DESIGN & ENGINEERING
A.Y. 2023/2024

POLITECNICO DI MILANO

Kinematics of the particle

Binbin Liu

DIPARTIMENTO DI MECCANICA

mecc

Outline

- **Basic concepts**
- The particle model
- Kinematics of a particle
- Motion of a particle

Kinematics

Kinematics is the branch of mechanics dealing with the motion of objects like particles, bodies, or systems of bodies, without taking into consideration the causes (forces, moments) originating the motion.

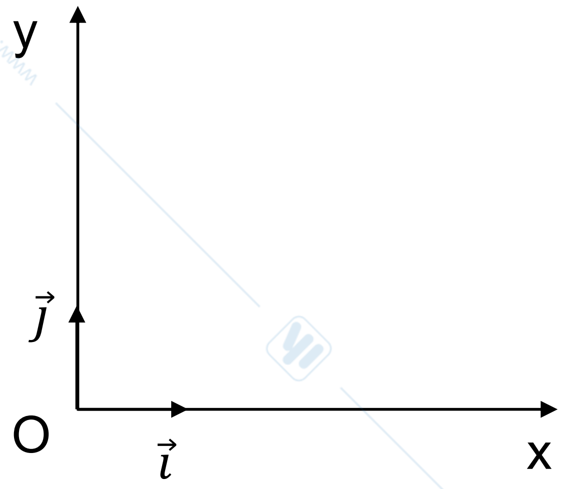
	particles
KINEMATICS	bodies
	systems of bodies

It is important to underline that all the description of motion is always based on what is seen by an observer.

Different observers (for example characterized by a relative motion between them) will describe the motion of the same point (or body, system of bodies, etc) in a different way (an example will be shown later).

Reference frame

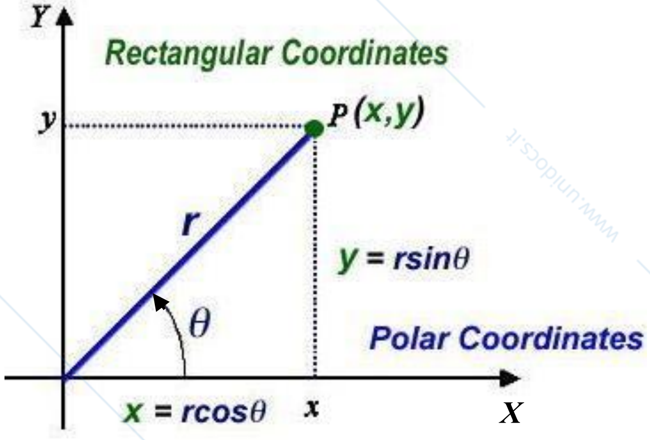
For the reason explained, it is important to remember that we define the motion of a system with respect to a reference frame, identified by *an origin* **O**, two (or three in space) *orthogonal axes* **x** and **y** (and **z** in space), whose direction and orientation are identified by two unit vectors conventionally called $\vec{\imath}$ and $\vec{\jmath}$ *(and* $\vec{k}$ *in space)*



Planar motion (2D): we assume that the motion of any point occurs in a plane described by a reference system e.g. the one shown above. This assumption remains true throughout the course.

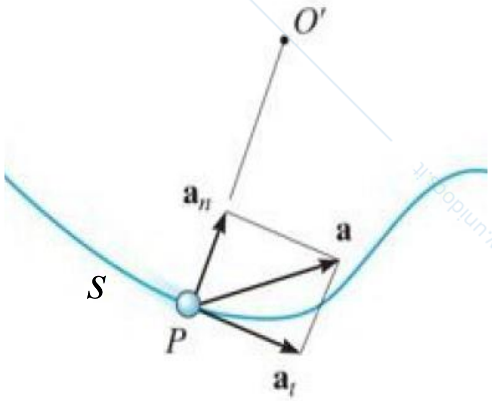
Coordinates (2D)

Rectangular/Cartesian coordinate(x, y): only 2 coordinates are needed to identify the position of a particle (2 DoF) in a plane.



Polar coordinate (r, θ): defined by the radial coordinate r, which is measured positive outward from a fixed origin to the particle, and the transverse coordinate θ is measured counterclockwise from a fixed reference line to r.

Normal and tangential coordinate (n, t): body fixed coordinate attached to and moved with a particle along the path s. The coordinate t is tangent to the path and positive in the direction of increasing s, while the coordinate n is normal to the path, and is positive when directed toward the center of curvature.



Outline

- Basic concepts
- **The particle model**
- Kinematics of a particle
- Motion of a particle

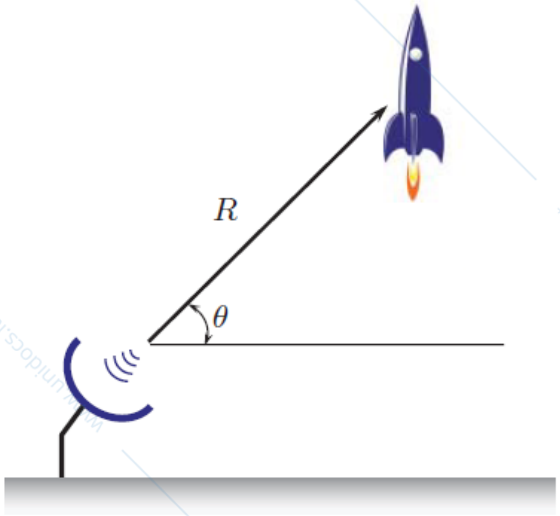
The particle model

The particle model is the simplest possible. A particle has the following features:

- It has **no dimensions** => it can just be in a specific position in the plane (or in space). We need only **2 variables** to describe its position (3 in space).
- The overall mass of the model is coincident with the particle itself and it's the only inertial property of the system.

only have mass for a particle

- Having no dimensions, the particle can't rotate.



