both P_1 = (-2;2) and P_2 = (2;0) are mapped into the same point P = (1;2).
We can obtain algebrically the expressions of the two inverse maps:

$$T_1^{-1}: \begin{cases} x = -\sqrt{y'-b} \\ y = x' + a\sqrt{y'-b} \end{cases} \quad ; \quad T_2^{-1}: \begin{cases} x = \sqrt{y'-b} \\ y = x' - a\sqrt{y'-b} \end{cases}$$

A 2D example

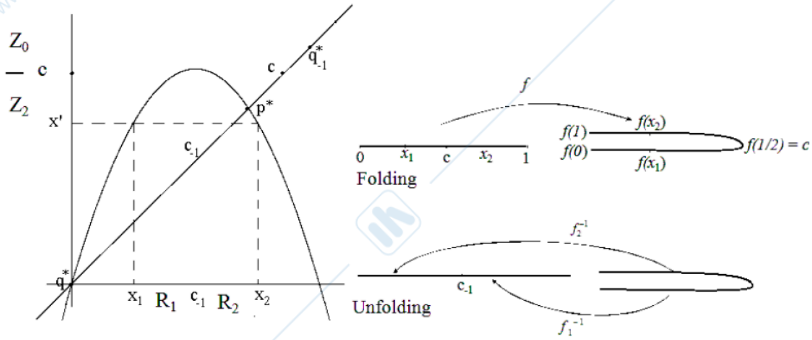


Z_k regions and critical sets

- The state space S of a noninvertible map can be subdivided into regions Z_k , k > o = 0, whose points have k distinct rank-1 preimages.
- Generally, as the point x' varies in R^n, pairs of preimages appear or disappear as it crosses the boundaries separating different regions.
- Such boundaries are characterized by the presence of at least two coincident (merging) preimages.

Critical Set
The critical set CS of a continuous map T is defined as the locus of points having at least two coincident rank-1 preimages, located on a set CS_{-1}, called set of merging preimages.

The action of the map and its inverses

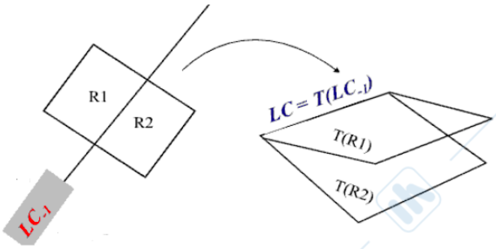


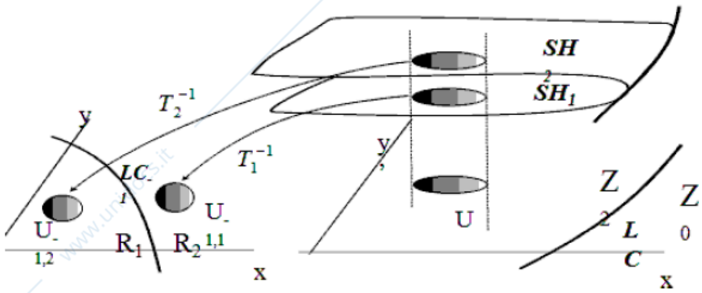
The noninvertible map "folds and pleats" the space S, while the inverses "unfold" S.

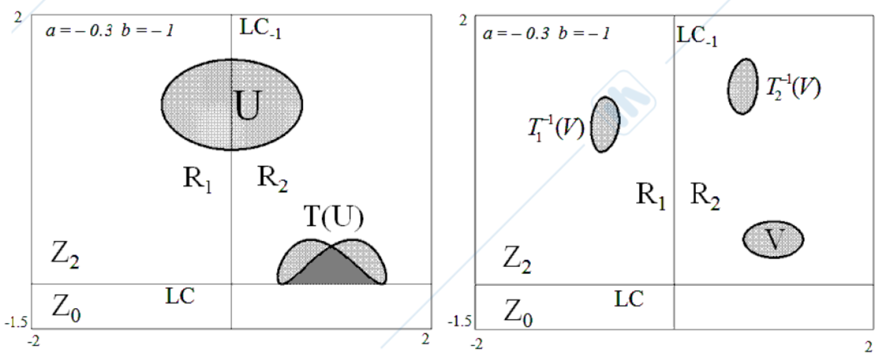
Critical sets

- The critical set CS is the n-dimensional generalization of the notion of local minimum or maximum value of a 1D, while the set CS_{-1} is the generalization of the local extremum point;
- In 2D CS and CS_{-1} they are named critical curve LC and fold curve LC_{-1}, respectively;
- To find points characterized by vanishing derivative can be useful to characterize Z_k regions in 1D, but this is a not sufficient condition.

