Microeconomics, Winter 2019
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SOLUTION Part A

Problem 1 (50 %)

The price of the output is p while the two inputs have prices $w_1 > 0, w_2 > 0$. Firm 1 has the production function $\phi(z_1, z_2) = [az_1 + bz_2]^s$ while firm 2 has the production function $\phi(z_1, z_2) = [2\min\{az_1, bz_2\}]^s$ with $a > 0, b > 0$ and $s > 0$

(I) For each of these production functions ($i = 1, 2$): (**a**) Present the isoquants. (**b**) How does the returns to scale depend on s?

Solution: (**a**) See figure file for illustrations.
(**b**) Let $t > 1$ and $z_1 > 0, z_2 > 0$. Then $\phi_1(tz_1, tz_2) = [atz_1 + btz_2]^s = [az_1 + bz_2]^s t^s > [az_1 + bz_2]^s t = t\phi_1(z_1, z_2)$ for $s > 1$, while we have equality for $s = 1$ and the opposite inequality for $s < 1$. Hence we have increasing returns to scale for $s > 1$, constant returns to scale for $s = 1$ and decreasing returns to scale for $s < 1$.
$\phi_2(tz_1, tz_2) = [2\min\{atz_1, btz_2\}]^s = t^s [2\min\{az_1, bz_2\}]^s = t^s\phi_2(z_1, z_2)$. So, as for the other production function, we have increasing returns to scale for $s > 1$, constant returns to scale for $s = 1$ and decreasing returns to scale for $s < 1$.

(II) (**a**) Show that the cost function can be written as $C_i(w_1, w_2, q) = C_i(w_1, w_2, 1)q^{1/s}$ where $C_i(w_1, w_2, 1)$ is the cost of producing one unit of output using the technology of firm i. (**b**) Which, if any, of the two firms is more cost efficient? Provide a brief intuition for your answer.

Solution: (**a**) For the first production function, since $MRTS = \frac{a}{b}$ cost minimization implies that $z_1 = 0$ for $\frac{w_1}{w_2} > \frac{a}{b}$ while $z_2 = 0$ for $\frac{w_1}{w_2} < \frac{a}{b}$. As a consequence, the cost function is

$$C_1(w_1, w_2, q) = q^{1/s}\frac{w_1}{a} \text{ for } \frac{w_1}{w_2} \leq \frac{a}{b} \text{ and } C_1(w_1, w_2, q) = q^{1/s}\frac{w_2}{b} \text{ for } \frac{w_1}{w_2} > \frac{a}{b}$$

which can be written as

$$C_1(w_1, w_2, q) = \min\{\frac{w_1}{a}, \frac{w_2}{b}\}q^{1/s} = C_1(w_1, w_2, 1)q^{1/s} \text{ where } C_1(w_1, w_2, 1) = \min\{\frac{w_1}{a}, \frac{w_2}{b}\}$$

For the second production function, since in a cost minimum $(2az_1)^s = (2bz_2)^s = q$, it is

$$C_2(w_1, w_2, y) = q^{1/s}\frac{1}{2}\left[\frac{w_1}{a} + \frac{w_2}{b}\right] = C_2(w_1, w_2, 1)q^{1/s} \text{ where}$$

$C_2(w_1, w_2, 1) = \frac{1}{2}[\frac{w_1}{a} + \frac{w_2}{b}]$.
(**b**) If $\frac{w_1}{a} = \frac{w_2}{b}$, $C_1(w_1, w_2, q) = C_2(w_1, w_2, q), \forall q$. On the other hand, for $\frac{w_1}{a} \neq \frac{w_2}{b}$, $\min\{\frac{w_1}{a}, \frac{w_2}{b}\} < \frac{1}{2}\frac{w_1}{a} + \frac{w_2}{b}$, hence $C_1(w_1, w_2, q) < C_2(w_1, w_2, q), \forall q > 0$. Therefore, in general technology 1 is more cost efficient, which is due to the possibility of (perfectly) substituting one input with the other for this technology. For technology 2, such a substitution is not possible.

(III) (**a**) For which s is supply well defined and what is then individual and aggregate supply? (**b**) Provide an example in terms of s, p, w_1 and w_2 where the supply of one firm is zero while it is positive for the other firm.

Solution: (**a**) We first show that for $s > 1$, there is no solution to the profit maximization problem for any of the two firms.

In both cases, $i = 1, 2$, the profit maximization problem can be written as

$$\max_q pq - C_i(w_1, w_2, 1)q^{1/s} = \max_q q\left[p - C_i(w_1, w_2, 1)q^{1/s-1}\right]$$

If $s > 1$, $C_i(w_1, w_2, 1)q^{1/s-1} \to 0$, as $q \to \infty$, hence profits tend to ∞ as $q \to \infty$, i.e. there is no solution to the profit maximization problem. If $s = 1$ and $p \leq C_i(w_1, w_2, 1)$ there is a solution to the profit maximization problem, else there is not.
Finally, for $s < 1$ the solution is defined by the FOC $p = C_i(w_1, w_2, 1)q^{1/s-1}$ i.e.

$$q^* = \left[\frac{p}{C_i(w_1, w_2, 1)}\right]^{\frac{s}{1-s}}$$

(**b**) If $s = 1$ and $\frac{w_2}{a} > p$ and $\frac{w_1}{a} = p$ firm 2 produces nothing but firm 1 may choose a positive output.

(IV) Suppose for firm 1 that input 2 is fixed at $\bar{z}_2 > 0$. (**a**) Find the short run cost function and compare with the long run cost function. (**b**) Is there a situation where long run costs are always lower than short run costs? Illustrate.