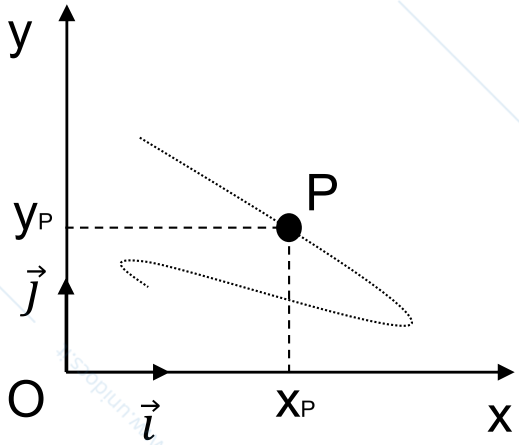
Kinematics of a particle

3 aspect: position, velocity, accelleration

Particle: a point mass with negligible size and shape (unnecessary to be small).

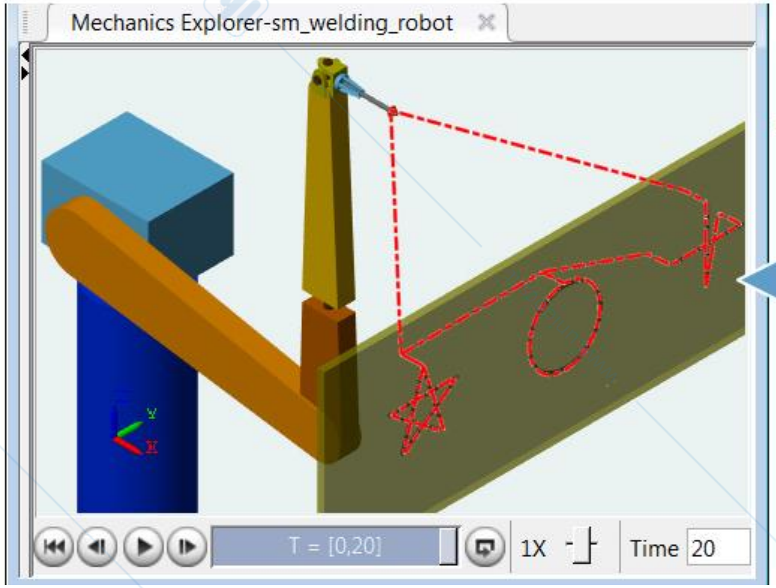
Kinematics of a particle: the description of motion in terms of geometric aspects of a point mass in space (or plane 2D)

Need to specify a **reference frame**(and a **coordinate system** in it to actually write the vector expressions).



Applications

Robot arm



Desired Tool Path

30-100/min Frenquency

4-40m/s Speed

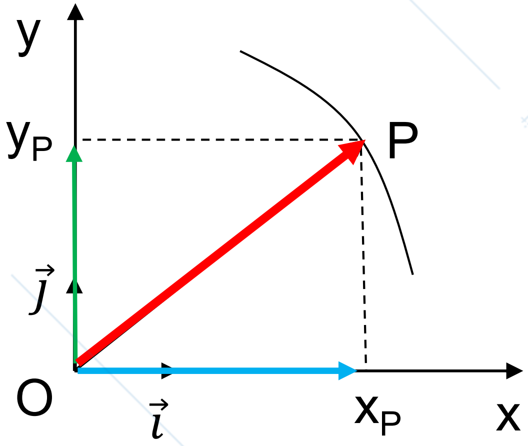
Auto Table Tennis

Outline

- Basic concepts
- The particle model
- **Kinematics of a particle** → describe the motion
- Motion of a particle

Position (Cartesian coord.)

Let’s consider a point P moving on the plane with respect to the reference frame:



Position vector $\vec{P} = (P - O) = x_P\vec{\imath} + y_P\vec{j}$

where $\vec{\imath}$ and $\vec{j}$ are unit vectors

Law of motion $\begin{cases} x_P = x(t) \\ y_P = y(t) \end{cases}$

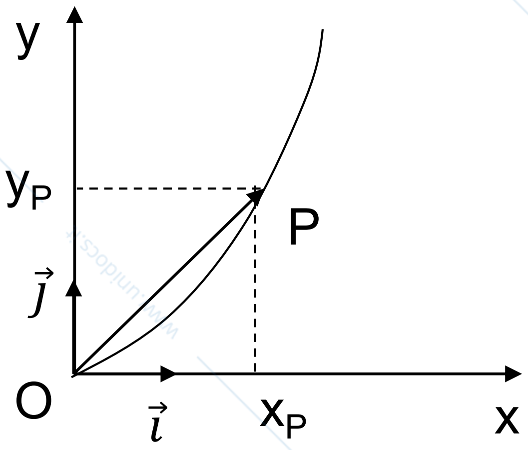
The vector $\vec{P}$ and, consequently, the coordinates x_P and y_P are functions of time

We can define the **trajectory (path)** of $\vec{P}$ by eliminating the dependancy of time from the expression, thus writing $y_P(x_P)$.

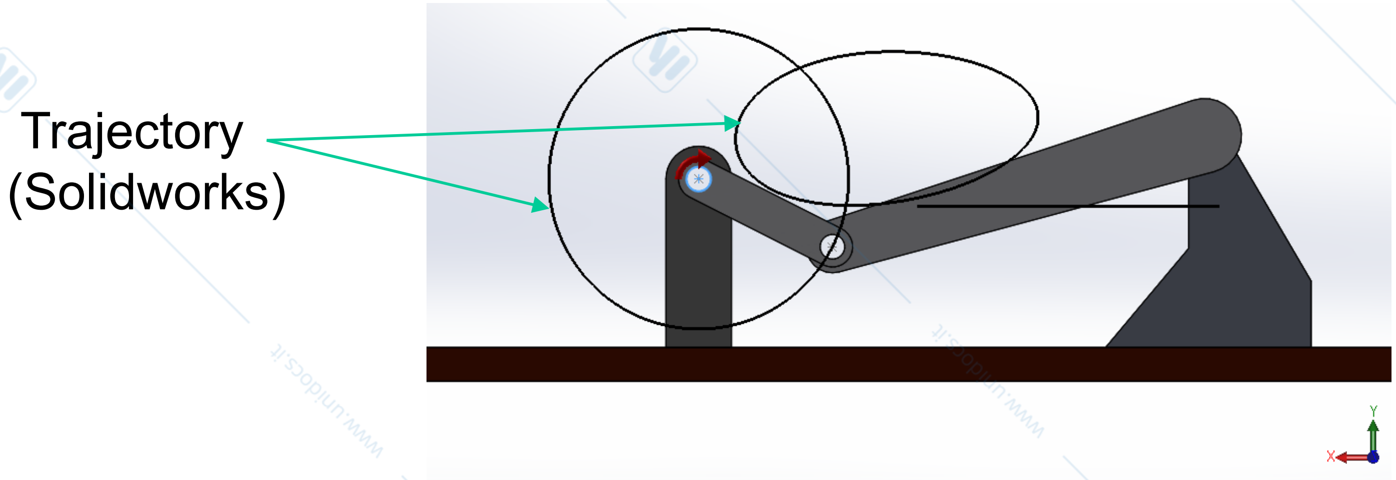
Position (Trajectory)