Focus on the 2D case







Calculus of LC_{-1} for 2D maps

Fold line

For a continuously differentiable map, the fold line LC_{-1} is the locus of points where the Jaobian matrix vanishes. Considering again the map:

$$T : \begin{cases} x' = ax + y \\ y' = x^2 + b \end{cases}$$

we have that:

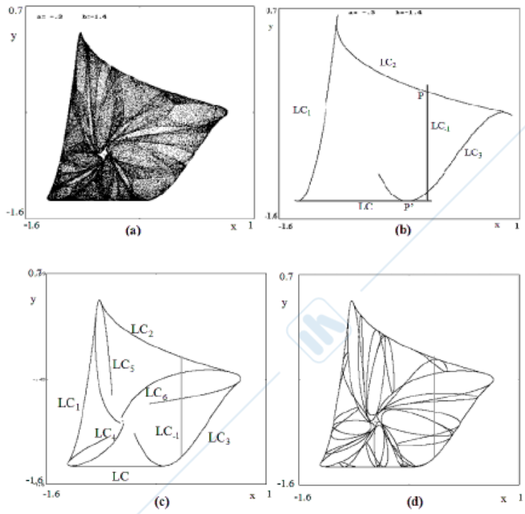
$$J : \begin{bmatrix} a & 1 \\ 2x & 0 \end{bmatrix}$$

that vanishes in LC_{-1}: $x = 0$ from which we obtain $LC : y' = b$.

In fact: $Z_0 = \{(x,y) | y < b\}$ and $Z_2 = \{(x,y) | y > b\}$

Boundaries of trapping regions

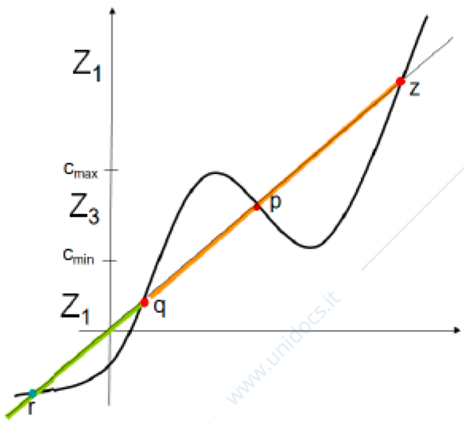
- The critical set CS and its images $CS_k = T^k(CS)$ can be used to obtain the boundaries of trapping regions where the asymptotic dynamics of the iterated points of a noninvertible map are confined.
- If a chaotic attractor exists which fills up a whole absorbing region, then the boundary of the chaotic attractor is formed by portions of critical sets.

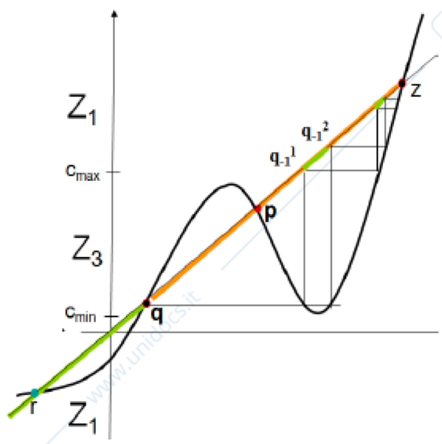


Contact bifurcations

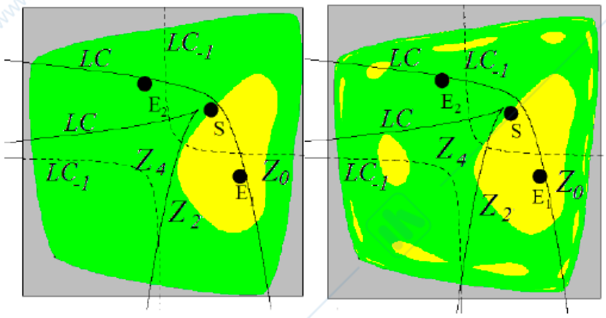
- When several attracting sets coexist the long run dynamics of the system depends on the starting condition;
- Each attracting set is characterized by its own basin of attraction;
- The global dynamical properties of the dynamical system cannot be studied through a linear approximation of the map;
- Global bifurcations cause sudden qualitative changes in the properties of the attracting sets and can be detected by observing contacts of critical curves with invariant sets (contact bifurcations);
- Basin boundaries are invariant sets and when they have a contact with a critical set complicated topological structures of the basins may arise.

