Exams-2012 - exam

Quantitative Finance And Derivatives - Module 1 (Università Commerciale Luigi Bocconi)

Quantitative Finance and Derivatives I
Finanza Quantitativa e Derivati I

code 20188

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Give a detailed justification to all your answers!

EXERCISE 1

Consider a multiperiod discrete market with $t = 0, 1, 2$ with the following information structure:

$$
\begin{array}{ccccc}
t=0 & & t=1 & & t=2 \\
 & & & \nearrow & \omega_1 \\
 & & f_1^1 & & \\
 & \nearrow & & \searrow & \omega_2 \\
f_1^0 & & & & \\
 & \searrow & & \nearrow & \omega_3 \\
 & & f_2^1 & & \\
 & & & \searrow & \omega_4
\end{array}
$$

Two securities are traded in the market. The first is the *locally risk-free asset* B that provides the locally riskless interest rate

$$
\begin{aligned}
r(0) &= 5\% \\
r(1)(f_1^1) &= 8\% \\
r(1)(f_2^1) &= 0\%.
\end{aligned}
$$

The second security is the *risky asset* S, whose price at time 0 is

$$S(0) = 10,$$

whereas at $t = 1$ its prices are

$$
\begin{gathered}
S(1)(f_1^1) = 11.5 \\
S(1)(f_2^1) = 9.5
\end{gathered}
$$

and at the final date $T = 2$

$$
\begin{gathered}
S(2)(\omega_1) = 13.57 \\
S(2)(\omega_2) = 11.27 \\
S(2)(\omega_3) = 10.925 \\
S(2)(\omega_4) = 9.025
\end{gathered}
$$

1. Write the prices of the locally riskless asset B in each node of the information structure for $t = 1, 2$.
2. Is the market dynamically complete?
3. Determine the set of the risk neutral probabilities $\mathbf{Q}$ for the market, specifying $\mathbf{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
4. Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *European put option* on S with maturity $T = 2$ and strike $K = 12.27$.

5. Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *European knock-in put option* on S with maturity $T = 2$ and strike $K = 12.27$ and barrier $b = 11$. The put option is activated only if the stock S exceeds the barrier $b = 11$ during the life of the option. The terminal payoff is therefore
$$X(2) = \begin{cases} (K - S(2))^+ & \text{if } \max_{t=0,1,2} S(t) \geq b \\ 0 & \text{otherwise} \end{cases}$$

6. Consider a *repo* (repurchasement agreement) with collateral S and maturity $T = 2$. The holder of S sells the stock at the initial date $= 0$, receives the initial price $S(0)$ and must buy back the stock at maturity $T = 2$ at the price F. The price F is settled so that the no-arbitrage price of the contract at date 0 is null. F therefore coincides with the forward price for S at T and hence it must satisfy
$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(2) - F}{B(2)}\right] = 0.$$

   Compute F.

   The repo is a collateralized loan: the stock holder receives $S(0)$ at the initial date (by selling the stock), and has to pay F at maturity, to buy back the stock. Find the *repo rate*, i.e. find r_{repo} such that
$$S(0) \cdot (1 + r_{repo})^T = F.$$

EXERCISE 2

Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 1$, $\delta = 2\%$, $\mu = 5\%$ and $\sigma = 12\%$.

1. Determine the *historical probability* $\mathbb{P}$ that a European *put* option on S with strike price $K = 1$ closes at maturity $T = 4$ *in the money.*

2. Consider the European derivative on S whose payoff X at maturity $T = 4$ is
$$X = (S(T))^2.$$

   The derivative trades at time $t \in [0; T]$ at the price
$$\begin{aligned} V_X(t) &= F(t, S(t)) \\ \text{with } F(t, S(t)) &= (S(t))^2 \cdot e^{\alpha(T-t)} \end{aligned}$$

   for some $\alpha \in \Re$. Apply the Black-Scholes PDE to determine α such that the market extended to the new derivative is arbitrage-free.

