

EXERCISES WITH SOLUTIONS

EXERCISE 1

Consider the following dynamical system in discrete time:

$$\begin{cases} x'_1 = 0.42x_1 - 0.04x_2 \\ x'_2 = -0.04x_1 + 0.48x_2 \end{cases} \quad (1)$$

Find its fixed points and study their stability properties.

SOLUTION

The given dynamical system is a 2D linear system in discrete time. Hence, it admits only one fixed point. Since the system is homogeneous, this fixed point is the point $E_0(0,0)$.

To study the stability properties of E_0 , we rewrite the dynamical system in the matrix form

$$\mathbf{x}' = A\mathbf{x}$$

with

$$A = \begin{pmatrix} 0.42 & -0.04 \\ -0.04 & 0.48 \end{pmatrix}.$$

The trace and the determinant of matrix A are

$$\text{tr}A = 0.9 \quad \text{and} \quad \det A = 0.2.$$

Determine stability: method 1

The stability conditions read as

$$\begin{aligned} 1 - \text{tr}A + \det A &= 1 - 0.9 + 0.2 = 0.3 > 0, \\ 1 + \text{tr}A + \det A &= 1 + 0.9 + 0.2 = 2.1 > 0, \\ 1 - \det A &= 1 - 0.2 = 0.8 > 0. \end{aligned}$$

All three conditions are satisfied, which means that the fixed point E_0 is stable. It can be either a stable node or a stable focus. To determine which one of the two, we compute the discriminant

$$\Delta = (\text{tr}A)^2 - 4 \det A = 0.9^2 - 4 \cdot 0.2 = 0.01.$$

The discriminant is positive, and hence, the fixed point E_0 is a stable node.

Determine stability: method 2

Let us determine the stability of E_0 by using the stability triangle (see Fig. 1). The point related to the pair $(\text{tr}A, \det A)$ is located inside the stability triangle, which means that the fixed point E_0 is stable. It can be either a stable node or a stable focus. To determine which one of the two, we compute the discriminant

$$\Delta = (\text{tr}A)^2 - 4 \det A = 0.9^2 - 4 \cdot 0.2 = 0.01.$$

The discriminant is positive, and hence, the fixed point E_0 is a stable node.

Determine stability: method 3

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - \text{tr}A\lambda + \det A = \lambda^2 - 0.9\lambda + 0.2 = 0.$$

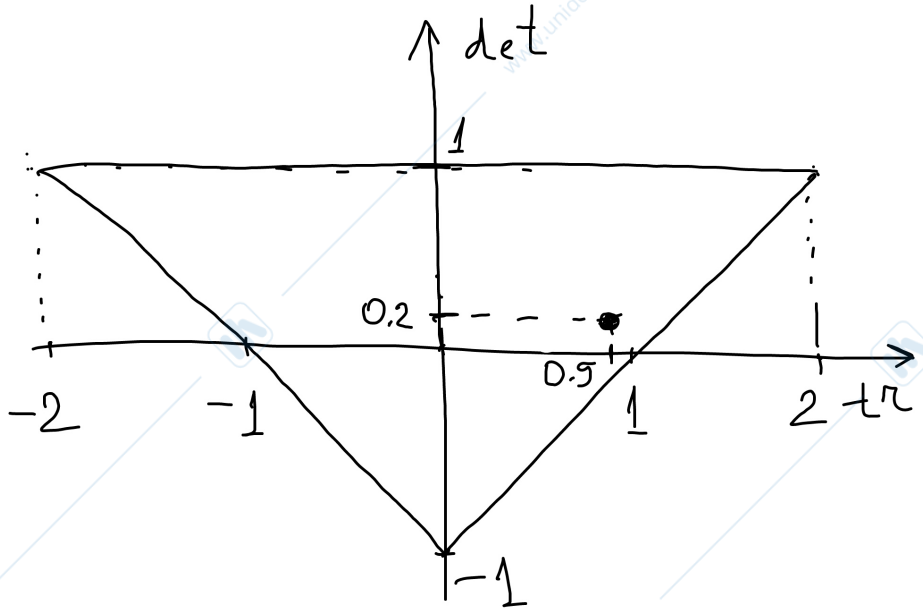


Figure 1: Stability triangle

Solving it, we get

$$\lambda_{1,2} = \frac{0.9 \pm \sqrt{0.9^2 - 4 \cdot 0.2}}{2} \Rightarrow \lambda_1 = 0.4, \lambda_2 = 0.5.$$

The fixed point E_0 has two real distinct eigenvalues, both located inside the unit circle. Hence, the point E_0 is a stable node.