A 1D example

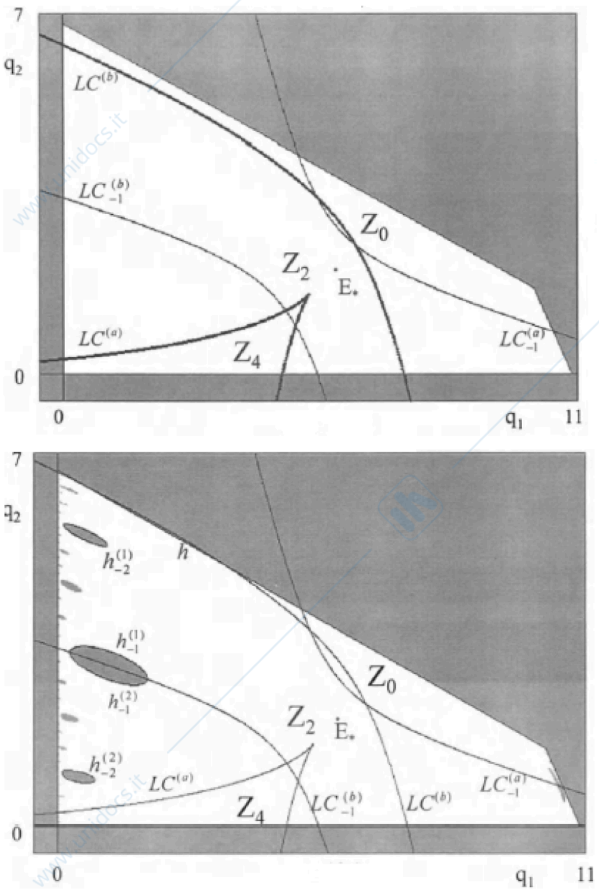




A first 2D example



A second 2D example



PIECEWISE-LINEAR MAPS - BORDER-COLLISION BIFURCATIONS

Piecewise-linear maps

A 1D map x' = f (x) is called Piecewise-defined if it takes different definitions for different subsets of the domain:

$$f(x) = \begin{cases} f_1(x) & \text{for} \quad x \in D_1 \\ f_2(x) & \text{for} \quad x \in D_2 \\ \dots & \\ f_n(x) & \text{for} \quad x \in D_n \end{cases}$$

where:

- $n \in \mathbb{N}$, $n \geq 2$;
- if $f_k(x)$ with $k = 1, \dots, n$ are all linear/affine functions then we talk about a *piecewise-linear map*;
- D_k with $k = 1, \dots, n$ are non-overlapping intervals of real numbers (to each boundary point only one function can be applied);
- D_1 and D_n can be unbounded.

Our restrictions

We will limit our analysis to piecewise-linear maps with only two definitions:

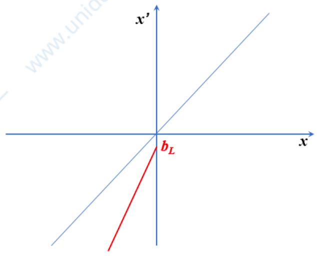
$$x' = f(x) = \begin{cases} a_L x + b_L & \text{for} \quad x < 0 \\ a_R x + b_R & \text{for} \quad x > 0 \end{cases}$$

and in particular we will focus on two subcases:

- $a_L > 0$; $a_R > 0$ (increasing/increasing case);
- $a_L > 0$; $a_R < 0$ (increasing/decreasing case);
- $a_L < 0$; $a_R < 0$ (decreasing/decreasing case).

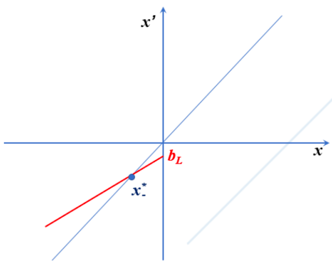
and we look for interesting cases (i.e. oscillations between the two regions).

