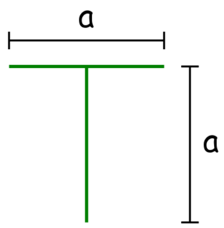
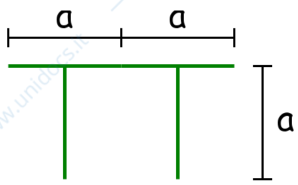
ESERCIZI DI INERZIA
(TUTORATO DEL 09/04/2018)

Esercizio 1. Ciascuno dei seguenti corpi piani ha densità ρ costante. Per ognuno di essi:

(1) individua il centro di massa C e gli assi principali d'inerzia in C;
(2) scrivi il tensore centrale d'inerzia $\mathbf{I}_C$.



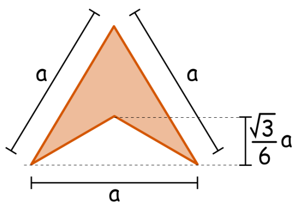
(i)



(ii)

Esercizio 2. Il seguente corpo piano *"a coda di rondine"* ha densità costante ρ. Trovane

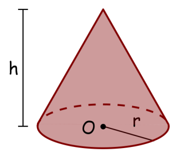
(1) il centro di massa C;
(2) il tensore centrale d'inerzia $\mathbf{I}_C$,
(3) il momento centrale d'inerzia relativo alla retta complanare al corpo e inclinata di $\frac{\pi}{4}$ rispetto all'asse orizzontale.



Esercizio 3. Considera il cono di massa totale m uniformemente distribuita, raggio di base r e altezza h. Scelta una base opportuna, scrivi

(1) il vettore $C - O$, dove C è il centro di massa;
(2) il tensore d'inerzia $\mathbf{I}_O$;
(3) il tensore centrale d'inerzia $\mathbf{I}_C$.

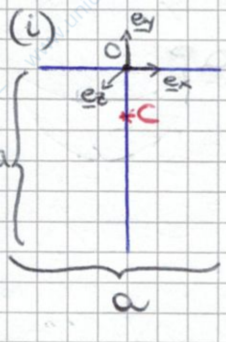
Quali sono gli assi principali d'inerzia in O e in C?



INERZIA - TUTORATO 09/04/2018

Es. 1:

(i)



densità: ρ

1) $C-O=?$ $\quad (C-O = C_x \underline{e}_x + C_y \underline{e}_y + C_z \underline{e}_z)$

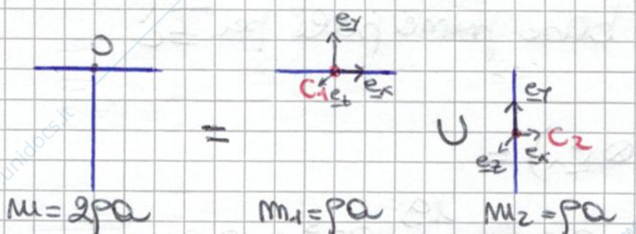
2) $\underline{\underline{I}}_c = ?$

TEOREMA DI COMPOSIZIONE

$$C-O = \frac{m_1(C_1-O)+m_2(C_2-O)}{m_1+m_2}$$

$$\underline{\underline{I}}_c = \underline{\underline{I}}_{c_1} + \underline{\underline{I}}_{c_2} + \frac{m_1 m_2}{m_1+m_2}|C_2-C_1|^2 (\underline{\underline{I}} - \underline{e}\otimes\underline{e})$$

1) Per simmetria: $C_x = C_z = 0$



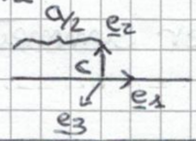
$C_1 - O = \underline{0}$

$C_2 - O = -\frac{a}{2}\underline{e}_y$

$$\Rightarrow \frac{m_1 m_2}{m_1+m_2} = \frac{\rho a}{2}$$

ASTA OMOGENEA:

$$\underline{\underline{I}}_c = \frac{1}{12} m a^2 (\underline{\underline{I}} - \underline{e}_1 \otimes \underline{e}_1)$$



SIMMETRIE

C ∈ piani di simmetria

O ∈ piano di simmetria

$\Rightarrow \underline{e} \perp$ piano è principale per O

$$C_y = \frac{m_1 C_{1y} + m_2 C_{2y}}{m_1+m_2} = \frac{0 - \frac{a}{2}\rho a}{2\rho a} = -\frac{1}{4}a$$

$$\Rightarrow C-O = -\frac{1}{4} a \underline{e}_y$$

Per simmetria: $\underline{e}_x$ ed $\underline{e}_z$ sono principali in C

$\Rightarrow$ anche $\underline{e}_y$ lo è

$\Rightarrow (\underline{e}_x, \underline{e}_y, \underline{e}_z)$ base principale per $\underline{\underline{I}}_c$.