Example:

$$\begin{cases} x_P = 2t \\ y_P = 4t^2 \end{cases} \Longrightarrow \begin{cases} t = \dfrac{x_P}{2} \\ y_P = 4\dfrac{{x_P}^2}{4} = x_P^2 \end{cases}$$



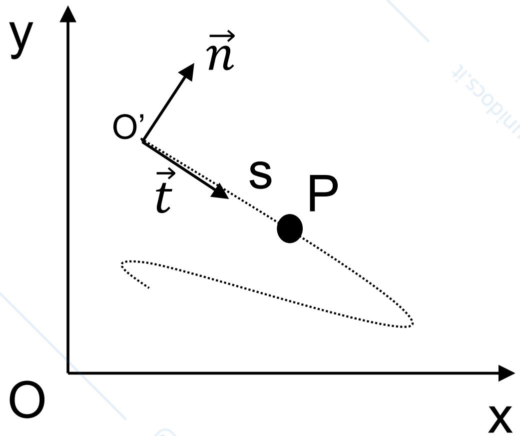
The trajectory of P, $y_P(x_P)$ is obtained by eliminating the time dependence

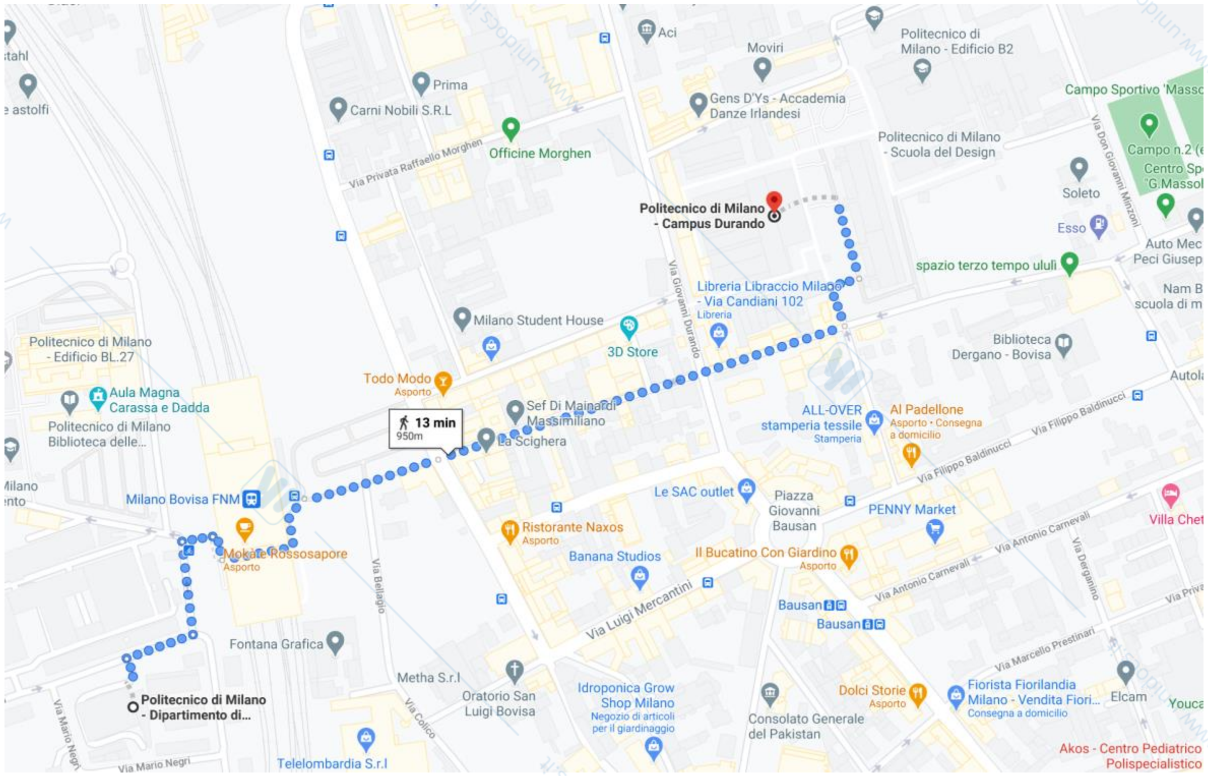


Position (Trajectory)

The trajectory can also be a function of time in the normal and tangential coordinates, so the position of P can also be defined using the trajectory function s(t).

time





Position vector

$$\vec{P} = = (P - O') = s\vec{t}$$

tangent unit vector! → tells the direction

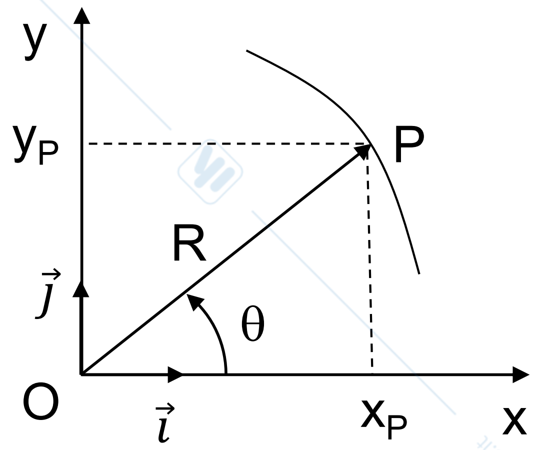
Position (Polar coord.)