Subcase 1a: $a_L > 1$, $b_L < 0$



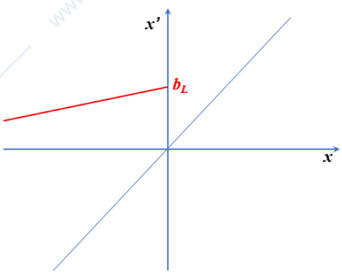
any initial condition $x_0 < 0$ leads to divergence ($-\infty$)

Subcase 2a: $0 < a_L < 1$, $b_L < 0$



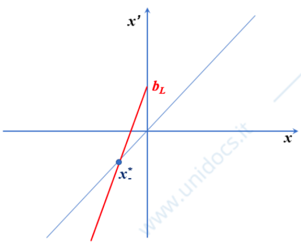
any initial condition $x_0 < 0$ leads to convergence to the equilibrium (x^*_-)

Subcase 3a: $0 < a_L < 1$, $b_L > 0$



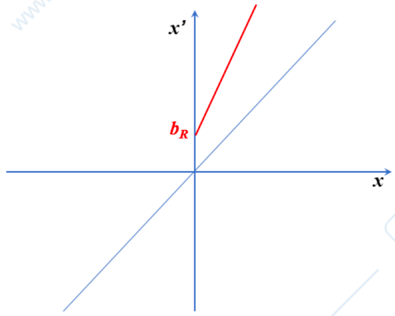
any initial condition $x_0 < 0$ leads to trajectories entering the right region in a finite number of iterations

Subcase 4a: $a_L > 1$, $b_L > 0$



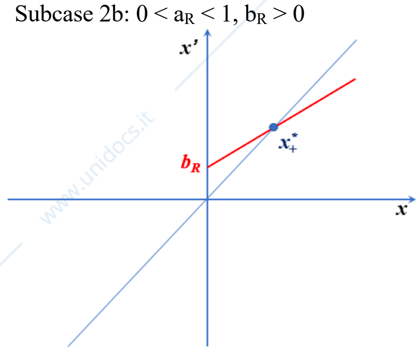
initial conditions $x_0 < x^*_-$ leads to divergence ($-\infty$)
initial conditions $x^*_- < x_0 < 0$ generate trajectories entering the right region in a finite number of iterations

Subcase 1b: $a_R > 1$, $b_R > 0$



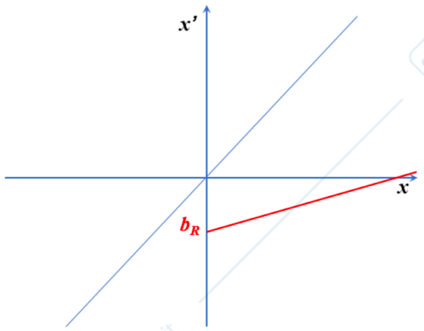
any initial condition $x_0 > 0$ leads to divergence ($+\infty$)

Subcase 2b: $0 < a_R < 1$, $b_R > 0$



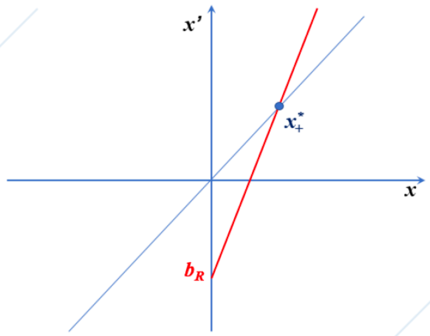
any initial condition $x_0 > 0$ leads to convergence to the equilibrium (x^*_+)

Subcase 3b: $0 < a_R < 1$, $b_R < 0$



any initial condition $x_0 > 0$ leads to trajectories entering the left region in a finite number of iterations

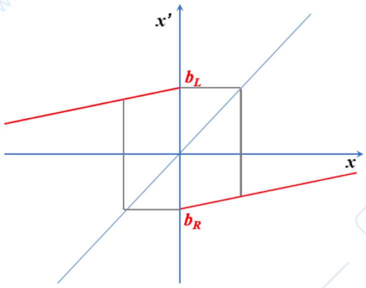
Subcase 4b: $a_R > 1$, $b_R < 0$



initial conditions $x_0 > x^*_+$) leads to divergence ($+\infty$)
initial conditions $0 < x_0 < x^*_+$) generate trajectories entering the left region in a finite number of iterations

Summarizing
The interesting subcases are 3a, 4a, 3b, 4b
Interesting subcases of the increasing/increasing case
In the four cases characterized by $b_L > 0$ and $b_R < 0$ trajectories may oscillate between the two regions.