2) $\underline{\underline{I}}_{c_1} = \frac{1}{12} m_1 a^2 (\underline{\underline{I}} - \underline{e}_x \otimes \underline{e}_x) = \frac{1}{12}\rho a^3 (\underline{\underline{I}} - \underline{e}_x \otimes \underline{e}_x)$

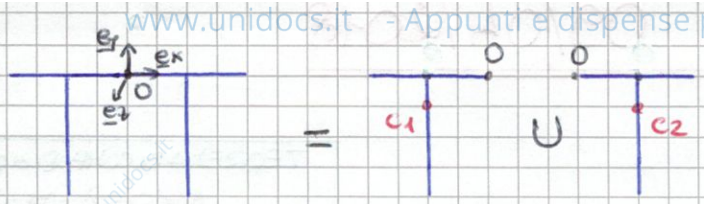
$\underline{\underline{I}}_{c_2} = \frac{1}{12}\rho a^3 (\underline{\underline{I}} - \underline{e}_y \otimes \underline{e}_y)$

$|C_2 - C_1|^2 = \frac{a^2}{4}$; $\quad \underline{e} = \underline{e}_y \quad$ ($\underline{e}_y$ o $-\underline{e}_y$ non cambia nulla)

$$\Rightarrow \underline{\underline{I}}_c = \frac{1}{12}\rho a^3 (2\underline{\underline{I}} - \underline{e}_x\otimes\underline{e}_x - \underline{e}_y\otimes\underline{e}_y) + \frac{1}{8}\rho a^3 (\underline{\underline{I}} - \underline{e}_y\otimes\underline{e}_y) =$$

$$= \frac{5}{24}\rho a^3 \underline{e}_x\otimes\underline{e}_x + \frac{1}{12}\rho a^3 \underline{e}_y\otimes\underline{e}_y + \frac{7}{24}\rho a^3 \underline{e}_z\otimes\underline{e}_z$$

(ii)



$m = 4a\rho$ $\quad m_1 = 2a\rho$ $\quad m_2 = 2a\rho$

$|c_2 - c_1|^2 = a^2$

$\underline{e} = \underline{e}_x$

$\frac{m_1 m_2}{m_1 + m_2} = a\rho$

$\underline{\underline{I}}_{c_1} = \underline{\underline{I}}_{c_2}$

1) Per simmetria: $c_z = c_x = 0$

Inoltre $c \in c_1 c_2 \Rightarrow c_y = c_{1y} = c_{2y} = -\frac{a}{4}$

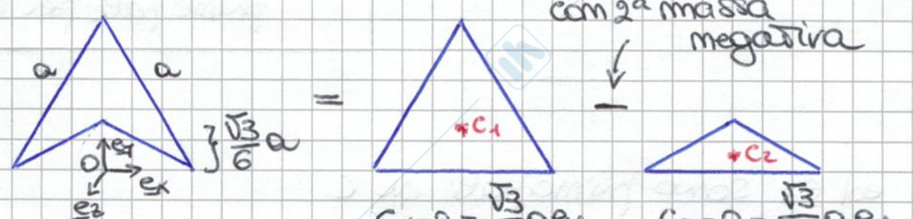
$\Rightarrow c - O = -\frac{1}{4} a \underline{e}_y$

Per simmetria: $(\underline{e}_x, \underline{e}_y, \underline{e}_z)$ è base principale per $\underline{\underline{I}}_c$

2) $\underline{\underline{I}}_c = \underline{\underline{I}}_{c_1} + \underline{\underline{I}}_{c_2} + a^3\rho\,(\underline{\underline{I}} - \underline{e}_x \otimes \underline{e}_x) =$

$= \frac{5}{12}\rho a^3 \underline{e}_x \otimes \underline{e}_x + \frac{7}{6}\rho a^3 \underline{e}_y \otimes \underline{e}_y + \frac{19}{12}\rho a^3 \underline{e}_z \otimes \underline{e}_z$

Es. 2:



$c_1 - O = \frac{\sqrt{3}}{6} a \underline{e}_y$ $\quad c_2 - O = \frac{\sqrt{3}}{18} a \underline{e}_y$