Another useful expression to represent the position is obtained imagining that the **X axis** corresponds to the **Real axis** and the **Y axis** to the **Imaginary axis** of a complex plane. This allows a simple representation of the position in *Polar coordinates*

θ describing the rotation of R (positive counterclockwise).

Position vector $\quad \vec{P} = (P - O) = R\boxed{e^{i\theta}} = x_P + iy_P$

Generally, both *R* and θ are functions of time, *R* is **positive outward**, and θ is **positive counterclockwise** from a fixed reference line to *R*.



From Polar to Cartesian

$$x_p = R\cos(\theta)$$
$$y_P = R\sin(\theta)$$

From Cartesian to Polar

$$R = \sqrt{{x_p}^2 + {y_p}^2}$$
$$\theta = \operatorname{atan}\left(\frac{y_P}{x_P}\right)$$

Note: $\vec{\imath}$ is the unit vector along x-axis,
i in $e^{i\theta}$ is imaginary unit = $\sqrt{-1}$,
$e^{i\theta} = \cos\theta + i\,\sin\theta$ (Euler's formula)

Velocity (Cartesian coord.)

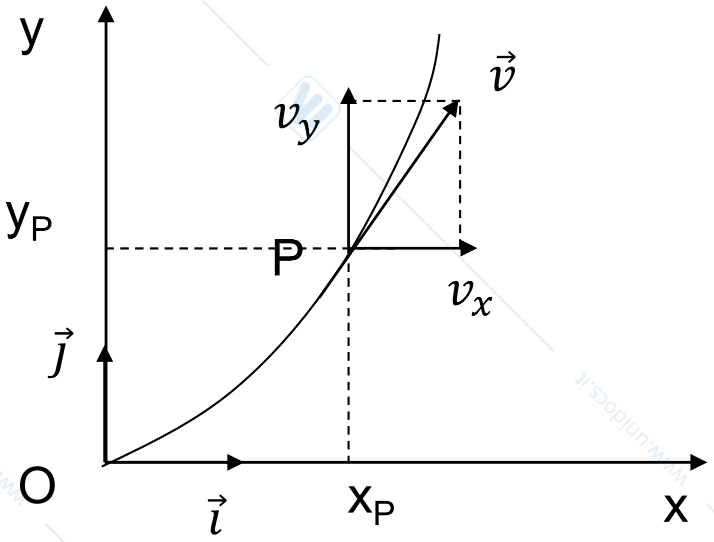
Velocity (instantaneous) is a **vector** representing how the position vector changes with respect to time

$$\overrightarrow{P} = (P - O) = x_P\vec{i} + y_P\vec{j} \longrightarrow \vec{v} = \frac{d\vec{P}}{dt} = \dot{x}_P\,\vec{i} + \dot{y}_P\vec{j}$$

$$\begin{cases} v_x = \dot{x}_P(t) \\ v_y = \dot{y}_P(t) \end{cases}$$

v_x and v_y are the components of velocity vector along x and y axis

we calculate them by time derivative of x_P and y_P respectively



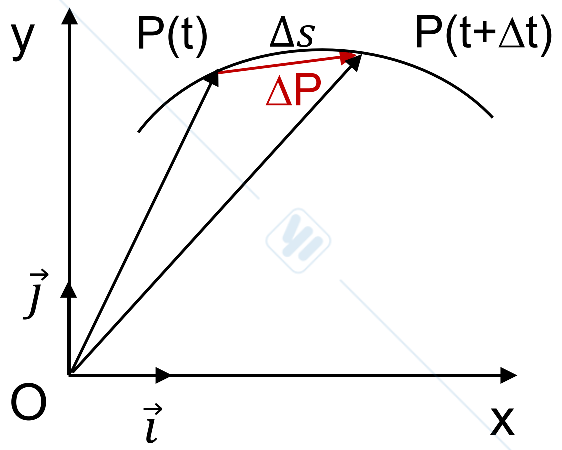
Velocity (Cartesian coord.)

The time derivative can be seen as the limit for Δt going to zero and since ΔP approaches the arc length Δ*s* as Δt → 0, we have

tangent unit vector!

$$\vec{v} = \frac{d\vec{P}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{P}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s\vec{t}}{\Delta t} = \frac{ds}{dt}\vec{t} = \dot{s}\,\vec{t}$$

time



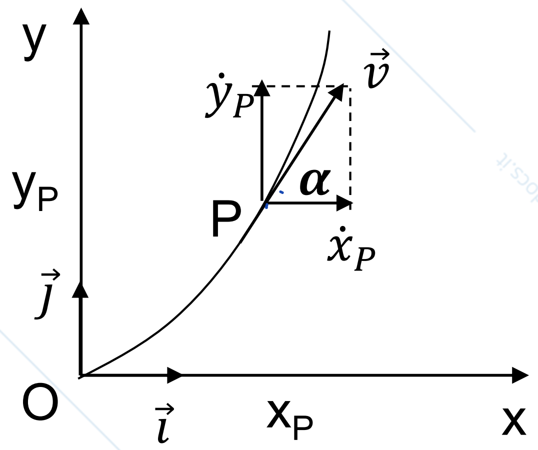
The **velocity vector** is always **tangent to the trajectory** (normal component is null)

Velocity

Example:

$$\begin{cases} x_P = 2t \\ y_P = 4t^2 \end{cases} \Longrightarrow \begin{cases} \dot{x}_P(t) = 2 \\ \dot{y}_P(t) = 8t \end{cases}$$

position velocity



Scalar quantity

Magnitude (speed):

$$|\vec{v}| = \sqrt{\dot{x}_P^2 + \dot{y}_P^2}$$

Angle:

$$tan(\alpha) = \frac{\dot{y}_P}{\dot{x}_P}$$

Tangent versor: $\vec{t} = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}$

It can be seen that the velocity represents how fast the position of point P changes along the trajectory and can be written by using magnitude and tangent unit vector as:

$$\vec{v} = |\vec{v}|\vec{t} \quad \text{or} \quad \boxed{\vec{v} = v\vec{t}}$$

Velocity (Polar coord.)