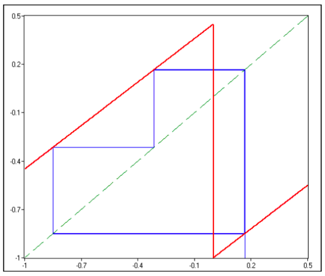
The absorbing interval



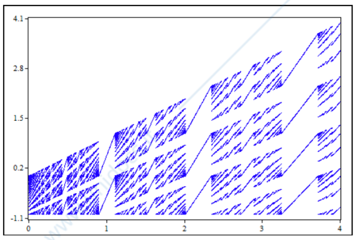
Absorbing interval
Dynamics are bounded in the interval $[b_R;b_L]$

Dynamics
- If $0 < a_L < 1$ and $0 < a_R < 1$ then dynamics are periodic (any cycle is stable);
- If $a_L > 1$ and $a_R > 1$ then dynamics are chaotic (any cycle is unstable);
- If $0 < a_L < 1$ and $a_R > 1$ (or the opposite) then dynamics can be both periodic or chaotic (cycles can be both stable and unstable)

Subcase $0 < a_L < 1$ and $0 < a_R < 1$
In this case we may only have periodic cycles, that are necessarily stable;



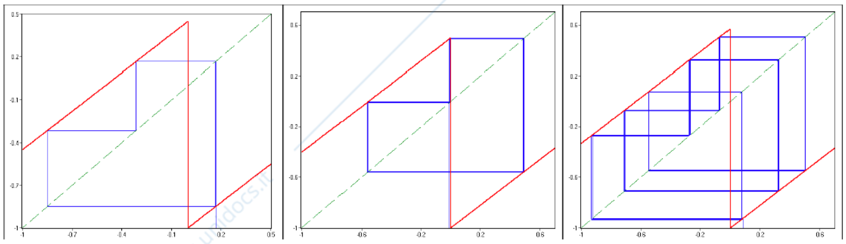
Subcase $0 < a_L < 1$ and $0 < a_R < 1$
The bifurcation diagram:



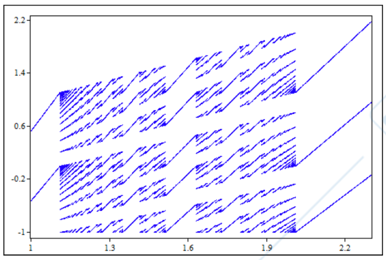
But, which kind of bifurcation occurs passing from one cycle to another one?

Border-Collision Bifurcations
Piecewise defined maps $f(x;\ \mu)$ are characterized by this new kind of bifurcation:
BCB
If a cycle C of the map f which exists for $\mu < \mu_0$ has one point that at $\mu = \mu_0$ collides with a border point (in our case $x = 0$) then a Border-Collision bifurcation occurs and for $\mu > \mu_0$ a different stable cycle exists.

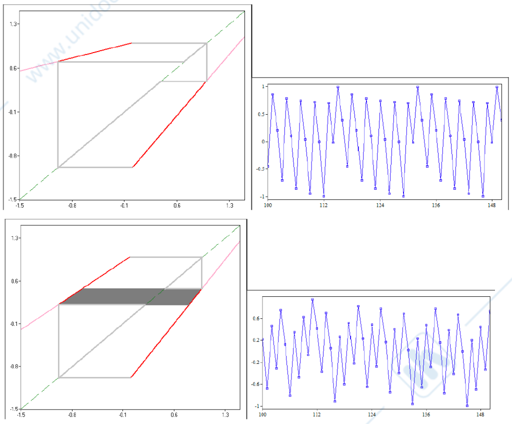
Border-Collision Bifurcations
Piecewise defined maps $f(x;\ \mu)$ are characterized by this new kind of bifurcation:



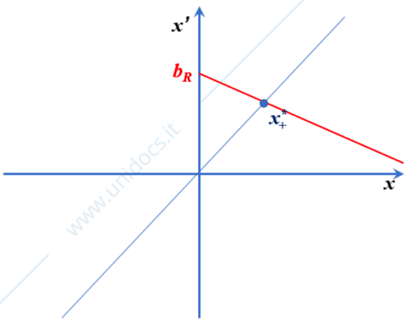
Period-adding structure
Increasing/increasing maps are characterzed by period cycles organized through the so-called period-adding structure:
Period-Adding
considering the parameters'space, between a region characterzing of periodic cycle of period k and another region characterizing a periodic cycle of period m, there exists a region characterizing a periodic cycle of period k +m.



