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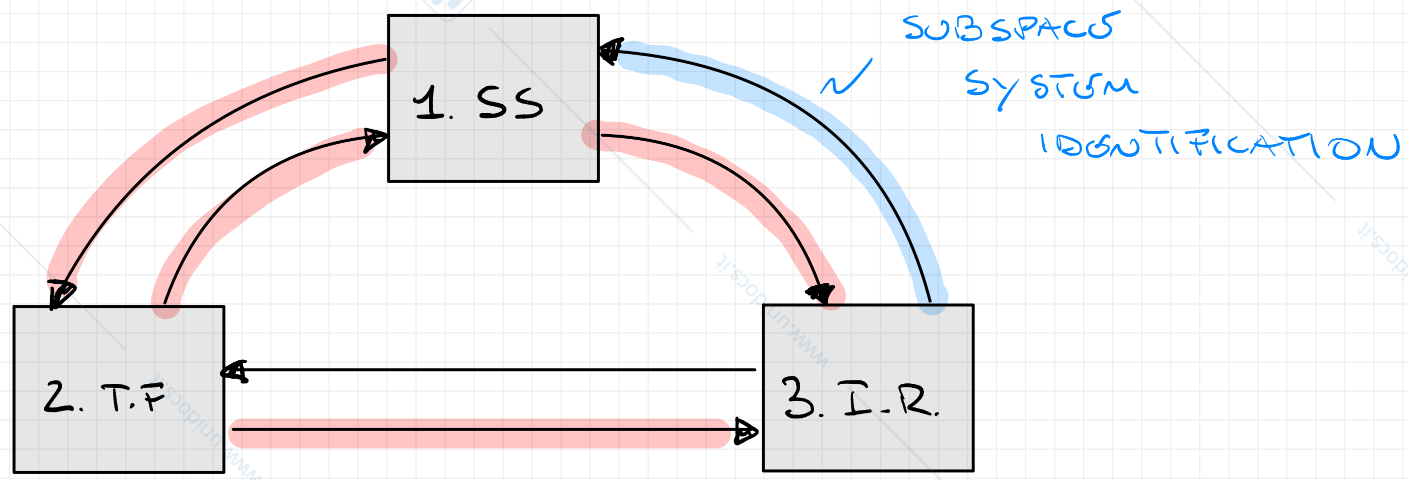
DYNAMIC SYSTEMS REPRESENTATIONS



MAIN TOPICS/SKILLS:

- HANKEL MATRIX CALCULATION
  - S.S. IFORMATION
  - I.R. DATA
- HANKEL MATRIX FACTORIZATION
- USE THE HANKEL MATRIX FOR SYSTEM IDENTIFICATION
  - SYSTEM'S ORDER ESTIMATE
  - SYSTEM'S S.S. MATRICES CALCULATION

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EXERCISE #1

CONSIDER A SECOND ORDER DYNAMIC SYSTEM DESCRIBED IN S.S:

$$F = \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \quad G = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \underbrace{D = \begin{bmatrix} 0 \end{bmatrix}}_{\text{STRICTLY PROPER}}$$

⇒ COMPUTE THE HANKEL MATRIX OF ORDER 3

$$\underbrace{H_p}_{p \times p} = O_p \cdot R_p \quad \text{WHERE} \quad O_p = \underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ H \cdot F^{p-1} \end{bmatrix}}_{p \times m} \quad \underbrace{R_p}_{m \times p} = \begin{bmatrix} G & FG & --- & F^{p-1} \cdot G \end{bmatrix}$$

LET'S COMPUTE H_3:

$$H_3 = \begin{bmatrix} \overset{H}{\boxed{1 \quad 0}} \\ \underbrace{[1 \quad 0] \cdot \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}}_{H \cdot F} \\ \overset{H \cdot F \cdot F}{[1 \; 0] \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}} \end{bmatrix} \begin{bmatrix} \underbrace{\begin{matrix} 1/2 \\ 1/2 \end{matrix}}_{G} & \underbrace{\begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}_{F \cdot G} & \underbrace{\begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}_{F \cdot F \cdot G} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ [0 \quad 2] \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1/2 & 1 & \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 7/4 \end{bmatrix} \\ 1/2 & 7/4 & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1/2 & 1 & 7/2 \\ 1/2 & 7/4 & 23/4 \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 & 1 & 7/2 \\ 1 & 7/2 & 23/2 \\ 7/2 & 23/2 & 38 \end{bmatrix} = H_3}$$

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⇒ COMPUTE H_3 USING THE SYSTEM IMPULSE RESPONSE DATA

$$H_3 = \begin{bmatrix} w(1) & w(2) & w(3) \\ w(2) & w(3) & w(4) \\ w(3) & w(4) & w(5) \end{bmatrix}$$