Remembering the expression of position in polar coordinates, it is possible to calculate the corresponding velocity by applying the time-derivative of the position expression. Remembering that in general both *R* and θ depend on time, the time derivative is the sum of two components

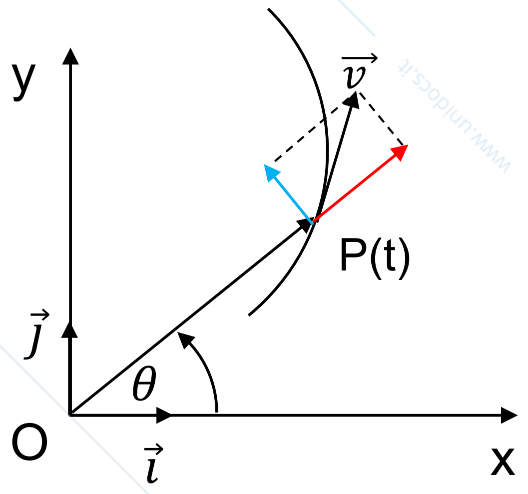
The total velocity will be the vector sum of the two components

Position vector $\overrightarrow{P} = (P - O) = Re^{i\theta} = x_P + iy_P$

Velocity vector $\overrightarrow{v} = \frac{d(P-O)}{dt} = \dot{R}e^{i\theta} + iR\dot{\theta}e^{i\theta} = \dot{R}e^{i\theta} + R\dot{\theta}e^{i(\theta+\frac{\pi}{2})}$

Velocity (Polar coord.)

Velocity vector $\vec{v} = \frac{d(P-O)}{dt} = \dot{R}e^{i\theta} + i R\dot{\theta}e^{i\theta} = \dot{R}e^{i\theta} + R\dot{\theta}e^{i(\theta+\frac{\pi}{2})}$



- The first component (red) represents how fast P changes its distance from the origin
- The second component (light blue) represents how fast the angle θ, associated with the position vector, is changing

15 min break! See you soon….

Acceleration (Cartesian coord.)

Acceleration is a **vector** considering the variation of velocity vector with respect to time

$$\overrightarrow{P} = (P - O) = x_P\vec{\imath} + y_P\vec{\jmath}$$

$$\vec{v} = \frac{d\vec{P}}{dt} = \dot{x}_P\ \vec{\imath} + \dot{y}_P\vec{\jmath}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}_P\ \vec{\imath} + \ddot{y}_P\vec{\jmath}$$

$$\begin{cases} a_x = \ddot{x}_P(t) \\ a_y = \ddot{y}_P(t) \end{cases}$$

a_x and a_y are the components of acceleration vector along x and y axis.

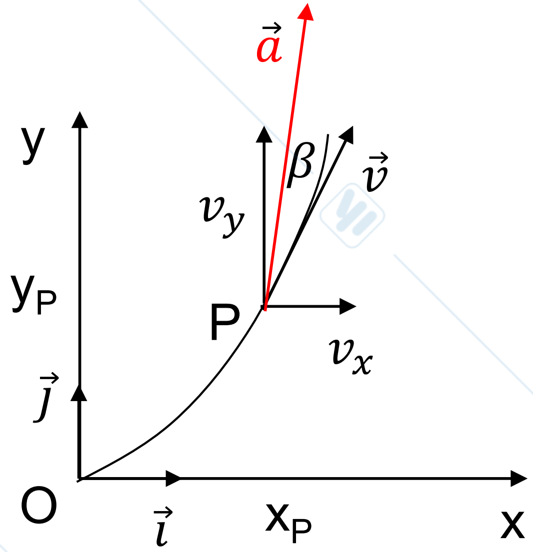
We calculate them by time derivative of $\dot{x}_P$ and $\dot{y}_P$ respectively

Acceleration

Acceleration is a **vector** considering the variation of velocity vector with respect to time

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}_P \, \vec{\imath} + \ddot{y}_P \vec{\jmath}$$

Total acceleration <u>is not</u> necessarily tangent to velocity.
In the example velocity and acceleration create β angle

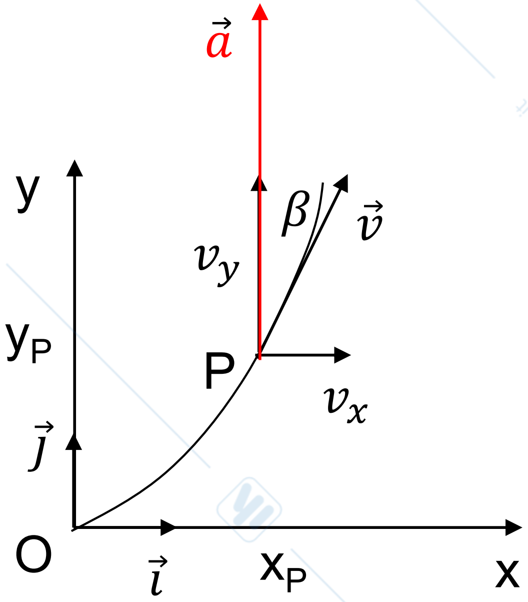


Acceleration – an example

$$\begin{cases} x_P = 2t \\ y_P = 4t^2 \end{cases} \Rightarrow \begin{cases} \dot{x}_P(t) = 2 \\ \dot{y}_P(t) = 8t \end{cases} \Rightarrow \begin{cases} \ddot{x}_P(t) = 0 \\ \ddot{y}_P(t) = 8 \end{cases}$$

position velocity acceleration

Catch me outside, how bout that



As explained before, it is possible to see that the acceleration is not tangent to the trajectory, but we can identify an angle between velocity and acceleration.

