Università degli Studi di Pavia - Corso di Laurea in Ingegneria Edile e Architettura - A.A.2019/2020

Esercitazioni di

MECCANICA RAZIONALE

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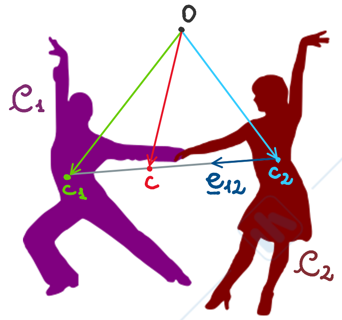
Esercitazione di Inerzia:

COMPLEMENTI DI TEORIA

1. Rappresentazione matriciale del tensore d'inerzia
2. Estensione al caso continuo
3. Riepilogo sui "trucchi di calcolo" del tensore d'inerzia
    - simmetrie materiali e basi principali
    - sistemi piani
    - Teorema di Huygens-Steiner
    - Teorema di Composizione (e Teorema della Lacuna)
4. Ricerca delle direzioni principali di inerzia

ESERCIZI

④ TEOREMA DI COMPOSIZIONE

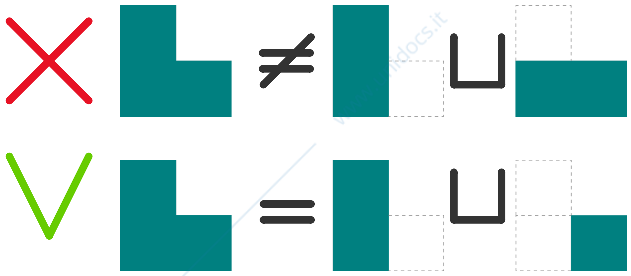


- $$\boxed{c - O = \frac{m_1(c_1 - O) + m_2(c_2 - O)}{m_1 + m_2}}$$
- $$\boxed{\underline{\underline{I}}_c = \underline{\underline{I}}_{c_1} + \underline{\underline{I}}_{c_2} + \frac{m_1 m_2}{m_1 + m_2} |c_1 - c_2|^2 (\underline{\underline{I}} - \underline{e}_{12} \otimes \underline{e}_{12})}$$

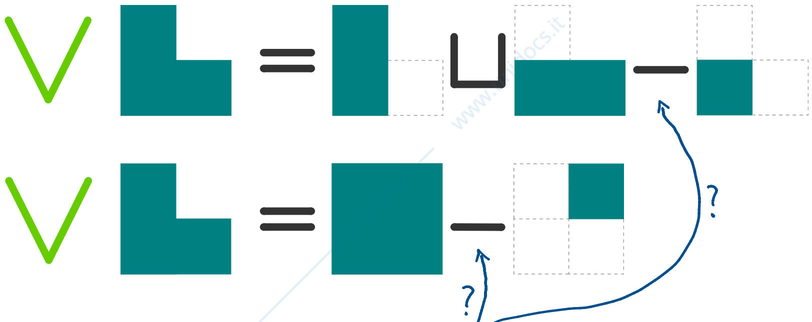
$$\left.\begin{array}{l} \text{con } m(\mathcal{C}_1 \cap \mathcal{C}_2) = 0 \\ \text{e } \mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \end{array}\right\} \begin{array}{l} \text{lo indicherò} \\ \text{con} \\ \mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2 \end{array}$$

Nota: $m(\mathcal{C}_1 \cap \mathcal{C}_2) = 0$ ⇝ i corpi $\mathcal{C}_1$ e $\mathcal{C}_2$ al massimo SI TOCCANO MA NON SI SOVRAPPONGONO

Esempio:



sto prendendo 2 volte l'intersezione!



Come faccio a "sottrarre" corpi tra loro?

⇝ uso il Teorema di Composizione in un modo particolare:

$$\underbrace{(m_1+m_2)}_{m}(c - \underbrace{c_2}_{c_2 \equiv O}) = m_1(c_1 - c_2) + m_2(c_2 - c_2) \quad \text{e} \quad \underline{\underline{I}}_c = \underline{\underline{I}}_{c_1} + \underline{\underline{I}}_{c_2} + \frac{m_1 m_2}{m_1 + m_2} |c_1 - c_2|^2 (\underline{\underline{I}} - \underline{e}_{12} \otimes \underline{e}_{12})$$

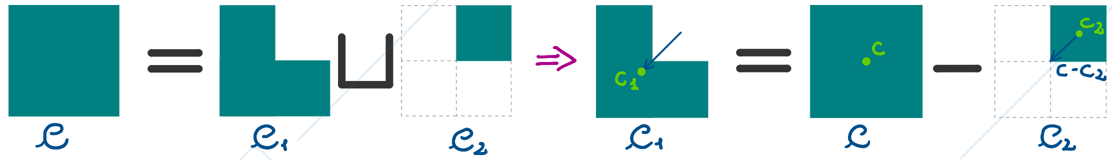
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