$m_1 + m_2 = 2m$ $\quad m_1 = 3m$ $\quad m_2 = -m$

TRIANGOLO OMOGENEO

$c - O = \frac{1}{3} h \underline{e}_y$

$\underline{\underline{I}}_c = \frac{1}{18} m h^2 \underline{e}_x \otimes \underline{e}_x + \frac{1}{24} m b^2 \underline{e}_y \otimes \underline{e}_y + \left(\frac{1}{18}h^2 + \frac{1}{24}b^2\right) m\, \underline{e}_z \otimes \underline{e}_z$

con $m := \rho \frac{\sqrt{3}}{12} a^2$

1) $c - O = \dfrac{3m \frac{\sqrt{3}}{6} a \underline{e}_y - m \frac{\sqrt{3}}{18} a \underline{e}_y}{2m} = \frac{2\sqrt{3}}{9} a \underline{e}_y$

Per simmetria: $(\underline{e}_x, \underline{e}_y, \underline{e}_z)$ principale per $\underline{\underline{I}}_c$

2) $\underline{\underline{I}}_{c_1} = \frac{1}{18} 3m \frac{3}{4} a^2 \underline{e}_x \otimes \underline{e}_x + \frac{1}{24} 3m\, a^2 \underline{e}_y \otimes \underline{e}_y + \left(\frac{1}{8} + \frac{1}{8}\right) m a^2 \underline{e}_z \otimes \underline{e}_z =$

$= \frac{1}{8} m a^2 \underline{e}_x \otimes \underline{e}_x + \frac{1}{8} m a^2 \underline{e}_y \otimes \underline{e}_y + \frac{1}{4} m a^2 \underline{e}_z \otimes \underline{e}_z$

[Sensato: è un triangolo equilatero ⇒ tutto il piano xy è principale per $\underline{\underline{I}}_{c_1}$]

$$\underline{\underline{I}}_{C_2} = -\frac{1}{18} m \frac{3}{36} a^2 \underline{e}_x \otimes \underline{e}_x - \frac{1}{24} m a^2 \underline{e}_y \otimes \underline{e}_y - \left(\frac{1}{24 \cdot 9} + \frac{1}{24}\right) m a^2 \underline{e}_z \otimes \underline{e}_z =$$

$$= -\frac{1}{18 \cdot 12} m a^2 \underline{e}_x \otimes \underline{e}_x - \frac{1}{24} m a^2 \underline{e}_y \otimes \underline{e}_y - \frac{5}{12 \cdot 9} m a^2 \underline{e}_z \otimes \underline{e}_z$$

$$|C_2 - C_1|^2 = \frac{3}{81} a^2 = \frac{1}{27} a^2 \quad ; \quad \underline{e} = \underline{e}_y$$

$$\frac{m_1 m_2}{m_1 + m_2} = -\frac{3m^2}{2m} = -\frac{3}{2} m$$

$$\Rightarrow \underline{\underline{I}}_C = \left(\frac{1}{8} - \frac{1}{8 \cdot 27} - \frac{3}{2 \cdot 27}\right) m a^2 \underline{e}_x \otimes \underline{e}_x + \left(\frac{1}{8} - \frac{1}{24}\right) m a^2 \underline{e}_y \otimes \underline{e}_y$$

$$+ \left(\frac{1}{8} - \frac{5}{4 \cdot 27} - \frac{3}{2 \cdot 27}\right) m a^2 \underline{e}_z \otimes \underline{e}_z =$$

$$= \frac{7}{108} m a^2 \underline{e}_x \otimes \underline{e}_x + \frac{1}{12} m a^2 \underline{e}_y \otimes \underline{e}_y + \frac{4}{27} m a^2 \underline{e}_z \otimes \underline{e}_z =$$

$$= \frac{7\sqrt{3}}{1296} \rho a^4 \underline{e}_x \otimes \underline{e}_x + \frac{\sqrt{3}}{144} \rho a^4 \underline{e}_y \otimes \underline{e}_y + \frac{\sqrt{3}}{81} \rho a^4 \underline{e}_z \otimes \underline{e}_z$$

3) $\underline{u} = \frac{\sqrt{2}}{2} (\underline{e}_x + \underline{e}_y)$

$$\Rightarrow I_{C\underline{u}} = \underline{u} \cdot \underline{\underline{I}}_C \underline{u} = \frac{1}{2} (\underline{e}_x + \underline{e}_y) \cdot \underline{\underline{I}}_C (\underline{e}_x + \underline{e}_y) =$$

$$= \frac{1}{2} (\underline{e}_x + \underline{e}_y) \cdot \left(\frac{7}{108} m a^2 \underline{e}_x + \frac{1}{12} m a^2 \underline{e}_y\right) =$$

$$= \frac{1}{2} \left(\frac{7}{108} + \frac{1}{12}\right) m a^2 = \frac{2}{27} m a^2 = \frac{\sqrt{3}}{162} \rho a^4$$