3. Find the replicating portfolio of the European derivative of the previous point, i.e. compute $\vartheta = (\vartheta_0(t), \vartheta_1(t))$ for $t \in [0; T]$
$$\begin{aligned} \vartheta_1(t) &= \frac{\partial}{\partial S} F(t, S(t)) \\ \vartheta_0(t) &= (F(t, S(t)) - \vartheta_1(t) \cdot S(t)) \cdot e^{-\delta t} \end{aligned}$$

   where α takes the no-arbitrage value of the previous point. To replicate the derivative do you need to go long or short the stock?

4. Consider now the *buy-and-hold* portfolio $\vartheta^{BH} = \left(\vartheta_0^{BH}, \vartheta_1^{BH}\right)$ constituted with the *initial positions* of the replicating strategy ϑ of the previous point. The components of the *buy-and-hold* portfolio ϑ^{BH} are therefore

$$\begin{aligned} \vartheta_0^{BH}(t) &= \vartheta_0(0) \\ \vartheta_1^{BH}(t) &= \vartheta_1(0) \end{aligned}$$

for all $t \in [0; T]$. Determine the *historical probability* $\mathbb{P}$ that the terminal value of this buy-and-hold strategy superreplicates the terminal payoff of the derivative of the previous point at maturity.

(For questions 1 and 4 *you only need to determine the formula for this probability, expressed in terms of the distribution function* $N(\cdot)$ *of a standard Normal random variable*).

QUESTION

In a one-period arbitrage-free market state the equivalent conditions that the initial price of a redundant derivative security has to satisfy in order to preserve no-arbitrage in the extended market.

SOLUTIONS TO EXERCISES

Exercise 1

1. The prices of security B are
$$B(1)(f_1^1) = B(1)(f_2^1) = 1.05$$
and at the final date $T = 2$
$$\begin{array}{c} B(2)(\omega_1) = B(2)(\omega_2) = 1.05 \cdot 1.08 = 1.134 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.05 \end{array}$$

2. The market is dynamically complete, because each one-period submarket is complete (in your exam check explicitly that the rank of the *terminal* payoff matrix of each one-period submarket has rank 2).

3. We look for risk neutral probabilities $\mathbf{Q}$ for the market. We have to solve the systems
$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbf{Q}[f_1^1] + S(1)(f_2^1)\mathbf{Q}[f_2^1] \right\} \\ \mathbf{Q}[f_1^1] + \mathbf{Q}[f_2^1] = 1 \\ \mathbf{Q}[f_1^1], \mathbf{Q}[f_2^1] > 0 \end{cases} \quad (1)$$
for m_0,
$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbf{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbf{Q}[\omega_2|f_1^1] \right\} \\ \mathbf{Q}[\omega_1|f_1^1] + \mathbf{Q}[\omega_2|f_1^1] = 1 \\ \mathbf{Q}[\omega_1|f_1^1], \mathbf{Q}[\omega_2|f_1^1] > 0 \end{cases} \quad (2)$$
for $m_{1,1}$, and
$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbf{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbf{Q}[\omega_4|f_2^1] \right\} \\ \mathbf{Q}[\omega_3|f_2^1] + \mathbf{Q}[\omega_4|f_2^1] = 1 \\ \mathbf{Q}[\omega_3|f_2^1], \mathbf{Q}[\omega_4|f_2^1] > 0 \end{cases} \quad (3)$$
System (1) can be rewritten as
$$\begin{cases} 10 = \frac{1}{1.05} \left\{ 11.5 \cdot \mathbf{Q}[f_1^1] + 9.5 \cdot \mathbf{Q}[f_2^1] \right\} \\ \mathbf{Q}[f_1^1] + \mathbf{Q}[f_2^1] = 1 \\ \mathbf{Q}[f_1^1], \mathbf{Q}[f_2^1] > 0 \end{cases}$$
and is solved by
$$\begin{array}{rcl} \mathbf{Q}[f_1^1] & = & 0.5 \\ \mathbf{Q}[f_2^1] & = & 0.5 \end{array}$$
System (2) can be rewritten as
$$\begin{cases} 11.5 = \frac{1}{1.08} \left\{ 13.57 \cdot \mathbf{Q}[\omega_1|f_1^1] + 11.27 \cdot \mathbf{Q}[\omega_2|f_1^1] \right\} \\ \mathbf{Q}[\omega_1|f_1^1] + \mathbf{Q}[\omega_2|f_1^1] = 1 \\ \mathbf{Q}[\omega_1|f_1^1], \mathbf{Q}[\omega_2|f_1^1] > 0 \end{cases}$$
and is solved by
$$\begin{array}{rcl} \mathbf{Q}[\omega_1|f_1^1] & = & 0.5 \\ \mathbf{Q}[\omega_2|f_1^1] & = & 0.5 \end{array}$$