Teorema della Lacuna:

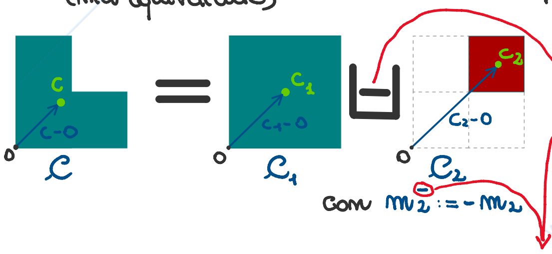
$$\boxed{c_1 - c_2 = \frac{m}{m - m_2}(c - c_2)} \quad \text{e} \quad \boxed{\underline{\underline{I}}_{c_1} = \underline{\underline{I}}_c - \underline{\underline{I}}_{c_2} - \frac{(m - m_2) m_2}{m} |c_1 - c_2|^2 (\underline{\underline{I}} - \underline{e}_{12} \otimes \underline{e}_{12})}$$

$$= \underline{\underline{I}}_c - \underline{\underline{I}}_{c_2} - \frac{m\, m_2}{m - m_2} |c - c_2|^2 (\underline{\underline{I}} - \underline{e}_{12} \otimes \underline{e}_{12})$$

* Nell'Esempio:



* Formulazione alternativa: "sottrarre" = "unire" corpi di massa NEGATIVA
(ma equivalente)



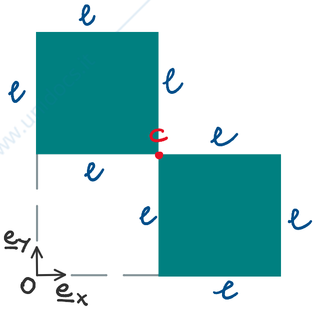
con $m_2^{\ominus} := -m_2$

$$\boxed{c - o = \frac{m_1(c_1 - o) + m_2^{\ominus}(c_2 - o)}{m_1 + m_2^{\ominus}}}$$

$$\boxed{\underline{\underline{I}}_c = \underline{\underline{I}}_{c_1} + \underline{\underline{I}}_{c_2}^{\ominus} + \frac{m_1 m_2^{\ominus}}{m_1 + m_2^{\ominus}} |c_1 - c_2|^2 (\underline{\underline{I}} - \underline{e}_{12} \otimes \underline{e}_{12})}$$

per ricordarmi che m_2 ha SEGNO NEGATIVO

Esercizi di Inerzia: FIGURE COMPOSTE



massa totale: $2m$ (uniformemente distribuita)

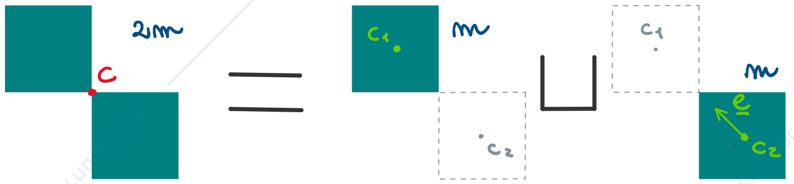
1) $c - o = ?$

2) $\underline{\underline{I}}_c = ?$

3) $\underline{\underline{I}}_o = ?$

1) Per simmetria: $\boxed{c - o = \ell\, \underline{e}_x + \ell\, \underline{e}_y}$

2)



$\underline{e} := \dfrac{-\underline{e}_x + \underline{e}_y}{\sqrt{2}}$

$|c_2 - c_1| = \sqrt{2}\,\ell$

$$\underline{\underline{I}}_{c_1}=\underline{\underline{I}}_{c_2}=\frac{1}{12}m\ell^2(\underline{\underline{I}}+\underline{e}_z\otimes\underline{e}_z)$$

$$\rightsquigarrow \boxed{\underline{\underline{I}}_c=}\ \frac{1}{6}m\ell^2(\underline{\underline{I}}+\underline{e}_z\otimes\underline{e}_z)$$

$$+m\,\ell^2\,(\underline{\underline{I}}-\underline{e}\otimes\underline{e})$$

$$=\frac{7}{6}m\ell^2\underline{\underline{I}}-m\ell^2\underbrace{\underline{e}}_{\text{direzione}}\otimes\underline{e}+\frac{1}{6}m\,\ell^2\underbrace{\underline{e}_z}_{\text{principale}}\otimes\underline{e}_z$$

direzione principale

$$=\frac{7}{6}m\,\ell^2\underline{\underline{I}}-\frac{1}{2}m\ell^2(\underline{e}_x\otimes\underline{e}_x-\underline{e}_x\otimes\underline{e}_y$$
$$-\underline{e}_y\otimes\underline{e}_x+\underline{e}_y\otimes\underline{e}_y)+\frac{1}{6}m\ell^2\underline{e}_z\otimes\underline{e}_z$$

$$=\boxed{\frac{2}{3}m\ell^2\underline{e}_x\otimes\underline{e}_x+\frac{1}{2}m\ell^2\underline{e}_x\otimes\underline{e}_y+\frac{1}{2}m\,\ell^2\underline{e}_y\otimes\underline{e}_x+\frac{2}{3}m\ell^2\underline{e}_y\otimes\underline{e}_y+\frac{4}{3}m\ell^2\underline{e}_z\otimes\underline{e}_z}$$