Es. 3:

$$\text{vol} = \frac{\pi r^2 h}{3} \Rightarrow \rho = \frac{3m}{\pi r^2 h}$$

coordinate:
$$\begin{cases} x = t\cos\theta \\ y = t\sin\theta \\ z = z \end{cases} \qquad \begin{array}{l} t \in [0, r] \\ \theta \in [0, 2\pi] \\ z \in [0, h(1 - t/r)] \end{array}$$

come per il triangolo

$\Downarrow$

jacobiano: $$J = \left| \det \begin{pmatrix} \cos\theta & -t\sin\theta & 0 \\ \sin\theta & t\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = t$$

1) Per simmetria, $c \in$ asse del cono $\Rightarrow c_x = c_y = 0$

$$c_z = \frac{\rho}{m}\int_0^r\int_0^{2\pi}\int_0^{h(1-\frac{t}{r})} t z \, dt\, d\theta\, dz = \frac{3}{\cancel{\pi} r^2 \cancel{h}}\cancel{2\pi}\int_0^r \frac{h^{\cancel{2}}}{\cancel{2}} t\left(1-\frac{t}{r}\right)^2 dt =$$

$$= \frac{3h}{r^2}\left[\frac{r^2}{2} - \frac{2}{3}r^2 + \frac{1}{4}r^2\right] = \frac{1}{4}h$$

$$\Rightarrow c - 0 = \frac{1}{4}h\,\underline{e}_z$$

$\Rightarrow$ Sia per 0 che per c $\underline{e}_z$ e tutte le direzioni nel piano xy sono principali $\Rightarrow$ entrambi $\underline{\underline{I}}_0$ e $\underline{\underline{I}}_c$ avranno simmetria cilindrica

2) $I_{0xx} = I_{0yy}$

$$I_{0yy} = \frac{3m}{\pi h r^2}\int_0^r\int_0^{2\pi}\int_0^{h(1-t/r)} t(t^2\sin^2\theta + z^2)\, dt\, d\theta\, dz =$$

$$= \frac{3m}{\pi h r^2}\left[\pi\int_0^r h\left(t^3 - \frac{t^4}{r}\right)dt + 2\pi\int_0^r \frac{h^3}{3}\left(t - \frac{3t^2}{r} + \frac{3t^3}{r^2} - \frac{t^4}{r^3}\right)dt\right] =$$

$$= \frac{3m}{r^2}\left[\frac{1}{4}r^4 - \frac{1}{5}r^4 + \frac{2h^2}{3}\left(\frac{1}{2}r^2 - r^2 + \frac{3}{4}r^2 - \frac{1}{5}r^2\right)\right] =$$

$$= \frac{3}{20}mr^2 + \frac{1}{10}mh^2$$

$$\left[\int \sin^2\theta = -\frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2}\right.$$

$$\int_0^{2\pi}\sin^2\theta = \pi$$

$$\left. e \int_0^{2\pi}\cos^2\theta = \int_0^{2\pi}(1-\sin^2\theta) = \pi\right.$$

$\hookrightarrow$ così in effetti $I_{0xx} = I_{0yy}$

$$I_{0zz} = \frac{3m}{\pi h r^2}\int_0^r\int_0^{2\pi}\int_0^{h(1-t/r)} t^3\, dt\, d\theta\, dz = \frac{6m}{r^2}\int_0^r\left(t^3 - \frac{t^4}{r}\right)dt = \frac{6m}{r^2}\left[\frac{1}{4}r^4 - \frac{1}{5}r^5\right] = \frac{3}{10}mr^2$$

$$\Rightarrow \underline{\underline{I}}_0 = \left(\frac{3}{20}mr^2 + \frac{1}{10}mh^2\right)\left(\underline{\underline{I}} - \underline{e}_z\otimes\underline{e}_z\right) + \frac{3}{10}mr^2\,\underline{e}_z\otimes\underline{e}_z$$

3) $|c-0|^2 = \frac{1}{16}h^2$; $\underline{e}_c = \underline{e}_z$

$$\Rightarrow \underline{\underline{I}}_c = \underline{\underline{I}}_0 - \frac{1}{16}mh^2\left(\underline{\underline{I}} - \underline{e}_z\otimes\underline{e}_z\right) =$$

$$= \left(\frac{3}{20}mr^2 + \frac{3}{80}mh^2\right)\left(\underline{\underline{I}} - \underline{e}_z\otimes\underline{e}_z\right) + \frac{3}{10}mr^2\,\underline{e}_z\otimes\underline{e}_z$$

simmetria cilindrica