From periodic to chaotic motion
By increasing the value of one of the two slopes (or both) dynamics turn from periodic to chaotic
Chaotic dynamics
Dynamics are periodic until the absorbing region of the map is noninvertible, then dynamics become chaotic.
Dynamics are for sure chaotic if $a_L > 1$ and $a_R > 1$.

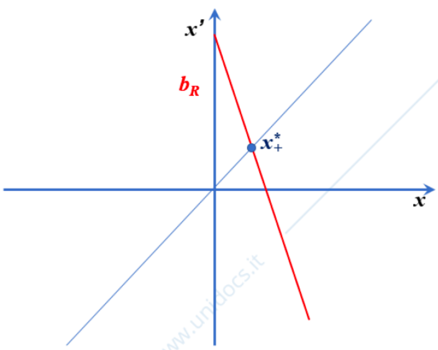


Subcase 1c: $-1 < a_R < 0$, $b_R > 0$



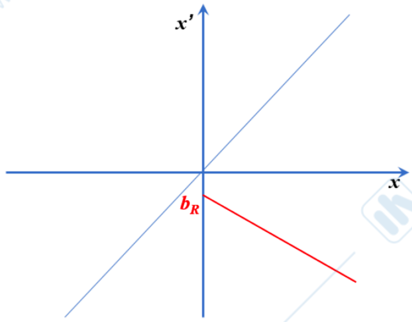
Initial conditions sufficiently close to x^*_0 generate converging trajectories with oscillations.

Subcase 2c: $a_R < -1$, $b_R > 0$



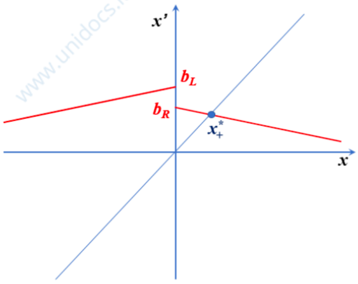
any initial condition $x_0 > 0$ leads, through oscillations, to the left side.

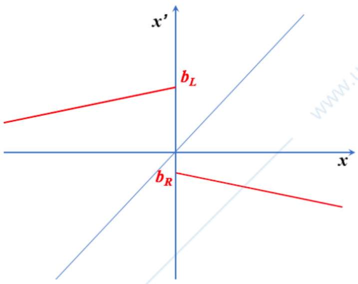
Subcase 3c: $a_R < 0$, $b_R < 0$



any initial condition $x_0 > 0$ leads to trajectories entering the left region in a single iteration

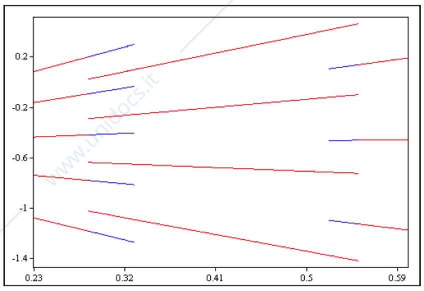
Summarizing
Interesting subcases of the increasing/decreasing case
The cases characterized by $b_L > 0$ and $b_R > 0$ or $b_L > 0$ and $b_R < 0$ may generate trajectories oscillating between the two regions.



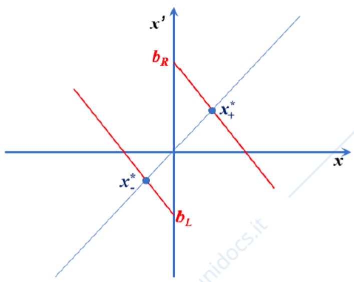


Period-incrementing structure
Increasing/decreasing maps are characterized by period cycles organized through the so-called period-incrementing structure:
Period-Incrementing
considering the parameters'space, close to a region characterzing a periodic cycle of period k, there is another region characterizing a periodic cycle of period k +v and then another with a cycle of period k +2v, and so on. Between two consecutive regions there is a portion of the parameter space where the two cycles coexist.

Period-incrementing structure
$a_L = 0{:}9$, $a_R = -0{:}9$, $b_R = -1$



Interesting cases
Combining the results of the previous subcases, we are interested in the following scenarios:
$b_L < 0$ and $b_R > 0$



EXAMPLES OF PIECEWISE-LINEAR MODELS

Financial market
Let us consider the market of a single asset.
The makert is populated by five types of agent:

- Fundamentalists traders of type 1, whose orders depend on the amount of mispricing;
- Fundamentalists traders of type 2, whose orders do not depend upon the amount of mispricing;
- Chartists (or Technical) of type 1, whose orders depend on the amount of mispricing;
- Chartists of type 2, whose orders do not depend upon the amount of mispricing;
- A market maker, who regulates the price according to the total excess demand.