* $\underline{e}_x,\underline{e}_y$ NON sono direzioni principali per $\underline{\underline{I}}_c$

* $I_{c_{zz}}=I_{c_{xx}}+I_{c_{yy}}$

TEOREMA DI COMPOSIZIONE

$$\underline{\underline{I}}_c=\underline{\underline{I}}_{c_1}+\underline{\underline{I}}_{c_2}+\frac{m_1m_2}{m_1+m_2}|c_1-c_2|^2(\underline{\underline{I}}-\underline{e}_{12}\otimes\underline{e}_{12})$$

QUADRATO

lato: ℓ

massa: m

$$\underline{\underline{I}}_c=\frac{1}{12}m\ell^2(\underline{\underline{I}}+\underline{e}_z\otimes\underline{e}_z)$$

TEOREMA DI HUYGENS-STEINER

$$\underline{\underline{I}}_o=\underline{\underline{I}}_c+m|c-o|^2(\underline{\underline{I}}-\underline{e}\otimes\underline{e})$$

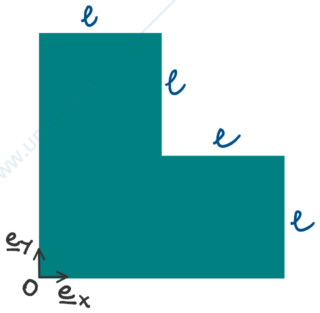
3) $$\boxed{\underline{\underline{I}}_o=}\ \frac{7}{6}m\ell^2\underline{\underline{I}}-m\ell^2\underline{e}\otimes\underline{e}+\frac{1}{6}m\ell^2\underline{e}_z\otimes\underline{e}_z$$
$$+2m\,2\ell^2(\underline{\underline{I}}-\underline{e}_\perp\otimes\underline{e}_\perp)\qquad \underline{e}_\perp:=\frac{\underline{e}_x+\underline{e}_y}{\sqrt{2}}\perp\underline{e}$$

$$=\boxed{\frac{25}{6}m\ell^2\underline{e}\otimes\underline{e}+\frac{7}{6}m\ell^2\underline{e}_\perp\otimes\underline{e}_\perp+\frac{16}{3}m\ell^2\underline{e}_z\otimes\underline{e}_z}$$

$\rightsquigarrow(\underline{e},\underline{e}_\perp,\underline{e}_z)$ principale per $\underline{\underline{I}}_o$

$$=\boxed{\frac{8}{3}m\ell^2\underline{e}_x\otimes\underline{e}_x-\frac{3}{2}m\ell^2\underline{e}_x\otimes\underline{e}_y-\frac{3}{2}m\ell^2\underline{e}_y\otimes\underline{e}_x+\frac{8}{3}m\ell^2\underline{e}_y\otimes\underline{e}_y+\frac{16}{3}m\ell^2\underline{e}_z\otimes\underline{e}_z}$$

Esercizi di Inerzia: FIGURE COMPOSTE



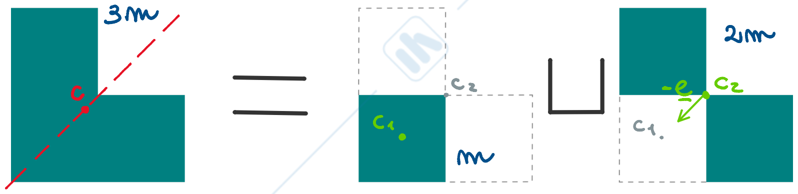
massa totale: 3m (uniformemente distribuita)

