Exam questions I – Discrete Optimization

Problem 1

Consider a transportation problem with a single type of product, m production plants ($1 \leq i \leq m$) and n clients ($1 \leq j \leq n$).

Let

- p_i be the production capacity of plant i (maximum amount that can be produced),
- d_j be the demand of client j,
- f_{ij} be the fixed transportation cost from plant i to client j.
- t^1_{ij} the unit transportation cost from plant i to client j for up to 100 units of product, and t^2_{ij} the unit transportation cost from plant i to client j for the units of product exceeding 100 units, with $t^1_{ij} > t^2_{ij}$,
- q_{ij} be the maximum amount of product that can be transported from plant i to client j.

To avoid excessive fragmentation of the deliveries, each client can be served from at most k plants.

Give a mixed integer linear programming formulation for the problem of determining a transporatation plan which minimizes the total transportation costs, while satisfying all client demands and all plant capacities.

Problem 2

The 118 Operations Center of the Niguarda Hospital, which coordinates the emergency medical services for the city of Milan and its province, must decide where to locate k ambulances so as to answer as well as possibile to the emergency calls/requests.

We assume to know:

- the set $S = \{1, \dots, m\}$ of the candidate sites where an ambulance can be stationed while waiting to be assigned a mission, with $|S| > k$,

- the set $C = \{1, \dots, n\}$ of the points which represent the possible emergency call locations (estimated based on demographic and historical data),

- the time t_{ij} (in minutes) that the ambulance needs to go from candidate site i to the call location j, for each pair $i \in S$ and $j \in C$.

In this simplified version of the problem we consider the answer to any next single call. Moreover, we suppose that every candidate site can host at most one ambulance and that for the given S, C and times t_{ij} a feasible solution exists.

a) Give an Integer Linear Programming formulation for the problem of determining in which candidate sites to locate the ambulances and which ambulance to send to each call location so as to minimize the maximum response time to the next single call (i.e., the time needed for the ambulance to arrive at the call location).

b) To partially account for the fact that an ambulance may already be busy (be assigned to another mission not yet completed), we have to identify two ambulances for each possible call location $j \in C$: a first ambulance that must be able to arrive at the call location within 8 minutes and a second ambulance that will be sent only in case the first ambulance is already busy.

Give an Integer Linear Programming formulation for the problem of determining at which candidate sites to locate the ambulances so as to minimize the maximum time needed for the second ambulance to arrive at any possible next single call location, while guaranteeing that in the case of any next single call an ambulance can arrive within 8 minutes.

Problem 3

A company must optimize its telecommunication network. Let V be the set of nodes and A the set of links (arcs). Let (s_k, t_k), with $k \in K$, denote the $|K|$ origin-destination pairs that requested the service and $d_k \geq 0$ the amount of data (in GB) to be sent from node s_k to node t_k. The network capacity, which is initially zero, can be increased by installing (activating) on each arc appropriate communication devices. Each such device has a capacity of 1 GB and a cost of c_{ij}, for each arc $(i, j) \in A$. Let u_{ij} be the maximum amount of capacity (in GB) that can be installed (activated) on each arc $(i, j) \in A$. Finally let s_{ij} be the cost for routing on arc $(i, j) \in A$ one unit amount of data.

a) Assume that, for each origin-destination pair (s_k, t_k), the data must be routed along a single path (to avoid delay issues at destination). Give a mixed integer linear programming formulation for the problem of determining how to install the capacity on the arcs of the network and how to route the demands so as to minimize the total routing and installation costs, while satisfying the demands of all origin-destination pairs.

b) To protect the network from failures, we require that there exist at least two link-disjoint paths, using only arcs with non zero capacity, from each origin s_k to each destination t_k, namely, the graph composed of arcs with non zero capacity must be biconnected. How can we modify the formulation to account for this robustness requirement? What is the size of the resulting formulation?

c) How can we extend the formulation of point b) to account for at least two node-disjoint paths?

Problem 4

Consider the minimum cost flow problem: given a directed graph $G = (V, A)$ with a capacity k_{ij} and a unit cost c_{ij} associated to each arc $(i, j) \in A$ and a demand/availability b_i for each node $i \in V$ (b_i positive for sources, negative for destinations, zero for intermediate nodes), determine a feasible flow of minimum total cost which satisfies all the demands. To guarantee feasibility, we assume that $\sum_{i \in V} b_i = 0$.