$w(k)$: I.R. DATA

LET'S COMPUTE $w(k)$:

$$\boxed{w(k \geq 1) = H \cdot F^{k-1} \cdot G}$$

$w(0) = 0$ SINCE THE SYSTEM IS STRICTLY PROPER

$$w(1) = HG = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1/2$$

$$w(2) = H \cdot F \cdot G = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 7/4 \end{bmatrix} = 1$$

$$w(3) = H \cdot F \cdot F \cdot G = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}}_{H \cdot F} \underbrace{\begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}_{F \cdot G}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7/4 \end{bmatrix} = 7/2$$

$$w(4) = HFFFG = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 2 \end{bmatrix}}_{H \cdot F} \overset{F}{\begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 7/4 \end{bmatrix}_{\sim F \cdot G} = \begin{bmatrix} 1 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 7/4 \end{bmatrix} = 23/2$$

$$w(5) = H \cdot FFFF \cdot G = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$= \overset{HFF}{\begin{bmatrix} 1 & 6 \end{bmatrix}} \overset{F}{\begin{bmatrix} 0 & 2 \\ 1/2 & 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 7/4 \end{bmatrix}^{\sim F \cdot G} = \begin{bmatrix} 1 & 6 \end{bmatrix} \begin{bmatrix} 7/2 \\ 23/4 \end{bmatrix} = 38$$

FINALLY WE CAN BUILD H_3 :

$$\boxed{H_3 = \begin{bmatrix} 1/2 & 1 & 7/2 \\ 1 & 7/2 & 23/2 \\ 7/2 & 23/2 & 38 \end{bmatrix}}$$

✓ IDENTICAL TO THE PREVIOUS ONE!

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⇒ **FACTORIZE THE HANKEL MATRIX:** $\underbrace{H_3}_{3\times 3} = \underbrace{\bar{O}_3}_{3\times 2} \cdot \underbrace{\bar{R}_3}_{2\times 3}$

NOTE! THE FACTORIZATION IS NOT UNIQUE!

IDEA:

$$H_3 = \underbrace{\left[\tilde{H}_1 \mid \tilde{H}_2 \mid \tilde{H}_3\right]}_{\text{COLUMNS}} = \underbrace{\overset{\bar{O}_3}{\left[\tilde{H}_1 \mid \tilde{H}_2\right]}}_{3\times 2} \cdot \underbrace{\overset{\bar{R}_3}{\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}}}_{2\times 3}$$

$$a \cdot \tilde{H}_1 + b \cdot \tilde{H}_2 = \tilde{H}_3 \qquad \text{CONSTRAINT}$$

⇒ IMPORTANT! : THIS PROCEDURE CAN BE APPLIED ONLY TO SINGULAR ($\det H_x = 0$) MATRIX.
INDEED WE ARE IMPOSING THAT THE COLUMN $\tilde{H}_3$ IS A LINEAR COMBINATION OF THE OTHER TWO COLUMNS OF H.

IN OUR CASE WE ARE SURE THAT $\det H_3 = 0$, SINCE THE SYSTEM'S ORDER IS $n = 2$

$$\det(H_{n+1}) = 0$$

LET'S FIND "a" AND "b":

$$a \begin{bmatrix} 1/2 \\ 1 \\ 7/2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 7/2 \\ 23/2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 23/2 \\ 38 \end{bmatrix}$$

$$a \cdot \tilde{H}_1 \quad + \quad b \cdot \tilde{H}_2 \quad = \quad \tilde{H}_3$$

. EQ #1: $\frac{1}{2} a + b = 7/2$

. EQ #2: $a + b \cdot 7/2 = 23/2$

. EQ #3: $a \cdot 7/2 + 23/2\, b = 38$

$$\left.\begin{aligned} a + b\cdot 7/2 &= 23/2 \\ a\cdot 7/2 + 23/2\, b &= 38 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 7/2 \\ 7/2 & 23/2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 23/2 \\ 38 \end{bmatrix}$$

$$\Downarrow$$

$$\frac{1}{\det \begin{bmatrix} 1 & 7/2 \\ 7/2 & 23/2 \end{bmatrix}} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \boxed{-\frac{4}{3}} \begin{bmatrix} 23/2 & -7/2 \\ -7/2 & 1 \end{bmatrix} \begin{bmatrix} 23/2 \\ 38 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

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CHECK USING EQ #1: $\frac{1}{2} \cdot \underset{a}{\textcircled{1}} + \underset{b}{\textcircled{3}} = \frac{7}{2}$ ✓

EVENTUALLY, THE FACTORIZATION OF THE HANKEL MATRIX IS:

$$\boxed{H_3 = \begin{bmatrix} 1/2 & 1 \\ 1 & 7/2 \\ 7/2 & 23/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}}$$