1) $c-o=?$

2) $\underline{\underline{I}}_c=?$

3) $\underline{\underline{I}}_o=?$

1)



$$\underline{e}:=\frac{-\underline{e}_x+\underline{e}_y}{\sqrt{2}}$$

$$\underline{e}_\perp:=\frac{\underline{e}_x+\underline{e}_y}{\sqrt{2}}\perp\underline{e}$$

$$|c_2-c_1|=\frac{\sqrt{2}}{2}\ell$$

$$c_1-O=\frac{\sqrt{2}}{2}\underline{e}_\perp\qquad c_2-O=\sqrt{2}\underline{e}_\perp$$

$$\boxed{c-O=}\ \frac{1}{3m}\left(m\frac{\sqrt{2}}{2}\underline{e}_\perp+2m\sqrt{2}\underline{e}_\perp\right)=\frac{5\sqrt{2}}{6}\ell\underline{e}_\perp=$$

$$=\boxed{\frac{5}{6}\ell\underline{e}_x+\frac{5}{6}\ell\,\underline{e}_y}$$

TEOREMA DI COMPOSIZIONE

$$c-O=\frac{m_1(c_1-O)+m_2(c_2-O)}{m_1+m_2}$$

$$\underline{\underline{I}}_c=\underline{\underline{I}}_{c_1}+\underline{\underline{I}}_{c_2}+\frac{m_1m_2}{m_1+m_2}|c_1-c_2|^2(\underline{\underline{I}}-\underline{e}_{12}\otimes\underline{e}_{12})$$

2) Primo modo

$$\underline{\underline{I}}_{c_1} = \frac{1}{12} m\ell^2 (\underline{\underline{I}} + \underline{e}_z \otimes \underline{e}_z) \qquad \underline{\underline{I}}_{c_2} = \frac{7}{6} m\ell^2 \underline{\underline{I}} - m\ell^2 \underline{e} \otimes \underline{e} + \frac{1}{6} m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

$$= \frac{7}{6} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{1}{6} m\ell^2 \underline{e} \otimes \underline{e} + \frac{4}{3} m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

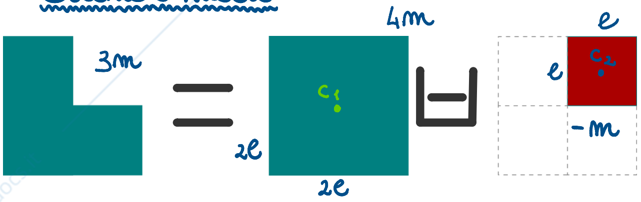
$$\rightsquigarrow \boxed{\underline{\underline{I}}_c} = \frac{5}{4} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{1}{4} m\ell^2 \underline{e} \otimes \underline{e} + \frac{3}{2} m\ell^2 \underline{e}_z \otimes \underline{e}_z + \frac{2}{3} m \frac{1}{2} \ell^2 (\underline{\underline{I}} - \underline{e}_\perp \otimes \underline{e}_\perp) =$$

$$= \boxed{\frac{5}{4} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{7}{12} m\ell^2 \underline{e} \otimes \underline{e} + \frac{11}{6} m\ell^2 \underline{e}_z \otimes \underline{e}_z}$$

$$= \frac{11}{12} m\ell^2 \underline{e}_x \otimes \underline{e}_x + \frac{1}{3} m\ell^2 \underline{e}_x \otimes \underline{e}_y + \frac{1}{3} m\ell^2 \underline{e}_y \otimes \underline{e}_x + \frac{11}{12} m\ell^2 \underline{e}_y \otimes \underline{e}_y + \frac{11}{6} m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

Nota:  ⇝ 3 oggetti, composti ② alla volta

Secondo modo



TEOREMA DELLA LACUNA

$$c - O = \frac{m_1(c_1 - O) + m_2^-(c_2 - O)}{m_1 + m_2^-}$$

$$\underline{\underline{I}}_c = \underline{\underline{I}}_{c_1} + \underline{\underline{I}}^-_{c_2} + \frac{m_1 m_2^-}{m_1 + m_2^-} |c_1 - c_2|^2 (\underline{\underline{I}} - \underline{e}_{12} \otimes \underline{e}_{12})$$

m_2^- NEGATIVA

$$c - O = \frac{1}{3m} \left(4m \sqrt{2} \ell \underline{e}_\perp - m \frac{3}{2} \sqrt{2} \ell \underline{e}_\perp\right) = \frac{5}{6} \sqrt{2} \ell \underline{e}_\perp$$

$$\underline{\underline{I}}_{c_1} = \frac{1}{12} 4m 4\ell^2 (\underline{\underline{I}} + \underline{e}_z \otimes \underline{e}_z) \qquad \underline{\underline{I}}^-_{c_2} = -\frac{1}{12} m\ell^2 (\underline{\underline{I}} + \underline{e}_z \otimes \underline{e}_z)$$

$$\underline{\underline{I}}_c = \frac{15}{12} m\ell^2 (\underline{\underline{I}} + \underline{e}_z \otimes \underline{e}_z) - \frac{4}{3} m \frac{1}{2} \ell^2 (\underline{\underline{I}} - \underline{e}_\perp \otimes \underline{e}_\perp)$$

$$= \frac{5}{4} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{7}{12} m\ell^2 \underline{e} \otimes \underline{e} + \frac{11}{6} m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