EXERCISE 2

Consider the dynamical system in discrete time of the form (1) (from the Exercise 1). Compute the eigenvectors for its fixed point $E_0(0,0)$, plot in the phase plane the respective eigenspaces and sketch at least one typical orbit.

SOLUTION

Let us first compute the eigenvalues of the fixed point $E_0(0,0)$. For this, we rewrite the dynamical system in the matrix form

$$\mathbf{x}' = A\mathbf{x}$$

with

$$A = \begin{pmatrix} 0.42 & -0.04 \\ -0.04 & 0.48 \end{pmatrix}.$$

The trace and the determinant of matrix A are

$$\text{tr}A = 0.9 \quad \text{and} \quad \det A = 0.2.$$

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - \text{tr}A\lambda + \det A = \lambda^2 - 0.9\lambda + 0.2 = 0.$$

Solving it, we get

$$\lambda_{1,2} = \frac{0.9 \pm \sqrt{0.9^2 - 4 \cdot 0.2}}{2} = 0 \Rightarrow \lambda_1 = 0.4, \lambda_2 = 0.5.$$

For computing the eigenvectors of E_0 , for each λ_i , $i = 1, 2$ we will solve the equation

$$(A - \lambda_i \mathbf{I}) \mathbf{v}_i = \mathbf{0} \Leftrightarrow \begin{pmatrix} 0.42 - \lambda_i & -0.04 \\ -0.04 & 0.48 - \lambda_i \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\mathbf{v}_i = (v_{i1}, v_{i2})$ is the eigenvector corresponding to λ_i . The equation written above is the system of dependent linear equations, which admits infinite number of solutions. Without losing generality, we can set $v_{i1} = 1$, and solve either the first or the second equation with respect to v_{i2} .

For $\lambda_1 = 0.4$, we get

$$v_{11} = 1, (0.42 - 0.4) - 0.04v_{12} = 0 \Rightarrow v_{12} = \frac{0.02}{0.04} = 0.5.$$

The eigenvector related to $\lambda_1 = 0.4$ is $\mathbf{v}_1 = (1, 0.5)$.

Similarly for $\lambda_2 = 0.5$, we get

$$v_{21} = 1, (0.42 - 0.5) - 0.04v_{22} = 0 \Rightarrow v_{22} = -\frac{0.08}{0.04} = -2.$$

The eigenvector related to $\lambda_2 = 0.5$ is $\mathbf{v}_2 = (1, -2)$.

The eigenvalues of E_0 are both real, positive and located inside the unit circle. It means that the fixed point is a stable node and a typical orbit converges monotonously to E_0 (see Fig. 2).

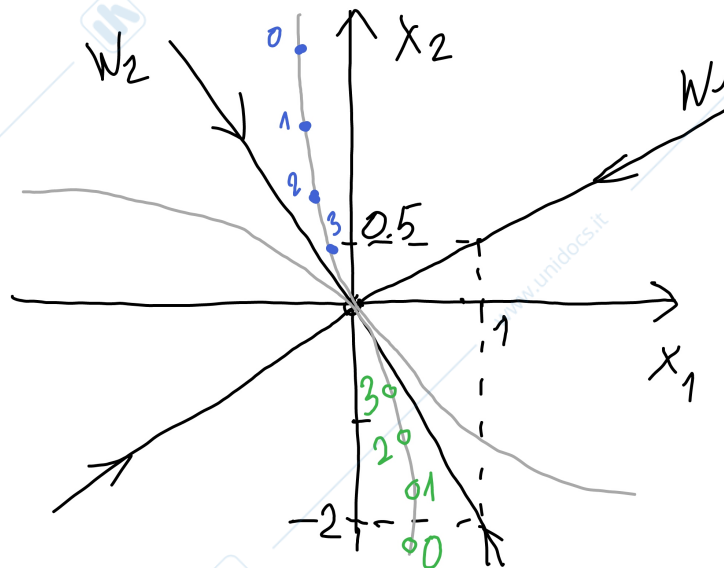


Figure 2: Two typical orbits in the phase space.

EXERCISE 3

Consider the following dynamical system in discrete time:

$$\begin{cases} x_1' = 2.5x_1 + x_2 + 0.5 \\ x_2' = 1.5x_1 + 2.5 \end{cases} \quad (2)$$

Find its fixed points, plot the stability triangle and mark on it the point corresponding to the current system.