Consequently, the acceleration vector can be expressed as the sum of one component parallel to the velocity (**tangent acceleration**) and one other component orthogonal to the velocity (**normal or centripetal acceleration**)

Acceleration (n,t coord.)

Describing total acceleration $\vec{a}$ as the sum of a_x and a_y terms is generally not useful. It's better to describe it as the sum of **tangential** and **normal** terms a_t and a_n

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}_P\,\vec{i} + \ddot{y}_P\vec{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(v\vec{t}\right)}{dt} = \frac{dv}{dt}\,\vec{t} + v\,\frac{d\vec{t}}{dt}$$

tangent unit vector!

time

$$\frac{d\vec{t}}{dt}\ ?$$

Acceleration (n,t coord.)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(v\vec{t}\right)}{dt} = \frac{dv}{dt}\,\vec{t} + v\,\frac{d\vec{t}}{dt}$$

tangent unit vector!

time

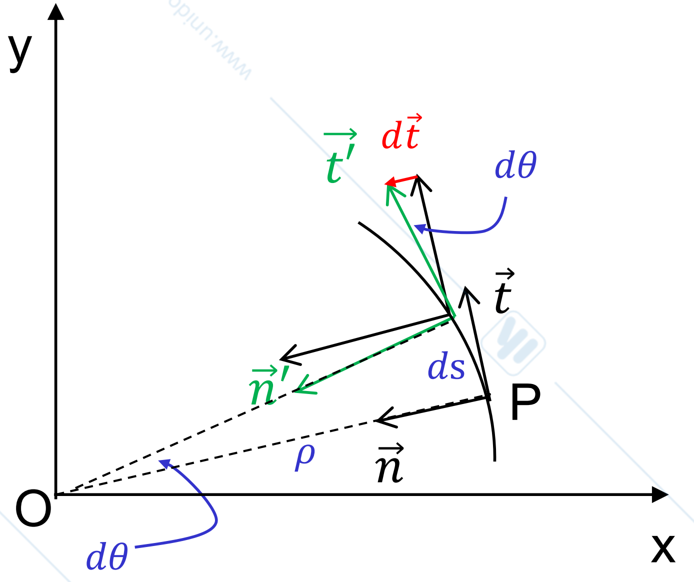
$$\frac{d\vec{t}}{dt}\,?$$

$$|d\vec{t}| = 1 \cdot d\theta$$

$$d\vec{t} = d\theta\,\vec{n} = \frac{ds}{\rho}\vec{n}$$

$$\frac{d\vec{t}}{dt} = \frac{1}{\rho}\frac{ds\vec{n}}{dt} = \frac{1}{\rho}v\,\vec{n}$$

$$\vec{a} = \frac{dv}{dt}\,\vec{t} + v\,\frac{d\vec{t}}{dt} = \dot{v}\vec{t} + \frac{v^2}{\rho}\vec{n}$$

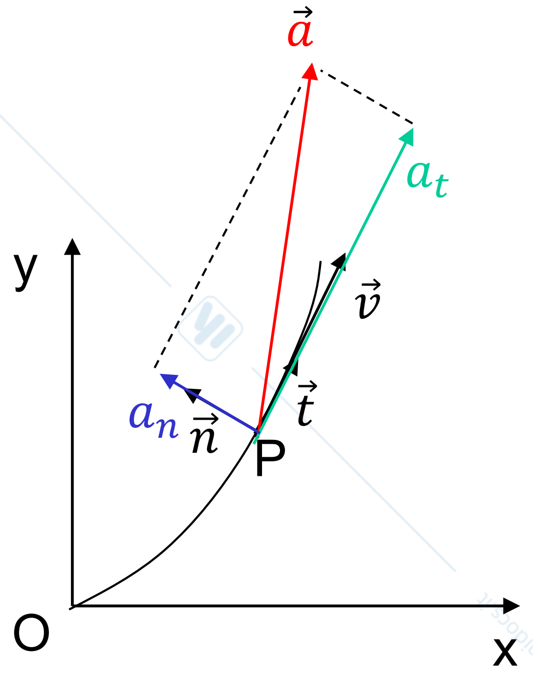


- $\vec{n}$ represents the unit vector orthogonal to the trajectory
- ρ represents the radius of the so-called ***osculating circle***, which is the circle approximating the trajectory in the neighborhood of point P

Acceleration (n,t coord.)

Total acceleration $\vec{a}$ can be seen as the sum of **tangential** and **normal** terms a_t and a_n

$$\vec{a} = \frac{dv}{dt}\vec{t} + v\frac{d\vec{t}}{dt} = \dot{v}\vec{t} + \frac{v^2}{\rho}\vec{n} = \ddot{s}\vec{t} + \frac{\dot{s}^2}{\rho}\vec{n} = a_t\vec{t} + a_n\vec{n}$$



Tangential acceleration $\dot{v}$ $(or\ \ddot{s})$ is **parallel** to the velocity (i.e. **tangent** to the trajectory) which indicates the variation of the **magnitude** of the velocity vector.

Normal acceleration $\frac{v^2}{\rho}$ (or $\frac{\dot{s}^2}{\rho}$) is **orthogonal** to the velocity which indicates the variation of the **direction** of the velocity vector.

Acceleration (Polar coord.)

While normal and tangential coordinates are more intuitive from a physical point of view, polar coordinates allow a simpler mathematical expression.