The fundamental value of the asset is exogenously given and known by all the agents: P = F

Fundamentalists of type 1
Fundamentalists believe in an immediate correction of misprincing, that is a price different from its fundamental value. Their reactivity is asymmetric between positve and negative mispricings.
As a consequence, their excess demand is given by:
Fundamentalists trading rule

$$D_t^{F,1} = \begin{cases} f^{1,a}(F-P(t)) & \text{if} \quad F-P(t)>0 \\ f^{1,b}(F-P(t)) & \text{if} \quad F-P(t)<0 \end{cases}$$

where $f^{1,a}>0$ and $f^{1,b}>0$ are the asymmetric *speeds of reaction*.

Fundamentalists of type 2
Fundamentalists believe in an immediate correction of misprincing, that is a price different from its fundamental value. Their reactivity is asymmetric between positve and negative mispricings.
As a consequence, their excess demand is given by:
Fundamentalists trading rule

$$D_t^{F,1} = \begin{cases} f^{2,a} & \text{if} \quad F-P(t)>0 \\ f^{2,b} & \text{if} \quad F-P(t)<0 \end{cases}$$

where $f^{2,a}>0$ and $f^{2,b}>0$ are the asymmetric *fixed orders*.

Chartists of type 1
Chartists, at the opposite, believe in the short run persistence of optimism/pessimism, identified as a high (resp. low) price. Their orders are proportional to the amount of the mispricing.
As a consequence, their excess demand is given by:
Chartists trading rule

$$D_t^{C,1} = \begin{cases} c^{1,a}(P(t)-F) & \text{if} \quad P(t)-F>0 \\ c^{1,b}(P(t)-F) & \text{if} \quad P(t)-F<0 \end{cases}$$

where $c^{1,a}>0$ and $c^{1,b}>0$ are the asymmetric *speeds of reaction*.

Chartists of type 2
Chartists, at the opposite, believe in the short run persistence of optimism/pessimism, identified as an high (resp. low) price. Their orders are fixed.
As a consequence, their excess demand is given by:
Chartists trading rule

$$D_t^{C,2} = \begin{cases} c^{2,a} & \text{if} \quad P(t)-F>0 \\ c^{2,b} & \text{if} \quad P(t)-F<0 \end{cases}$$

where $c^{2,a}>0$ and $c^{2,b}>0$ are the asymmetric *fixed orders*.

The market maker
The market maker adjusts the asset price according to the rule: P(t +1) = P(t)+D(t) where $D(t) = D^{F;1}(t)+D^{F;2}(t)+D^{C;1}(t)+D^{C;2}(t)$ is the total demand excess.
By substituting the trading rules of the four types of investors we get,any by using the auxiliary variable x(t) = P(t)-F:

$$x(t+1) = \begin{cases} s_R x(t)+m_R & \text{if} \quad x(t)>0 \\ s_L x(t)+m_L & \text{if} \quad x(t)<0 \end{cases}$$

where $s_R = 1+c^{1,a}-f^{1,b}$, $s_L = 1+c^{1,b}-f^{1,a}$, $m_R = c^{2,a}-f^{2,b}$ and $m_L = f^{2,a}-c^{2,b}$

This is a piecewise-linear map with slopes and intercepts that can be of any kind.

The Cobweb model
We consider the price formation in a partial market of a single commodity.
The quantity demanded by consumers is a decreasing function of the price (demand function): $Q^d(t) = D(p(t))$
Producers decide the amount of output on the basis of an increasing function of price (supply function): $Q^s(t) = S(p^e(t))$

where pe(t) represents the expected price for time time when they must decide how much to produce.
In equilibrium we have: $Q^d(t) = Q^s(t)$

Prospect Theory and Consumer Choice
Richard Thaler (1985) is one of the first economists who tried to incorporate features from Prospect Theory into a Theory of the Consumer Choice.
In particular Thaler postulates that from a transaction two kinds of utility may derive:

1. Acquisition Utility: it is the classical utility dependent upon the value of the good received compared to the outlay;
2. Transaction Utility: It depends on the perceived merits of the "deal".

Transaction Utility
In order to define the Transaction Utility we need to introduce the so-called Reference Price.