SOLUTION

The given dynamical system is a 2D linear system in discrete time. Hence, it admits only one fixed point. The system is nonhomogeneous and to find its fixed point we use the equilibrium conditions:

$$\begin{cases} x_1 = 2.5x_1 + x_2 + 0.5 \\ x_2 = 1.5x_1 + 2.5 \end{cases} \Leftrightarrow \begin{cases} x_1 = 4x_1 + 3 \\ x_2 = 1.5x_1 + 2.5 \end{cases} \Leftrightarrow \begin{cases} x_1 = -1 \\ x_2 = 1 \end{cases}$$

So, the system (2) has the fixed point $E(-1, 1)$.

The matrix of the system is

$$A = \begin{pmatrix} 2.5 & 1 \\ 1.5 & 0 \end{pmatrix}.$$

with the trace and determinant $\text{tr}A = 2.5$, $\det A = -1.5$. The stability triangle is shown in Fig. 3

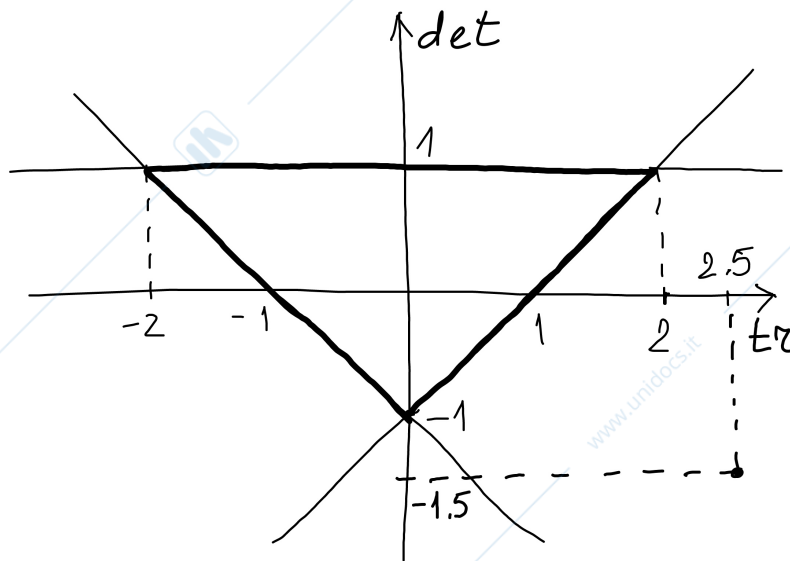


Figure 3: Stability triangle.

EXERCISE 4

Consider the dynamical system in discrete time of the form (2).

Find its fixed points, plot in the phase plane the respective eigenspaces and sketch at least one typical orbit.

SOLUTION

The given dynamical system is a 2D linear system in discrete time. Hence, it admits only one fixed point. The system is nonhomogeneous and to find its fixed point we use the equilibrium conditions:

$$\begin{cases} x_1 = 2.5x_1 + x_2 + 0.5 \\ x_2 = 1.5x_1 + 2.5 \end{cases} \Leftrightarrow \begin{cases} x_1 = 4x_1 + 3 \\ x_2 = 1.5x_1 + 2.5 \end{cases} \Leftrightarrow \begin{cases} x_1 = -1 \\ x_2 = 1 \end{cases}$$

So, the system (2) has the fixed point $E(-1, 1)$.

Let us compute the eigenvalues of E . For this, we rewrite the dynamical system in the matrix form

$$\mathbf{x}' = A\mathbf{x}$$

with

$$A = \begin{pmatrix} 2.5 & 1 \\ 1.5 & 0 \end{pmatrix}.$$

The trace and the determinant of matrix A are

$$\text{tr}A = 2.5 \quad \text{and} \quad \det A = -1.5.$$

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - \text{tr}A\lambda + \det A = \lambda^2 - 2.5\lambda - 1.5 = 0.$$

Solving it, we get

$$\lambda_{1,2} = \frac{2.5 \pm \sqrt{2.5^2 + 4 \cdot 1.5}}{2} \Rightarrow \lambda_1 = 3, \lambda_2 = -0.5.$$

For computing the eigenvectors of E , for each λ_i , $i = 1, 2$ we will solve the equation

$$(A - \lambda_i \mathbf{I}) \mathbf{v}_i = \mathbf{0} \Leftrightarrow \begin{pmatrix} 2.5 - \lambda_i & 1 \\ 1.5 & -\lambda_i \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\mathbf{v}_i = (v_{i1}, v_{i2})$ is the eigenvector corresponding to λ_i . The equation written above is the system of dependent linear equations, which admits infinite number of solutions. Without losing generality, we can set $v_{i1} = 1$, and solve either the first or the second equation with respect to v_{i2} .