Solution: (**a**) For $(b\bar{z}_2)^s \geq q$, $\tilde{Z}_1(w_1, w_2, q; \bar{z}_2) = 0$ and $\tilde{C}_1(w_1, w_2, q; \bar{z}_2) = w_2\bar{z}_2$. For $(b\bar{z}_2)^s \leq q$, observe that $[az_1 + b\bar{z}_2]^s = q \Leftrightarrow z_1 = \frac{q^{1/s} - b\bar{z}_2}{a}$ so that $\tilde{Z}_1(w_1, w_2, q; \bar{z}_2) = \frac{q^{1/s} - b\bar{z}_2}{a}$ and $\tilde{C}_1(w_1, w_2, q; \bar{z}_2) = w_2\bar{z}_2 + w_1\frac{q^{1/s} - b\bar{z}_2}{a}$.
Thus $\tilde{C}_1(w_1, w_2, q; \bar{z}_2) = w_2\bar{z}_2 + \max\left\{0, w_1\frac{q^{1/s} - b\bar{z}_2}{a}\right\}$
(**b**) If $\frac{w_1}{a} < \frac{w_2}{b}$ long run costs are $\frac{w_1}{a}q^{1/s} < w_2\bar{z}_2 + \max\left\{0, w_1\frac{q^{1/s} - b\bar{z}_2}{a}\right\}, \forall q > 0$.

Problem 2 (50 %)

(I) There is one good and money (which can also be "consumed" in negative amounts) and the price of the good is $p > 0$. There are L consumers, each having the utility function (over the quantity x of the good and the quantity N of money) $u(x, N) = x^{2/3} + N$, defined on $\Re_+ \times \Re$, and income $M > 0$. Find demand for the good as well as the inverse demand function. (Hint: there are no corner solutions)

Solution: The MRS is $\frac{2/3}{x^{1/3}}$ and setting equal to the price ratio, $p/1$, we get $x = \left[\frac{2/3}{p}\right]^3$, so that aggregate demand is $D(p)) = L\left[\frac{2/3}{p}\right]^3$ and inverse demand is $P(q) = \frac{\frac{2}{3}L^{1/3}}{q^{1/3}}$.

(II) A monopolist, who produces the good, has the production function $\phi(z) = z^{3/4}$. (**a**) With input price $w > 0$, find the cost function. (**b**) With aggregate demand from (I), set up and solve the monopolist's profit maximization problem. (**c**) Explain why the outcome is inefficient.

Solution: (**a**) Input demand is $z = q^{4/3}$, hence the cost function is $C(w, q) = wq^{4/3}$.

(**b**)

$$\max_q \frac{2}{3}\frac{L^{1/3}}{q^{1/3}}q - wq^{4/3} \text{ with FOC } \frac{4}{9}L^{1/3}q^{-1/3} = \frac{4}{3}wq^{1/3} \text{ i.e. } q^* = \left[\frac{L^{1/3}}{3w}\right]^{3/2} = \frac{L^{1/2}}{(3w)^{3/2}}$$

(**c**) Since $MC > p$ the outcome is inefficient.

(III) (**a**) Suppose in stead that, with $\alpha > 0, \beta > 0$, all consumers have the utility function $u(x, N) = x^\alpha N^\beta$. Show that there is never a solution to the monopolist's profit maximization problem in that case and explain the intuition behind this result. *NB: You do not need to derive demand, but can just state its format.* (**b**) Suppose that, as before, utility is $u(x, N) = x^{2/3} + N$ but that the monopoly in stead has the production function $\tilde{\phi}(z_1) = \sqrt{\max\{0, z - K\}}$ with $K > 0$. Find the cost function and illustrate and show that in this case the solution to the profit maximization problem of the monopoly may be to produce nothing. A carefully explained graphical argument suffices here. (HINT: You will most likely find that this is the (conceptually) most difficult question in Part 1.)

Solution: (**a**) In this case aggregate demand is

$$D(p) = \frac{L\alpha}{\alpha + \beta}\frac{M}{p} \text{ with inverse } p = \frac{L\alpha}{\alpha + \beta}\frac{M}{q}$$

hence total revenue is $\frac{L\alpha}{\alpha+\beta}M$ independently of the quantity supplied (price elasticity is -1). As the monopoly decreases its quantity towards 0, revenue stays the same, but cost decreases - thus there is no solution to the profit maximization problem.
(**b**) Now, for $q > 0$, $q = \sqrt{z - K} \Leftrightarrow z = q^2 + K$ and $C(w, q) = w(q^2 + K) = wq^2 + wK$, where the first part is variable costs and the second part fixed costs (non-sunk). On the other hand, $C(w, 0) = 0$. Thus average costs is $wq + w\frac{K}{q}$. and if this is everywhere higher than the price, the monopoly would not want to produce anything (i.e. set $q = 0$). So compare $P(q) = \frac{L^{1/3}}{q^{1/3}}$ with average costs:
(i) If $wK > L^{1/3}$, $w\frac{K}{q} > \frac{L^{1/3}}{q^{1/3}}$ for $q \leq 1$, i.e. $wq + w\frac{K}{q} > \frac{L^{1/3}}{q^{1/3}}$ for $q \leq 1$.
(ii) If $w > L^{1/3}$, $\frac{L^{1/3}}{q^{1/3}} < wq$ for all $q \geq 1$, i.e. $wq + w\frac{K}{q} > \frac{L^{1/3}}{q^{1/3}}$ for $q \geq 1$.
In conclusion, for $wK > L^{1/3}$ and $w > L^{1/3}$ the monopolist produces nothing. See figure file for an illustration.