Reference Price
A Reference Price is a price that the consumer considers fair.
It may depends upon past transactions, comparisons with similar products or considerations upon the costs of production.
Consumers compare actual price (p) and reference price (p*):
If the price is higher than the expected price than the consumer derives a disutility from the transaction. At the opposite, if the price is lower than the expect price, the transaction increases its value in the consumers' eyes.

A Production saturation effect dynamics
We assume static expectations (i.e. $p^e(t) = p(t-1)$), so we have: D(p(t)) = S(p(t □1))
We still assume a simple linear supply function, such as: S(p) = cp   where c > 0 is a positive parameter.
We consider a demand made up by two components, reflecting
Thaler's dichotomy (Acquisition and Transaction Utility):

$$D(p(t)) = \begin{cases} a - bp(t) + \omega\alpha_1(P^* - p(t)) & \text{if} \quad p(t) < P^* \\ a - bp(t) + \omega\alpha_2(P^* - p(t)) & \text{if} \quad p(t) > P^* \end{cases} \qquad (1)$$

P^* is the *exogenously given reference price*. $\omega \geq 0$ measures the importance of the Transaction Utility component, while $0 < \alpha_1 < \alpha_2$ permit to introduce *loss aversion* (in particular the ratio $\alpha_2/\alpha_1 > 1$ represents a measure of loss aversion).

The piecewise-linear map
The map is now the following:

$$p' = \begin{cases} \frac{a+\omega\alpha_1 P^*}{b+\omega\alpha_1} - \frac{c}{b+\omega\alpha_1} p & \text{if} \quad p < P^* \\ \frac{a+\omega\alpha_2 P^*}{b+\omega\alpha_2} - \frac{c}{b+\omega\alpha_2} p & \text{if} \quad p > P^* \end{cases}$$

and it is a piecewise-linear decreasing/decreasing map.

Results

Classic Cobweb

- The equilibrium is given by:

$$\bar{P}_{cc} = \frac{a}{b+c} \qquad (2)$$

- The equilibrium is stable provided that

$$c < b \qquad (3)$$

Behavioral Cobweb

- The equilibrium is given by:

$$\bar{P}_{bc} = \frac{a + \omega\alpha_i P^*}{b + c + \omega\alpha_i} \qquad (4)$$

with $i = 1$ or 2.

- The equilibrium is stable provided that

$$c < b + \omega\alpha_i \qquad (5)$$

Equilibrium Price

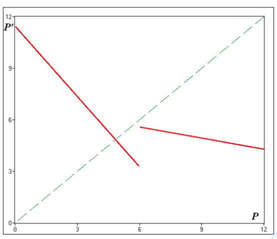
if $P^* > \frac{a}{b+c}$ then $\bar{P}_{bc} = \frac{a+\omega\alpha_1 P^*}{b+c+\omega\alpha_1}$ and $\bar{P}_{cc} < \bar{P}_{bc} < P^*$

otherwise

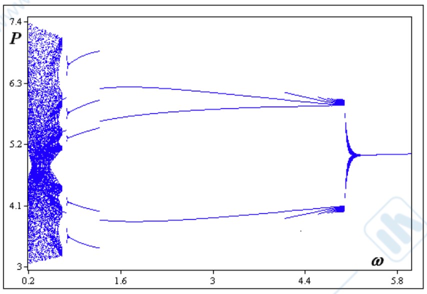
if $P^* < \frac{a}{b+c}$ then $\bar{P}_{bc} = \frac{a+\omega\alpha_2 P^*}{b+c+\omega\alpha_2}$ and $P^* < \bar{P}_{bc} < \bar{P}_{cc}$

The reference price adjusts the equilibrium price towards it.

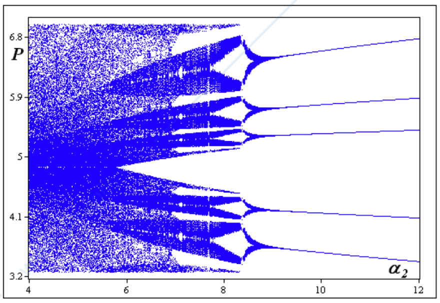
The map
a = 24, b = 2, c = 3, w = 1, P* = 6, a_1 = 0:2, a_2 = 12



The role of the behavioral component



The role of loss aversion



Summary
The behavioral component has a stabilizing effect;
The more consumers are loss averse, the more stable are the trajectories.
The equilibrium price is inuenced by the existence of a reference price.