Velocity vector

$$\vec{v}= \frac{d(P-O)}{dt} = \dot{R}e^{i\theta} + iR\dot{\theta}e^{i\theta} = \dot{R}e^{i\theta} + R\dot{\theta}e^{i(\theta+\frac{\pi}{2})}$$

$$\hookrightarrow \ddot{R}e^{i\theta} + \dot{R}i\dot{\theta}e^{i\theta} + i\dot{R}\dot{\theta}e^{i\theta} + iR(\ddot{\theta}e^{i\theta} + \dot{\theta}i\dot{\theta}e^{i\theta})$$

$$= \ddot{R}e^{i\theta} + i\dot{R}\dot{\theta}e^{i\theta} + i\dot{R}\dot{\theta}e^{i\theta} + iR\ddot{\theta}e^{i\theta} - i^2\dot{\theta}^2Re^{i\theta} =$$

Acceleration vector $= \ddot{R}e^{i\theta} + 2i\dot{R}\dot{\theta}e^{i\theta} + R\ddot{\theta}e^{i(\theta+\frac{\pi}{2})} + R\dot{\theta}^2e^{i(\theta+\pi)}$

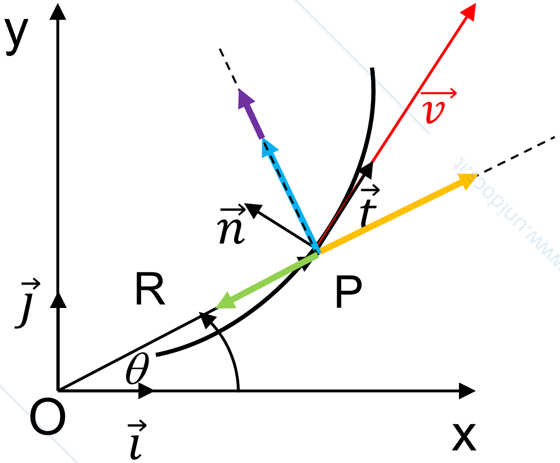
$$\vec{a}= \frac{d\vec{v}}{dt} = \ddot{R}e^{i\theta} + 2i\dot{R}\dot{\theta}e^{i\theta} + R\ddot{\theta}e^{i(\theta+\frac{\pi}{2})} + Ri\dot{\theta}^2e^{i(\theta+\frac{\pi}{2})}$$

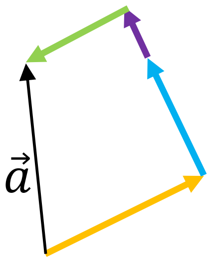
$$= \ddot{R}e^{i\theta} + 2\dot{R}\dot{\theta}e^{i(\theta+\frac{\pi}{2})} + R\ddot{\theta}e^{i(\theta+\frac{\pi}{2})} + R\dot{\theta}^2e^{i(\theta+\pi)}$$

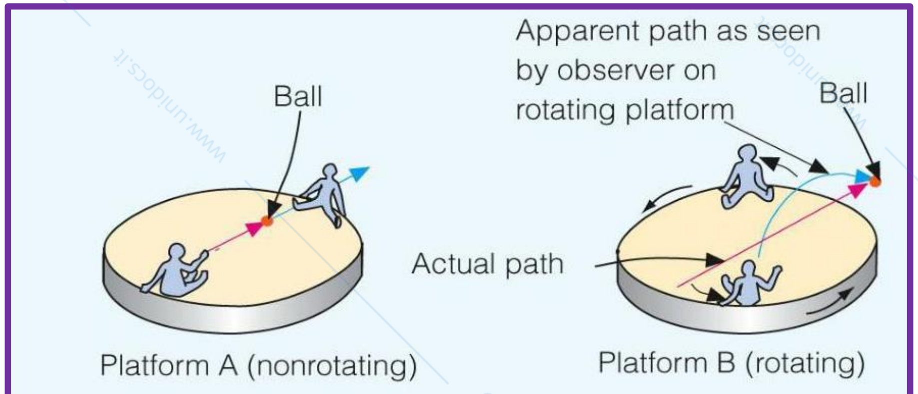
Acceleration (polar coord.)

Acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{R}e^{i\theta} + 2\dot{R}\dot{\theta}e^{i\left(\theta+\frac{\pi}{2}\right)} + R\ddot{\theta}e^{i\left(\theta+\frac{\pi}{2}\right)} + R\dot{\theta}^2 e^{i(\theta+\pi)}$$







Coriolis effect

© 2007 Thomson Higher Education

- $\ddot{R}e^{i\theta}$: represents the contribution due to the variation of the speed of the magnitude of the position vector of the point P
- $R\ddot{\theta}e^{i\left(\theta+\frac{\pi}{2}\right)}$: represents the variation of angular speed of the vector associated with the position of point P
- $R\dot{\theta}^2 e^{i(\theta+\pi)}$: represents the contribution due to the rotation of the vector associated with the position of point P
- $2\dot{R}\dot{\theta}e^{i\left(\theta+\frac{\pi}{2}\right)}$: called Coriolis acceleration, is due to the contemporary presence of a variation of the magnitude and the orientation

Motion of a particle

- Basic concepts
- The particle model
- Kinematics of a particle
- **Motion of a particle**

Motion of a particle

Rectilinear motion: refers to the motion of an object along a straight line.

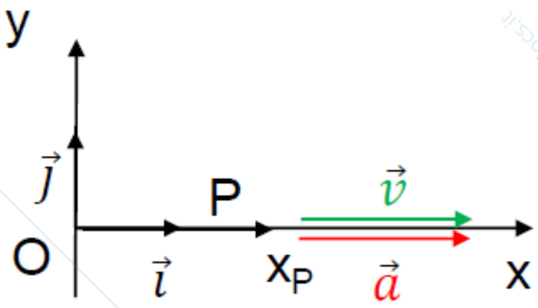
Source: Pin on Practical Exam (Physics) (pinterest.com)

Motion of a particle

Example

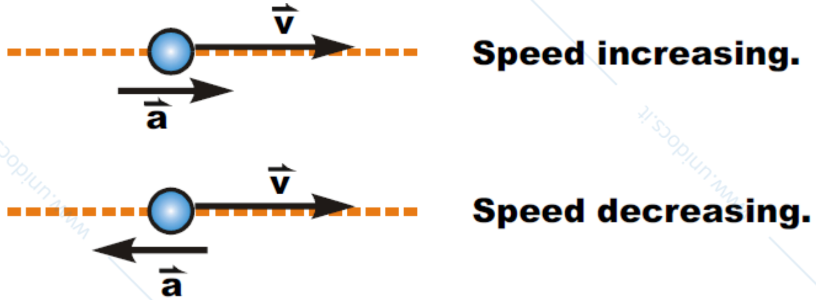
$$\begin{cases} x_P = 2t^3 + 24t + 6 \\ y_P = 0 \end{cases} \Longrightarrow \begin{cases} \dot{x}_P(t) = v_x = 6t^2 + 24 \\ \dot{y}_P(t) = v_y = 0 \end{cases} \Longrightarrow \begin{cases} \ddot{x}_P(t) = a_x = 12t \\ \ddot{y}_P(t) = a_y = 0 \end{cases}$$

position velocity acceleration



In this example, the vertical components of the velocity v_y and acceleration a_y are both null. The tangential component of the acceleration $a_t = a_x$.