and System (3) can be rewritten as

$$\begin{cases} 9.5 = \frac{1}{1+r(1)(f_2^1)}\left\{10.925 \cdot \mathbf{Q}[\omega_3|f_2^1] + 9.025 \cdot \mathbf{Q}[\omega_4|f_2^1]\right\} \\ \mathbf{Q}[\omega_3|f_2^1] + \mathbf{Q}[\omega_4|f_2^1] = 1 \\ \mathbf{Q}[\omega_3|f_2^1], \mathbf{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbf{Q}[\omega_3|f_2^1] &= 0.25 \\ \mathbf{Q}[\omega_4|f_2^1] &= 0.75 \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{Q}[\omega_1] &= 0.5 \cdot 0.5 = 0.25 \\ \mathbf{Q}[\omega_2] &= 0.5 \cdot 0.5 = 0.25 \\ \mathbf{Q}[\omega_3] &= 0.5 \cdot 0.25 = 0.125 \\ \mathbf{Q}[\omega_4] &= 0.5 \cdot 0.75 = 0.375 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

4. The terminal payoff of *European put option* on S with maturity $T = 2$ and strike $K = 12.27$ is in the nodes ω_k for $k = 1, ..., 4$

$$p(2) = \begin{cases} (12.27 - S(2)(\omega_1))^+ = (12.27 - 13.57)^+ = 0 \\ (12.27 - S(2)(\omega_2))^+ = (12.27 - 11.27)^+ = 1 \\ (12.27 - S(2)(\omega_3))^+ = (12.27 - 10.925)^+ = 1.345 \\ (12.27 - S(2)(\omega_4))^+ = (12.27 - 9.025)^+ = 3.245 \end{cases}$$

The no arbitrage prices of the *put* option at $t = 0$ and in the nodes f_1^1 and f_2^1 at $t = 1$ are

$$\begin{aligned} p(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(2)}{1+r(1)}\middle|\mathcal{P}_1\right](f_1^1) \\ &= \frac{0 \cdot 0.5 + 1 \cdot 0.5}{1.08} = 0.46296 \\ p(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(2)}{1+r(1)}\middle|\mathcal{P}_1\right](f_2^1) \\ &= \frac{1.345 \cdot 0.25 + 3.245 \cdot 0.75}{1.0} = 2.77 \\ p(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(1)}{1+r(0)}\right] \\ &= \frac{0.46296 \cdot 0.5 + 2.77 \cdot 0.5}{1.05} = 1.5395. \end{aligned}$$

5. The *knock-in* put option is activated only if the stock S exceeds the barrier $b = 11$ during the life of the option. The stock exceeds the barrier $b = 11$ at $t = 1$ in f_1^1 only, therefore the payoff is active at $T = 2$ in ω_1 (where the put is out of the money) and in ω_2 only. Hence

$$\begin{aligned} X(2)(\omega_1) &= (K - S(2)(\omega_1))^+ = (12.27 - 13.57)^+ = 0 \\ X(2)(\omega_2) &= (K - S(2)(\omega_2))^+ = (12.27 - 11.27)^+ = 1 \\ X(2)(\omega_3) &= X(2)(\omega_4) = 0 \text{ since the barrier is not activated} \end{aligned}$$