- Give an integer linear programming formulation for the problem.

- Show that it is an ideal formulation, after clarly explaining the meaning of such a statement.
- Explain why the shortest path problem (Given a directed graph $G = (V, A)$ with a cost c_{ij} for each arc $(i, j) \in A$, and two prescribed nodes s and t, determine a path of minimum toal cost from s to t) is a special case of the minimum cost flow problem.
- Explain why the the maximum flow problem (Given a directed graph $G = (V, A)$ with a capacity u_{ij} for each arc $(i, j) \in A$, a source s and a sink t, determine a feasible flow of maximum value from s to t) is also a special case of the minimum cost flow problem.

Problem 5

Consider the Symmetric Traveling Salesman Problem: given an undirected graph $G = (V, E)$ with a cost c_e associated to each edge $e \in E$, determine an Hamiltonian cycle, i.e., a cycle which visits each node exactly once, of minimum total cost.

- Give two integer linear programming formulations for the problem and indicate the size of the formulation in terms of the number of nodes in the graph.
- Which relation exists between the linear relaxations of these two formulations? Explain and motivate your answer.

Problem 6

Consider a set of n candidate sites where a depot can be open and a set of m clients distributed across a given area. Suppose there is a fixed opening cost f_j for each candidate site j, with $1 \leq j \leq n$, and a profit c_{ij} when the whole demand of client i is satisfied by the depot located in candidate site j, with $1 \leq i \leq m$ and $1 \leq j \leq n$. In the Uncapacitated Facility Location problem (UFL) variant considered here, we have to decide in which candidate sites to open depots and how to satisfy the demand of each client so as to maximize the total profit minus the total fixed costs.

- Give two integer linear programming formulations for the problem.
- Indicate which formulation is stronger (provides a tighter bound) and motivate your answer.
- Describe in detail the Lagrangian relaxation for this problem where the demand constraints are relaxed. Clearly explain how the Lagrangian subproblem and Lagrangian dual can be solved.
- Apply the Lagrangian relaxation method to the instance with $m = 6$ clients, $n = 5$ locations, the fixed location costs $f = (2, 4, 5, 3, 3)$, and the client-location profit

matrix

$$(c_{5ij}) = \begin{pmatrix} 6 & 2 & 1 & 3 & 5 \\ 4 & 10 & 2 & 6 & 1 \\ 3 & 2 & 4 & 1 & 3 \\ 2 & 0 & 4 & 1 & 4 \\ 1 & 8 & 6 & 2 & 5 \\ 3 & 2 & 4 & 8 & 1 \end{pmatrix}.$$

Start with the multiplier vector $\mathbf{u}_0^t = (5, 6, 3, 2, 6, 4)$.

Problem 7

Consider the feasible region of a generic binary knapsack problem, namely $X = \{\mathbf{x} \in \{0,1\}^n \,:\, \sum_{j=1}^n a_j x_j \leq b\}$ where all coefficients a_j e b are positive.

- State the definition of a cover inequality and explain why it is a valid inequality for X.
- Describe the separation problem for the cover inequalities and explain how it can be solved.
- Consider the specific feasible region $X = \{\mathbf{x} \in \{0,1\}^6 \,:\, 12x_1 + 9x_2 + 7x_3 + 5x_4 + 5x_5 + 3x_6 \leq 14\}$, list all the minimal cover inequalities that are valid for X, and apply the lifting procedure to the cover inequality including the variables x_3, x_5 and x_6.

Problem 8

- Describe the general idea of the column generation method for integer linear programming problems, including all components.
- Describe the method for the 1-D cutting stock problem or for another discrete optimization problem. Given a set of large rolls of width W and demands for b_i small rolls of width w_i, with $i \in I = \{1, \ldots, m\}$, the 1-D cutting stock problem consists in deciding how to cut the large rolls into small rolls so as to minimize the number of large rolls used, while satisfying the customer demand.
- Briefly discuss the main advantages and drawbacks of the method as well as a possibile extension.

Problem 9

- Describe the Lagrangian relaxation method for Integer Linear Programming problems, clearly specifying all the steps and how they can be carried out.
- Illustrate the method on an example.

- State the central result concerning the strength of the bound obtained by solving the Lagrangian dual problem and describe how the Lagrangian dual problem can be solved.

Problem 10

Similar open questions concerning the other methods covered in the course.

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