TEOREMA DI HUYGENS-STEINER

$$\underline{\underline{I}}_o = \underline{\underline{I}}_c + m|c - O|^2 (\underline{\underline{I}} - \underline{e} \otimes \underline{e})$$

3) Primo modo

$$\boxed{\underline{\underline{I}}_o} = \frac{5}{4} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{7}{12} m\ell^2 \underline{e} \otimes \underline{e} + \frac{11}{6} m\ell^2 \underline{e}_z \otimes \underline{e}_z + 3m \frac{25}{18} \ell^2 (\underline{\underline{I}} - \underline{e}_\perp \otimes \underline{e}_\perp)$$

$$= \boxed{\frac{5}{4} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{57}{12} m\ell^2 \underline{e} \otimes \underline{e} + 6 m\ell^2 \underline{e}_z \otimes \underline{e}_z}$$

$$= 3m\ell^2 \underline{e}_x \otimes \underline{e}_x - \frac{7}{4} m\ell^2 \underline{e}_x \otimes \underline{e}_y - \frac{7}{4} m\ell^2 \underline{e}_y \otimes \underline{e}_x + 3m\ell^2 \underline{e}_y \otimes \underline{e}_y + 6m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

Secondo modo

Infatti: $\underline{\underline{I}}_o^{\text{▙}} = \int (\underline{r}^2 \underline{\underline{I}} - \underline{r} \otimes \underline{r})\, dm$

▙ ↳ spezzo in $\int_{\text{▙}} \ldots + \int_{\text{▗}} \ldots$

$$\underline{\underline{I}}_o^{\text{▙}} = \underline{\underline{I}}_o^{\text{▙}} + \underline{\underline{I}}_o^{\text{▗}} =$$

$$= \frac{35}{6} m\ell^2 \underline{e} \otimes \underline{e} + \frac{7}{6} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{16}{3} m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

$$+ \frac{7}{12} m\ell^2 \underline{e} \otimes \underline{e} + \frac{1}{12} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + \frac{2}{3} m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

$$= \frac{57}{12} m\ell^2 \underline{e} \otimes \underline{e} + \frac{5}{4} m\ell^2 \underline{e}_\perp \otimes \underline{e}_\perp + 6m\ell^2 \underline{e}_z \otimes \underline{e}_z$$

QUADRATO

$$\underline{\underline{I}}_o = \frac{1}{12} m\ell^2 (\underline{\underline{I}} + \underline{e}_z \otimes \underline{e}_z)$$

$$+ \frac{1}{2} m\ell^2 (\underline{\underline{I}} - \underline{e}_\perp \otimes \underline{e}_\perp)$$

Inerzia | RICERCA DELLE DIREZIONI PRINCIPALI

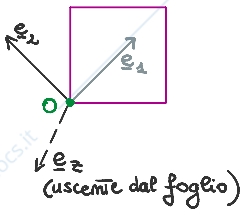
METODO ①: ricerca dei PIANI DI RIFLESSIONE

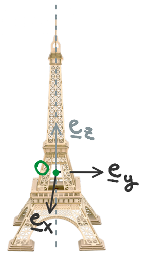
⤳ è un metodo usabile A PRIORI (prima di conoscere $\underline{\underline{I}}_o$)

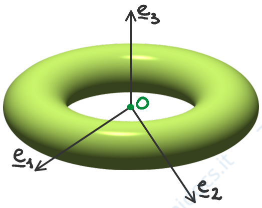
Se il PIANO ⊥ $\underline{e}$ passante per o è di RIFLESSIONE per $\mathcal{C}$ $(=\underline{\underline{R}}_{\underline{e}}\mathcal{C}=\mathcal{C})$

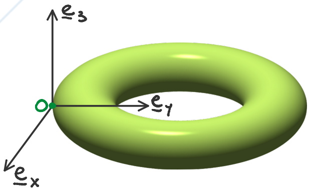
$\Rightarrow$ $\underline{e}$ è una DIREZIONE PRINCIPALE D'INERZIA per $\underline{\underline{I}}_o$

1) Trovo $\underline{e}_1 \perp \underline{e}_2$ principali usando le riflessioni interne
2) Completo la base con $\underline{e}_3 := \underline{e}_1 \times \underline{e}_2$ (che è sicuramente principale)









METODO ②: ricerca degli AUTOVETTORI

⤳ è un metodo usabile solo A POSTERIORI (dopo aver calcolato $\underline{\underline{I}}_o$)

Le DIREZIONI PRINCIPALI D'INERZIA sono gli AUTOVETTORI di $\underline{\underline{I}}_o$

$\underline{e}_i$ principale $\Rightarrow$ $\underline{\underline{I}}_o \underbrace{\underline{e}_i}_{\text{autovettore}} = \underbrace{I_{oii}}_{\text{autovalore}} \underline{e}_i$ con $I_{oii} = \underline{e}_i \cdot \underline{\underline{I}}_o \underline{e}_i$