The no-arbitrage price at $t=0$ and at $t=1$ of the *European knock-in put option* on S are

$$\begin{aligned} S_X(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_1^1) \\ &= \frac{0\cdot 0.5+1\cdot 0.5}{1.08}=0.462\,96 \\ S_X(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_2^1) \\ &= \frac{0\cdot 0.25+0\cdot 0.75}{1.0}=0 \\ S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1)}{1+r(0)}\right] \\ &= \frac{0.462\,96\cdot 0.5+0\cdot 0.5}{1.05}=0.220\,46. \end{aligned}$$

6. The *forward price* F for stock S at date 2 is such that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(2)-F}{B(2)}\right]=0$$

hence

$$F\cdot\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right]=\mathbb{E}^{\mathbb{Q}}\left[\frac{S(2)}{B(2)}\right]=S(0)$$

where the last equality follows by the martingality property of the discounted risky asset with respect to the risk-neutral measure. We compute

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right]=\frac{1}{1.134}\cdot 0.5+\frac{1}{1.05}\cdot 0.5=0.917\,11.$$

We get therefore

$$F=\frac{S(0)}{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right]}=\frac{10}{0.917\,11}=10.904$$

To determine the repo rate r_{repo} we solve the equation

$$10\cdot(1+r_{repo})^2=10.904$$

that yields

$$r_{repo}=\sqrt{\frac{10.904}{10}}-1=\sqrt{1.090\,4}-1=4.4\%$$

Exercise 2

1. The *historical probability* $\mathbb{P}$ that a European *put* option on S with strike price $K=1$ closes

at maturity $T = 4$ *in the money is*

$$\begin{aligned}
\mathbb{P}[S(4) < K] &= \mathbb{P}\left[e^{\left(\mu - \frac{\sigma^2}{2}\right)\cdot 4 + \sigma W_4} < \frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[\left(\mu - \frac{\sigma^2}{2}\right)\cdot 4 + \sigma W_4 < \ln\frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[Z < \frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 4\right)\right] = \\
&= N\left(\frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 4\right)\right) \\
&= N\left(\frac{1}{0.12\sqrt{4}}\left(\ln\frac{1}{1} - \left(0.05 - \frac{0.12^2}{2}\right)\cdot 4\right)\right) = \\
&= N(-0.713\,33)
\end{aligned}$$

where Z denotes a standard normal random variable with respect to the *historical probability* $\mathbb{P}$.

2. The price

$$\begin{aligned}
V_X(t) &= F(t, S(t)) \\
\text{with } F(t, S(t)) &= (S(t))^2 \cdot e^{\alpha(T-t)}
\end{aligned}$$

for the European derivative on S whose payoff X at maturity $T = 4$ is

$$X = (S(T))^2$$

with $\alpha \in \Re$ is a *no-arbitrage price* if and only if F satisfies the Black-Scholes PDE with the proper boundary condition.

The boundary condition at maturity is satisfied *for any* α, since

$$F(T, S(T)) = (S(T))^2 \cdot e^{\alpha(T-T)} = (S(T))^2 = X$$

for any possible value of the underlying at maturity $S(T)$. We then impose the Black-Scholes PDE on F. To do so we first compute the derivatives of $F(t, S) = S^2 \cdot e^{\alpha(T-t)}$.