For $\lambda_1 = 3$, we get

$$v_{11} = 1, (2.5 - 3) + v_{12} = 0 \Rightarrow v_{12} = 0.5.$$

The eigenvector related to $\lambda_1 = 3$ is $\mathbf{v}_1 = (1, 0.5)$.

Similarly for $\lambda_2 = -0.5$, we get

$$v_{21} = 1, (2.5 + 0.5) + v_{22} = 0 \Rightarrow v_{22} = -3.$$

The eigenvector related to $\lambda_2 = -0.5$ is $\mathbf{v}_2 = (1, -3)$.

The eigenvalues of E are both real, λ_1 being positive and located outside the unit circle, the other λ_2 being negative and located inside the unit circle. It means that the fixed point is a saddle and a typical orbit is monotonous with respect to the eigenspace W_1 and oscillatory with respect to the eigenspace W_2 (see Fig. 4).

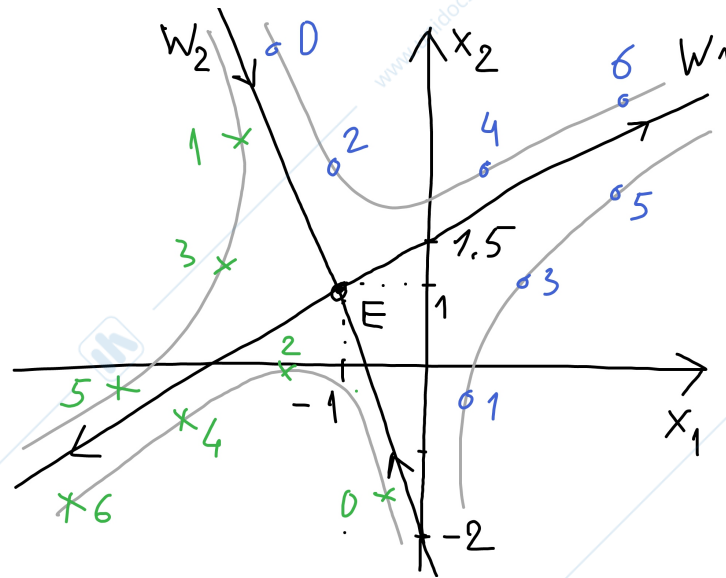


Figure 4: Two typical orbits in the phase space.

EXERCISE 5

Consider the following dynamical system in discrete time:

$$\begin{cases} x'_1 = -1.75x_1 - 0.25x_2 + 2.75 \\ x'_2 = -0.25x_1 - 1.75x_2 + 0.25 \end{cases} \quad (3)$$

Find its fixed points and study their stability properties.

SOLUTION

The given dynamical system is a 2D linear system in discrete time. Hence, it admits only one fixed point. The system is nonhomogeneous and to find its fixed point we use the equilibrium conditions:

$$\begin{cases} x_1 = -1.75x_1 - 0.25x_2 + 2.75 \\ x_2 = -0.25x_1 - 1.75x_2 + 0.25 \end{cases} \Leftrightarrow \begin{cases} x_1 = -0.25 \frac{-0.25x_1 + 0.25}{2.75} + 2.75 \\ x_2 = \frac{-0.25x_1 + 0.25}{2.75} \end{cases} \Leftrightarrow \begin{cases} x_1 = 1 \\ x_2 = 0 \end{cases}$$

So, the system (2) has the fixed point $E(1, 0)$.

The matrix of the system is

$$A = \begin{pmatrix} -1.75 & -0.25 \\ -0.25 & -1.75 \end{pmatrix}.$$

with the trace and determinant $\text{tr}A = -3.5$, $\det A = 3$.