1) Risolvo $\underline{\underline{I}}_o \underline{v} = \lambda \underline{v}$:
   - $\det(\underline{\underline{I}}_o - \lambda \underline{\underline{I}}) = 0$ ⤳ trovo gli autovalori $\lambda_1, \lambda_2, \lambda_3$
   - per $i=1,2,3$ $(\underline{\underline{I}}_o - \lambda_i \underline{\underline{I}})\underbrace{\underline{v}_i}_{v_x\underline{e}_x+v_y\underline{e}_y+v_z\underline{e}_z} = \underline{0}$ ⤳ trovo gli autovettori $\underline{v}_1, \underline{v}_2, \underline{v}_3$

     e trovo le relazioni che legano v_x, v_y, v_z
     (poi scelgo dei valori numerici che le soddisfano)
2) Normalizzo (per ottenere dei VERSORI): $\underline{e}_i := \dfrac{\underline{v}_i}{|\underline{v}_i|}$
3) Ordino $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ in modo che $\underline{e}_3 = \underline{e}_1 \times \underline{e}_2$

(oppure trovo solo $\underline{e}_1, \underline{e}_2$ – controllando che $\underline{e}_2 \perp \underline{e}_1$ – e poi aggiungo $\underline{e}_3 := \underline{e}_1 \times \underline{e}_2$)

METODO ③: ricerca dei PUNTI STAZIONARI di $\underline{e} \mapsto \underline{\underline{I}}_{o_{\underline{e}}}$ ⇝ metodo A POSTERIORI

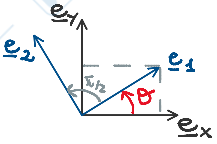
Le DIREZIONI PRINCIPALI D'INERZIA sono i PUNTI STAZIONARI della funzione ⇝ minimi, selle, massimi

$$f(\underline{e}) := I_{o_{\underline{e}}} = \underline{e} \cdot \underline{\underline{I}}_o \underline{e}$$

Vediamo il caso 2D (C e O $\in$ piano xy)
⇝ in 3D il conto è simile, ma più complicato

1) Scrivo $\underline{\underline{I}}_o = I_{oxx}\, \underline{e}_x \otimes \underline{e}_x + I_{oxy}\, \underline{e}_x \otimes \underline{e}_y$
$+ I_{oxy}\, \underline{e}_y \otimes \underline{e}_x + I_{oyy}\, \underline{e}_y \otimes \underline{e}_y + I_{ozz}\, \underline{e}_z \otimes \underline{e}_z$

* $\underline{e}_z$ è principale ⇝ mi basta $\underline{e}_1$ nel piano xy
poi $\underline{e}_2 := \underline{e}_z \times \underline{e}_1$



2) Scrivo $\underline{e}$ generico : $\underline{e} := \cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y$

* cerco l'angolo θ

* mi aspetto di trovare coppie di soluzioni : $\underbrace{\theta}_{\underline{e}_1}$ e $\underbrace{\theta + \frac{\pi}{2}}_{\underline{e}_2}$

3) Impongo la STAZIONARIETÀ:

$$\frac{\partial}{\partial\theta}\left(\underline{e} \cdot \underline{\underline{I}}_o \underline{e}\right) = \frac{\partial}{\partial\theta} f(\underline{e}) = 0$$

* $\underline{\underline{I}}_o \underline{e} = \cos\theta \left(I_{oxx}\, \underline{e}_x + I_{oxy}\, \underline{e}_y\right) + \sin\theta \left(I_{oxy}\, \underline{e}_x + I_{oyy}\, \underline{e}_y\right)$

* $\underline{e} \cdot \underline{\underline{I}}_o \underline{e} = \cos^2\theta\, I_{oxx} \underbrace{+ \sin\theta\cos\theta\, I_{oxy} + \sin\theta\cos\theta\, I_{oxy}}_{+\sin(2\theta) I_{oxy}} + \sin^2\theta\, I_{oyy}$

* $\frac{\partial}{\partial\theta}\left(\underline{e} \cdot \underline{\underline{I}}_o \underline{e}\right) = 0 \Rightarrow \underbrace{-2\sin\theta\cos\theta}_{\sin(2\theta)} I_{oxx} + 2\cos(2\theta) I_{oxy} + \underbrace{2\sin\theta\cos\theta}_{\sin(2\theta)} I_{oyy} = 0$

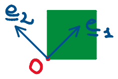
4) Risolvo $2\cos(2\theta) I_{oxy} = \sin(2\theta)\left(I_{oxx} - I_{oyy}\right)$

$\boxed{\text{CASO 1}}$ $I_{oxx} = I_{oyy} \Rightarrow I_{oxy} \cos(2\theta) = 0$