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EXERCISE # 2

CONSIDER THE SAMPLES OF A LTI SYSTEM I.R.:

$$w(0) = 0, \quad w(1) = 0, \quad w(2) = 1, \quad w(3) = -1/2, \quad w(4) = 1/4$$

$$w(5) = -1/8, \quad w(6) = 1/16$$

⇒ FIND ONE POSSIBLE STATE-SPACE REPRESENTATION OF THE SYSTEM

STEP # 1 : ESTIMATE SYSTEM'S ORDER

WE COMPUTE INCREASINGLY LARGER HANKEL MATRIX, UNTIL THE RANK STOPS INCREASING

$$H_1 = [w(1)] = 0 \rightarrow \text{RANK} = 0$$

$$H_2 = \begin{bmatrix} w(1) & w(2) \\ w(2) & w(3) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1/2 \end{bmatrix} \rightarrow \det = -1 \neq 0 \rightarrow \text{RANK} = 2$$

$$H_3 = \begin{bmatrix} w(1) & w(2) & w(3) \\ w(2) & w(3) & w(4) \\ w(3) & w(4) & w(5) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & -1/2 & 1/4 \\ -1/2 & 1/4 & -1/8 \end{bmatrix}$$

$$\det H_3 = 0 - 1 \cdot \det \begin{bmatrix} 1 & -1/2 \\ 1/4 & -1/8 \end{bmatrix} - 1/2 \cdot \det \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/4 \end{bmatrix}$$

$$= -1 \cdot [-1/8 + 1/8] - 1/2 \cdot [1/4 - 1/4] = 0 \rightarrow \text{RANK} = 2$$

THE RANK IS NON-INCREASING, SO $H_3 = H_{m+1} \Rightarrow m = 2$

STEP # 2 : FACTORIZE HANKEL MATRIX

$$H_3 = \begin{bmatrix} \bar{H}_1 \\ \hline \bar{H}_2 \\ \hline \bar{H}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ \hline \bar{H}_3 \end{bmatrix}$$

$$a \cdot \bar{H}_1 + b \cdot \bar{H}_3 = \bar{H}_2 \qquad \text{CONSTRAINT}$$

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* SEE END OF NOTES

$$H_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1/2 \\ -1/2 & 1/4 & -1/8 \end{bmatrix}$$

STEP #3: FIND S.S. MATRICES

$$H_{m+1} = \underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ HF^m \end{bmatrix}}_{O_{m+1}} \underbrace{\begin{bmatrix} G & FG & \cdots & F^m \cdot G \end{bmatrix}}_{R_{m+1}}$$

- $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ FIRST ROW OF O_3
- $G = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$ FIRST COLUMN OF R_3
- $O_3(1:\text{end}-1, :) \cdot F = O_3(2:\text{end}, :) \qquad F = O_3^{-1} \cdot O_3$
- $F \cdot R_3(:, 1:\text{end}-1) = R_3(:, 2:\text{end}) \Rightarrow F = R_3 \cdot R_3^{-1}$

LET'S USE THE OBSERVABILITY MATRIX:

$$F = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} -2 & \\ & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0 & -2 \\ 0 & -1/2 \end{bmatrix}} = F$$

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* H_3 FACTORIZATION

$$H_3 = \begin{bmatrix} 0 & 1 & -1/2 \\ 1 & -1/2 & 1/4 \\ -1/2 & 1/4 & -1/8 \end{bmatrix} = \begin{bmatrix} \bar{H}_1 \\ \hline \bar{H}_2 \\ \hline \bar{H}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ \hline \bar{H}_3 \end{bmatrix}$$

WE HAVE TO IMPOSE THAT:

$$\bar{H}_2 = a \bar{H}_1 + b \bar{H}_3$$

$$\Downarrow$$

$$a \cdot \overset{\bar{H}_1}{\begin{bmatrix} 0 & 1 & -1/2 \end{bmatrix}} + b \overset{\bar{H}_3}{\begin{bmatrix} -1/2 & 1/4 & -1/8 \end{bmatrix}} = \overset{\bar{H}_2}{\begin{bmatrix} 1 & -1/2 & 1/4 \end{bmatrix}}$$

USING EQUATION # 1 WE HAVE: $-1/2\, b = 1 \Rightarrow b = -2$

USING EQUATION # 3 WE HAVE: $-1/2\, a + 1/4 = 1/4 \Rightarrow a = 0$

USING EQUATION # 2 TO CHECK: $a + 1/4\, b = -1/2$ ✓

SO THE FACTORIZATION IS

$$\boxed{H_3 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1/2 \\ -1/2 & 1/4 & -1/8 \end{bmatrix}}$$

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