Determine stability: method 1

The stability conditions read as

$$\begin{aligned} 1 - \text{tr}A + \det A &= 1 + 3.5 + 3 = 7.5 > 0, \\ 1 + \text{tr}A + \det A &= 1 - 3.5 + 3 = 0.5 > 0, \\ 1 - \det A &= 1 - 3 = -2 < 0. \end{aligned}$$

The third condition is violated, which means that the fixed point E is unstable. Taking into account that the first and the second conditions hold, E can be either an unstable node or an unstable focus. To determine which one of the two, we compute the discriminant

$$\Delta = (\operatorname{tr}A)^2 - 4 \det A = 3.5^2 - 4 \cdot 3 = 0.25.$$

The discriminant is positive, and hence, the fixed point E is an unstable node.

Determine stability: method 2

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - \operatorname{tr}A\lambda + \det A = \lambda^2 + 3.5\lambda + 3 = 0.$$

Solving it, we get

$$\lambda_{1,2} = \frac{-3.5 \pm \sqrt{3.5^2 - 4 \cdot 3}}{2} \Rightarrow \lambda_1 = -2, \lambda_2 = -1.5.$$

The fixed point E has two real distinct eigenvalues, both located outside the unit circle. Hence, the point E is an unstable node.

EXERCISE 6

Consider the dynamical system in discrete time of the form (3).

Find its fixed points, plot in the phase plane the respective eigenspaces and sketch at least one typical orbit.

SOLUTION

The given dynamical system is a 2D linear system in discrete time. Hence, it admits only one fixed point. The system is nonhomogeneous and to find its fixed point we use the equilibrium conditions:

$$\begin{cases} x_1 = -1.75x_1 - 0.25x_2 + 2.75 \\ x_2 = -0.25x_1 - 1.75x_2 + 0.25 \end{cases} \Leftrightarrow \begin{cases} x_1 = -0.25 \frac{-0.25x_1 + 0.25}{2.75} + 2.75 \\ x_2 = \frac{-0.25x_1 + 0.25}{2.75} \end{cases} \Leftrightarrow \begin{cases} x_1 = 1 \\ x_2 = 0 \end{cases}$$

So, the system (3) has the fixed point $E(1, 0)$.

The matrix of the system is

$$A = \begin{pmatrix} -1.75 & -0.25 \\ -0.25 & -1.75 \end{pmatrix}.$$

with the trace and determinant $\operatorname{tr}A = -3.5$, $\det A = 3$.

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - \operatorname{tr}A\lambda + \det A = \lambda^2 + 3.5\lambda + 3 = 0.$$

Solving it, we get

$$\lambda_{1,2} = \frac{-3.5 \pm \sqrt{3.5^2 - 4 \cdot 3}}{2} \Rightarrow \lambda_1 = -2, \lambda_2 = -1.5.$$

The fixed point E has two real distinct eigenvalues, both located outside the unit circle. Hence, the point E is an unstable node.

For computing the eigenvectors of E , for each λ_i , $i = 1, 2$ we will solve the equation

$$(A - \lambda_i \mathbf{I}) \mathbf{v}_i = \mathbf{0} \Leftrightarrow \begin{pmatrix} -1.75 - \lambda_i & -0.25 \\ -0.25 & -1.75 - \lambda_i \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $\mathbf{v}_i = (v_{i1}, v_{i2})$ is the eigenvector corresponding to λ_i . The equation written above is the system of dependent linear equations, which admits infinite number of solutions. Without losing generality, we can set $v_{i1} = 1$, and solve either the first or the second equation with respect to v_{i2} .

For $\lambda_1 = -2$, we get

$$v_{11} = 1, (-1.75 + 2) - 0.25v_{12} = 0 \Rightarrow v_{12} = 1.$$

The eigenvector related to $\lambda_1 = -2$ is $\mathbf{v}_1 = (1, 1)$.

Similarly for $\lambda_2 = -1.5$, we get

$$v_{21} = 1, (-1.75 + 1.5) - 0.25v_{22} = 0 \Rightarrow v_{22} = -1.$$

The eigenvector related to $\lambda_2 = -1.5$ is $\mathbf{v}_2 = (1, -1)$.