$\boxed{a}$ $I_{oxy} = 0 \Rightarrow 0 = 0$ ⇝ ogni θ va bene
⇓
ogni direzione nel piano xy è principale
(e $\underline{\underline{I}}_o$ era già in forma diagonale)

$$\underline{\underline{I}}_o = \underbrace{\frac{1}{2} I_{ozz}}_{I_{oxx} = I_{oyy}} \left(\underline{\underline{I}} - \underline{e}_z \otimes \underline{e}_z\right) + \underbrace{I_{ozz}}_{I_{oxx} + I_{oyy}} \underline{e}_z \otimes \underline{e}_z$$ ⇝ SIMMETRIA CILINDRICA

$\boxed{b}$ $I_{oxy} \neq 0 \Rightarrow \cos(2\theta) = 0$ ⇝ $\theta = \begin{cases} \pi/4 \\ 3\pi/4 \end{cases}$

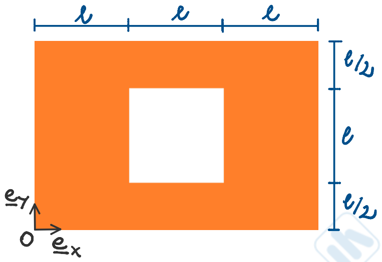


$\boxed{\text{CASO 2}}$ $I_{oxx} \neq I_{oyy} \Rightarrow \cos(2\theta) \neq 0$ ⇝ $\tan(2\theta) = \frac{2 I_{oxy}}{I_{oxx} - I_{oyy}}$

$$⇝ \boxed{\theta = \frac{1}{2} \arctan \frac{2 I_{oxy}}{I_{oxx} - I_{oyy}}}$$

Se $I_{oxy} = 0 \Rightarrow \theta = \begin{cases} 0 \\ \pi/2 \end{cases}$ ⇝ $\underline{e}_x$ ed $\underline{e}_y$ erano già principali

Esercizi di Inerzia: FIGURE COMPOSTE E DIREZIONI PRINCIPALI



massa totale : 5 m (uniformemente distribuita)

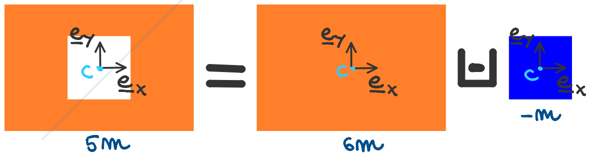
1) C-O = ? 2) $\underline{\underline{I}}_C$ = ?

3) $\underline{\underline{I}}_O$ = ? 4) $I_{O_{\underline{n}}}$ = ? con $\underline{n} \parallel C-O$

5) base principale di $\underline{\underline{I}}_O$ = ?

1) Per simmetria $\boxed{C-O = \frac{3}{2}\ell\,\underline{e}_x + \ell\,\underline{e}_y}$ e $(\underline{e}_x, \underline{e}_y, \underline{e}_z)$ principale per $\underline{\underline{I}}_C$

2)



TEOREMA DELLA LACUNA

$$C-O = \frac{m_1(C_1-O)+m_2^-(C_2-O)}{m_1+m_2^-}$$

$$\underline{\underline{I}}_C = \underline{\underline{I}}_{C_1} + \underline{\underline{I}}_{C_2}^- + \frac{m_1 m_2^-}{m_1+m_2^-}|C_1-C_2|^2(\underline{\underline{I}} - \underline{e}_{12}\otimes\underline{e}_{12})$$

m_2^- NEGATIVA

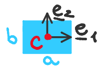
$C_1 \equiv C_2 \equiv C \Rightarrow C_1 - C_2 = \underline{0}$

$$\underline{\underline{I}}_{C_1} = \frac{1}{12}6m\,4\ell^2\underline{e}_x\otimes\underline{e}_x + \frac{1}{12}6m\,9\ell^2\underline{e}_y\otimes\underline{e}_y + \frac{1}{12}6m\,13\ell^2\underline{e}_z\otimes\underline{e}_z$$

$$\underline{\underline{I}}_{C_2} = -\frac{1}{12}m\ell^2\underline{e}_x\otimes\underline{e}_x - \frac{1}{12}m\ell^2\underline{e}_y\otimes\underline{e}_y - \frac{1}{6}m\ell^2\underline{e}_z\otimes\underline{e}_z$$

$$\boxed{\underline{\underline{I}}_C = \frac{23}{12}m\ell^2\underline{e}_x\otimes\underline{e}_x + \frac{53}{12}m\ell^2\underline{e}_y\otimes\underline{e}_y + \frac{19}{3}m\ell^2\underline{e}_z\otimes\underline{e}_z}$$

RETTANGOLO



massa: m

$$\underline{\underline{I}}_C = \frac{1}{12}mb^2\underline{e}_1\otimes\underline{e}_1 + \frac{1}{12}ma^2\underline{e}_2\otimes\underline{e}_2 + \frac{1}{12}m(a^2+b^2)\underline{e}_3\otimes\underline{e}_3$$