Rectilinear motion: both velocity and acceleration are tangent to the **trajectory**. The normal component of the acceleration $a_n = 0$, which means the direction of the velocity does not change with time.



Speed increasing.

Speed decreasing.

Motion of a particle

Curvilinear motion: refers to the motion of a particle along a curve (the trajectory is curved).

circular motion

General curvilinear motion

Motion of a particle

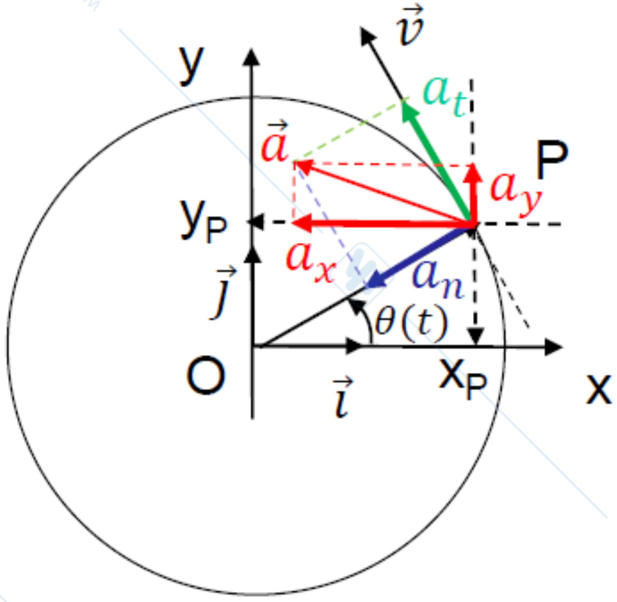
Example

$$\begin{cases} x_P = \cos\theta \\ y_P = \sin\theta \\ where\ \theta = t^2 \end{cases} \Longrightarrow \begin{cases} \dot{x}_P(t) = v_x = -\dot{\theta}\sin\theta \\ \dot{y}_P(t) = v_y = \dot{\theta}\cos\theta \end{cases} \Longrightarrow \begin{cases} \ddot{x}_P(t) = a_x = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta \\ \ddot{y}_P(t) = a_y = \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta \end{cases}$$

position velocity acceleration

$$where\ \theta = t^2 \Longrightarrow \begin{cases} a_t = \ddot{\theta} = 2 \\ a_n = \dot{\theta}^2 = 4t^2 \end{cases}$$

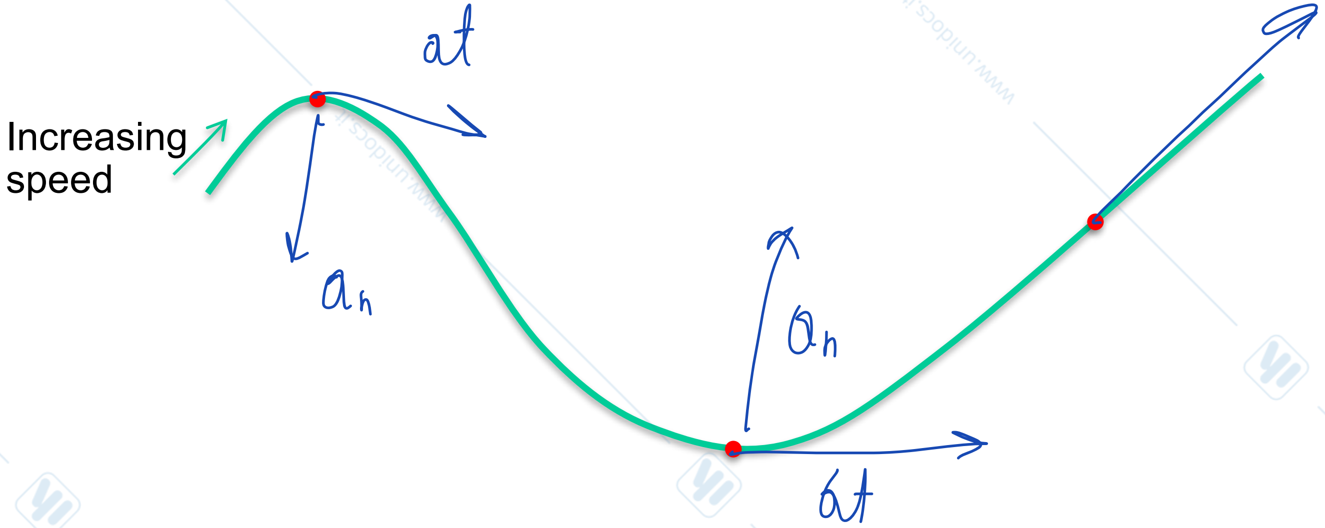
In this example, both **tangential** and **normal** components of the acceleration are non-zero, and the **magnitude** and **direction** of the velocity are both changing with time.



Curvilinear motion: The normal component of the acceleration $a_n \neq 0$, which means the direction of the velocity changes with time.

Practice

Indicate the directions of acceleration a_t, a_n of the red particle on its trajectory.



Hints:

Tangential acceleration *a_t is **parallel** to the velocity (i.e. **tangent** to the trajectory) which indicates the variation of the **magnitude** of the velocity vector.*

Normal acceleration *a_n is **orthogonal** to the velocity which indicates the variation of the **direction** of the velocity vector.*