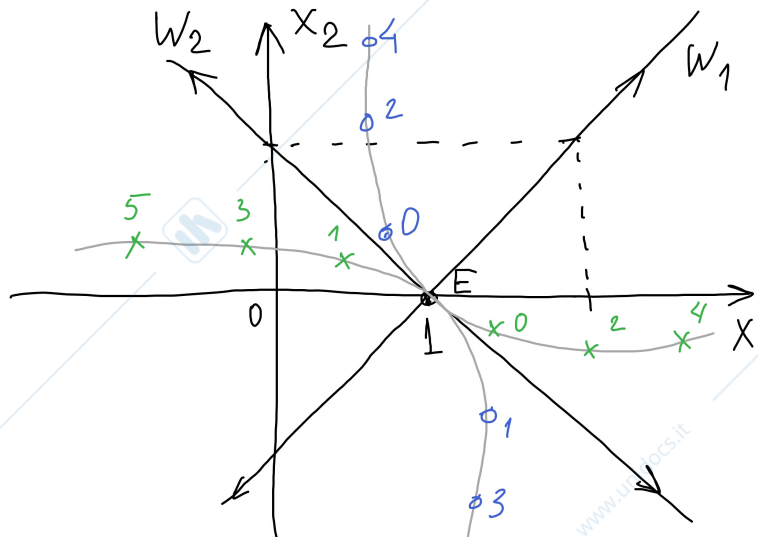


Figure 5: Two typical orbits in the phase space.

The eigenvalues of E are both real, negative and located outside the unit circle. It means that the fixed point is an unstable node and a typical orbit is oscillatory with respect to both eigenspaces, W_1 and W_2 (see Fig. 5).

EXERCISE 7

Consider the following dynamical system in discrete time:

$$\begin{cases} x'_1 = x_1 + x_2 \\ x'_2 = -x_1 + x_2 \end{cases} \quad (4)$$

Find its fixed points and compute their eigenvalues. Draw the unit circle with showing these eigenvalues on it. Define the types of the determined fixed points.

SOLUTION

The given dynamical system is a 2D linear system in discrete time. Hence, it admits only one fixed point. Since the system is homogeneous, this fixed point is the point $E_0(0,0)$.

To study the stability properties of E_0 , we rewrite the dynamical system in the matrix form

$$\mathbf{x}' = A\mathbf{x}$$

with

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The trace and the determinant of matrix A are

$$\text{tr}A = 2 \quad \text{and} \quad \det A = 2.$$

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - \text{tr}A\lambda + \det A = \lambda^2 - 2\lambda + 2 = 0.$$

Solving it, we get

$$\lambda_{1,2} = \frac{2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

The fixed point E has two complex conjugate eigenvalues, which are plotted on the unit circle in Fig. 6. From the plot we see that the eigenvalues are located outside the unit circle, and hence, the

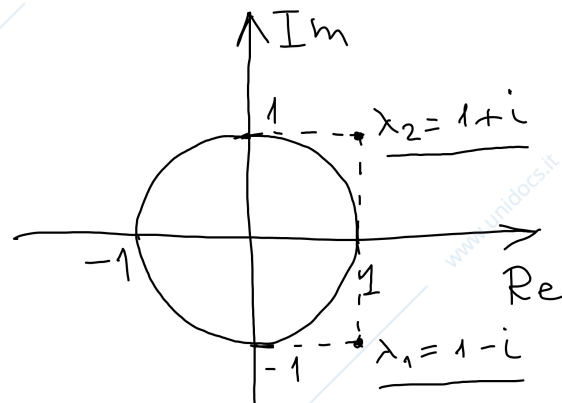


Figure 6: Unit circle and the eigenvalues.

point E is an unstable focus.

EXERCISE 8

Consider the dynamical system in discrete time of the form (4). Sketch in the phase plane its typical orbit.

SOLUTION

The given dynamical system is a 2D linear system in discrete time. Hence, it admits only one fixed point. Since the system is homogeneous, this fixed point is the point $E_0(0, 0)$.

To sketch a typical orbit we need determine the type of the fixed point E_0 . For this we rewrite the dynamical system in the matrix form

$$\mathbf{x}' = A\mathbf{x}$$

with

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The trace and the determinant of matrix A are

$$\text{tr}A = 2 \quad \text{and} \quad \det A = 2.$$

To determine the type of the fixed point we use the stability triangle (see Fig. 7). The related point (tr, \det) is located outside the triangle, above the parabola related to the zero discriminant $\Delta = \text{tr}^2 - 4\det = 0$. Hence, the fixed point E_0 is an unstable focus.