3) $\underline{n} = \underline{e} = \frac{C-O}{|C-O|} = \frac{\frac{3}{2}\underline{e}_x+\underline{e}_y}{\sqrt{13/4}\,\ell}\ell = \frac{3}{\sqrt{13}}\underline{e}_x + \frac{2}{\sqrt{13}}\underline{e}_y$

$|C-O|^2 = \frac{13}{4}\ell^2$ $\qquad = \frac{1}{\sqrt{13}}(3\underline{e}_x + 2\underline{e}_y)$

TEOREMA DI HUYGENS-STEINER

$$\underline{\underline{I}}_O = \underline{\underline{I}}_C + m|C-O|^2(\underline{\underline{I}} - \underline{e}\otimes\underline{e})$$

$$\boxed{\underline{\underline{I}}_O} = \frac{23}{12}m\ell^2\underline{e}_x\otimes\underline{e}_x + \frac{53}{12}m\ell^2\underline{e}_y\otimes\underline{e}_y + \frac{19}{3}m\ell^2\underline{e}_z\otimes\underline{e}_z + 5m\frac{13}{4}\ell^2\left[\underline{\underline{I}} - \frac{1}{13}(3\underline{e}_x+2\underline{e}_y)\otimes(3\underline{e}_x+2\underline{e}_y)\right]$$

$$= \left(\frac{23}{12}+5\right)m\ell^2\underline{e}_x\otimes\underline{e}_x - 5\frac{3}{2}m\ell^2\underline{e}_x\otimes\underline{e}_y - 5\frac{3}{2}m\ell^2\underline{e}_y\otimes\underline{e}_x + \left(\frac{53}{12}+5\frac{9}{4}\right)m\ell^2\underline{e}_y\otimes\underline{e}_y + \left(\frac{19}{3}+5\frac{13}{4}\right)m\ell^2\underline{e}_z\otimes\underline{e}_z$$

$$= \boxed{\frac{83}{12}m\ell^2\underline{e}_x\otimes\underline{e}_x - \frac{15}{2}m\ell^2\underline{e}_x\otimes\underline{e}_y - \frac{15}{2}m\ell^2\underline{e}_y\otimes\underline{e}_x + \frac{188}{12}m\ell^2\underline{e}_y\otimes\underline{e}_y + \frac{271}{12}m\ell^2\underline{e}_z\otimes\underline{e}_z}$$

4) $\boxed{I_{O,\underline{n}}} = \underline{n}\cdot\underline{\underline{I}}_O\underline{n} = \frac{1}{13}(3\underline{e}_x+2\underline{e}_y)\cdot\left[\left(\frac{83}{4}-15\right)\underline{e}_x + \left(\frac{94}{3}-\frac{45}{2}\right)\underline{e}_y\right]m\ell^2$

$= \frac{1}{13}\left(\frac{69}{4}+\frac{53}{3}\right) = \frac{1}{13}\cdot\frac{419}{12} = \boxed{\frac{419}{166}}$

definizione

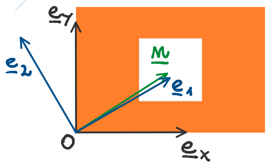
MOMENTO ASSIALE:

$I_{O,\underline{u}} = \underline{u}\cdot\underline{\underline{I}}_O\underline{u}$

5) $B = (\underline{e}_1, \underline{e}_2, \underline{e}_z)$ con $\underline{e}_1 = \cos\vartheta\,\underline{e}_x + \sin\vartheta\,\underline{e}_y$, $\underline{e}_2 = -\sin\vartheta\,\underline{e}_x + \cos\vartheta\,\underline{e}_y$

dove $\vartheta = \frac{1}{2}\arctan\frac{2I_{Oxy}}{I_{Oxx}-I_{Oyy}}$

$= \frac{1}{2}\arctan\frac{12}{7}$



DIREZIONI PRINCIPALI in 2D

$B = (\underline{e}_1, \underline{e}_2, \underline{e}_z)$

$\underline{e}_1 = \cos\vartheta\,\underline{e}_x + \sin\vartheta\,\underline{e}_y$, $\underline{e}_2 = \underline{e}_z \times \underline{e}_1$

$I_{Oxx} = I_{Oyy}$ e $I_{Oxy} = 0 \rightsquigarrow \forall\vartheta$

$I_{Oxx} = I_{Oyy}$ e $I_{Oxy} \neq 0 \rightsquigarrow \vartheta = \pi/4$ o $\vartheta = 3\pi/4$

$I_{Oxx} \neq I_{Oyy}$

$\hookrightarrow \vartheta = \frac{1}{2}\arctan\frac{2I_{Oxy}}{I_{Oxx}-I_{Oxy}}$