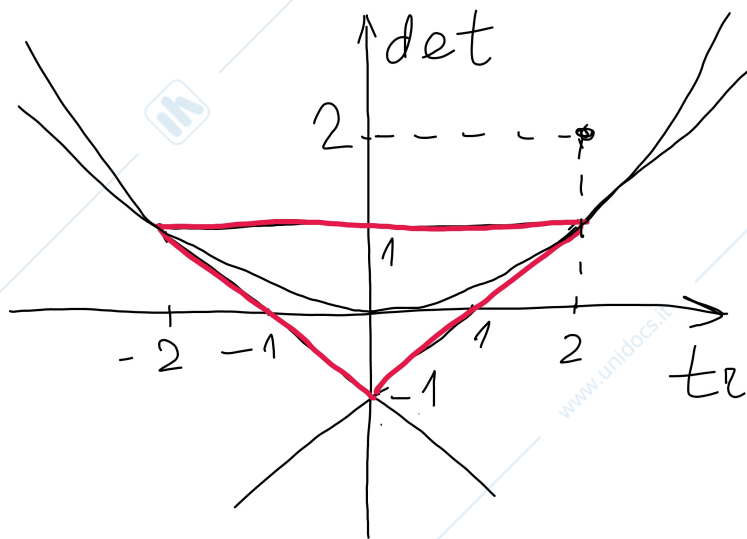


Figure 7: Stability triangle.

The final thing to determine is whether orbits roll around E_0 clockwise or counterclockwise. Let us take an arbitrary initial condition, say, $\mathbf{x}_0 = (0, 0.5)$, and compute several points of the orbit:

$$\mathbf{x}_0 = (0, 0.5) \rightarrow \mathbf{x}_1 = (0.5, 0.5) \rightarrow \mathbf{x}_2 = (1, 0) \rightarrow \mathbf{x}_3 = (1, -1) \dots$$

The points \mathbf{x}_i , $i = 0, \dots, 3$ are shown in Fig. 8. The orbit rolls around E_0 clockwise. The sketch of the phase space around E_0 is shown in Fig. 9.

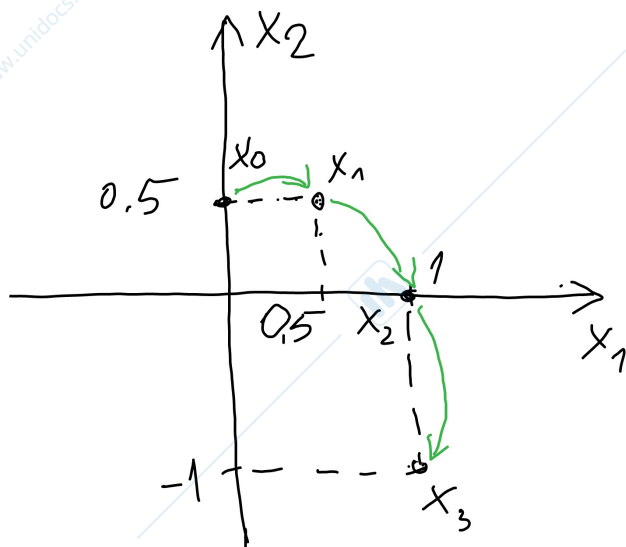


Figure 8: A sample orbit with an i.c. $\mathbf{x}_0 = (0, 0.5)$.

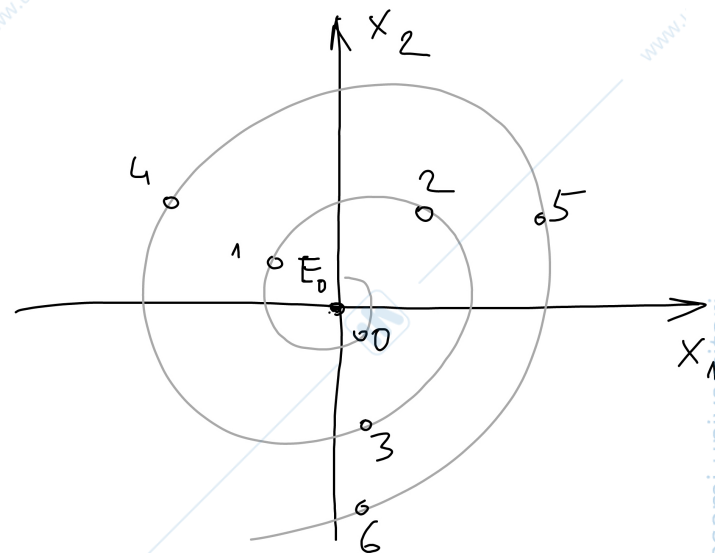


Figure 9: A sketch of a typical orbit in the neighborhood of E_0 .

EXERCISE 9

Consider the following dynamical system in discrete time:

$$\begin{cases} x'_1 = 0.5x_1 - 0.2x_2 \\ x'_2 = 0.2x_1 + 0.5x_2 \end{cases} \quad (5)$$

Define the type of its fixed point $E_0(0, 0)$ by using the stability conditions and sketch in the phase plane its typical orbit.

SOLUTION

Let us rewrite the dynamical system (5) in the matrix form

$$\mathbf{x}' = A\mathbf{x}$$

with

$$A = \begin{pmatrix} 0.5 & -0.2 \\ 0.2 & 0.5 \end{pmatrix}.$$

The trace and the determinant of matrix A are

$$\text{tr}A = 1 \quad \text{and} \quad \det A = 0.29.$$

The stability conditions read as

$$1 - \text{tr}A + \det A = 1 - 1 + 0.29 = 0.29 > 0,$$

$$1 + \text{tr}A + \det A = 1 + 1 + 0.29 = 2.29 > 0,$$

$$1 - \det A = 1 - 0.29 = 0.71 > 0.$$

Since all three conditions are satisfied, the fixed point E_0 is globally stable. Moreover, by computing the discriminant

$$\Delta = (\text{tr}A)^2 - 4\det A = 1 - 1.16 = -0.16 < 0$$

we conclude that the eigenvalues are complex conjugate. Hence, E_0 is a stable focus.

The final thing to determine is whether orbits roll around E_0 clockwise or counterclockwise. Let us take an arbitrary initial condition, say, $\mathbf{x}_0 = (0, 2)$, and compute several points of the orbit:

$$\mathbf{x}_0 = (0, 2) \rightarrow \mathbf{x}_1 = (-0.4, 1) \rightarrow \mathbf{x}_2 = (-0.4, 0.42) \dots$$

The points \mathbf{x}_i , $i = 0, \dots, 2$ are shown in Fig. 10. The orbit rolls around E_0 counterclockwise. The sketch of the phase space around E_0 is shown in Fig. 11.

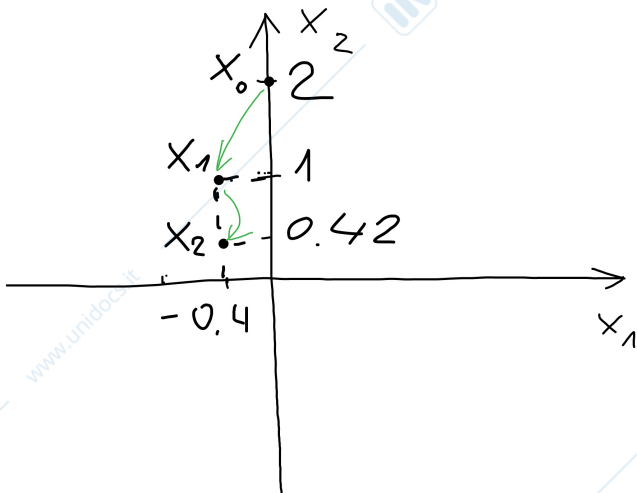


Figure 10: A sample orbit with an i.c. $\mathbf{x}_0 = (0, 2)$.

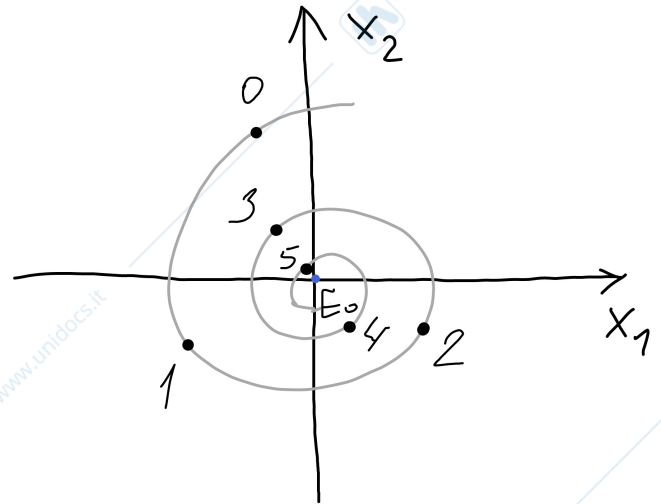


Figure 11: A sketch of a typical orbit in the neighborhood of E_0 .