We have

$$\frac{\partial F(t,S)}{\partial t} = -S^2\alpha e^{\alpha(T-t)}; \qquad \frac{\partial F(t,S)}{\partial S} = 2Se^{\alpha(T-t)}; \qquad \frac{\partial^2 F(t,S)}{\partial S^2} = 2e^{\alpha(T-t)}$$

The Black-Scholes PDE

$$\frac{\partial F(t,S)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F(t,S)}{\partial S^2} + \delta S \frac{\partial F(t,S)}{\partial S} - \delta F(t,S) = 0$$

applied to F is

$$-S^2\alpha e^{\alpha(T-t)} + \frac{1}{2}\sigma^2 S^2\left(2e^{\alpha(T-t)}\right) + \delta S\left(2Se^{\alpha(T-t)}\right) - \delta S^2 \cdot e^{\alpha(T-t)} = 0$$

that can be written as

$$-\alpha + \frac{1}{2}\sigma^2(2) + \delta(2) - \delta = 0$$

i.e.

$$-\alpha + \sigma^2 + \delta = 0$$

Therefore α must satisfy

$$\alpha = \sigma^2 + \delta = 0.12^2 + 0.02 = 0.034\,4$$

to preserve no-arbitrage in the market.

3. The replicating portfolio of the European derivative of the previous point is for $t \in [0;T]$

$$\begin{aligned}\vartheta_1(t) &= \frac{\partial}{\partial S}F(t,S(t)) = 2S(t)e^{\alpha(T-t)} > 0\\ \vartheta_0(t) &= (F(t,S(t)) - \vartheta_1(t)\cdot S(t))\cdot e^{-\delta t} = \left((S(t))^2\cdot e^{\alpha(T-t)} - 2S(t)e^{\alpha(T-t)}\cdot S(t)\right)\cdot e^{-\delta t}\\ &= -(S(t))^2\cdot e^{\alpha(T-t)}\cdot e^{-\delta t} < 0\end{aligned}$$

where $\alpha = 0.0344$. Due to the signs of $\vartheta_1(t)$ and $\vartheta_0(t)$, the replicating strategy is *long* S and *short* B.

4. The *buy-and-hold* portfolio $\vartheta^{BH} = \left(\vartheta_0^{BH}, \vartheta_1^{BH}\right)$ is constituted with the *initial positions* of the replicating strategy ϑ of the previous point. The components of the *buy-and-hold* portfolio ϑ^{BH} are therefore

$$\begin{aligned}\vartheta_0^{BH}(t) &= \vartheta_0^{BH} = \vartheta_0(0) = -(S(0))^2\cdot e^{\alpha(T-0)}\cdot e^{-\delta\cdot 0} = -(S(0))^2\cdot e^{\alpha T} = -(1)^2\cdot e^{0.0344\cdot 4} = -1.1475\\ \vartheta_1^{BH}(t) &= \vartheta_1^{BH} = \vartheta_1(0) = 2S(0)\cdot e^{\alpha(T-0)} = 2\cdot 1\cdot e^{0.0344(4-0)} = 2.295\end{aligned}$$

unchanged for all $t \in [0;T]$. The *historical probability* $\mathbb{P}$ that the terminal value of this buy-and-hold strategy superreplicates the terminal payoff of the derivative at maturity is

$$\begin{aligned}\mathbb{P}\left[V_{\vartheta^{BH}}(T) \geq (S(T))^2\right] &= \mathbb{P}\left[\underbrace{-(S(0))^2\cdot e^{\alpha T}}_{\vartheta_0^{BH}}\cdot\underbrace{e^{\delta T}}_{B(T)} + \underbrace{2S(0)\cdot e^{\alpha(T-0)}}_{\vartheta_1^{BH}}\cdot S(T) \geq (S(T))^2\right] =\\ &= \mathbb{P}\left[-(1)^2\cdot e^{\alpha T}\cdot e^{\delta T} + 2\cdot 1\cdot e^{\alpha T}\cdot S(T) \geq (S(T))^2\right] =\\ &= \mathbb{P}\left[(S(T))^2 - 2e^{\alpha T}S(T) + e^{\alpha T}\cdot e^{\delta T} \leq 0\right]\end{aligned}$$

The inequality

$$(y)^2 - 2e^{\alpha T}y + e^{\alpha T}\cdot e^{\delta T} \leq 0$$

is satisfied for any $y \in [y_1; y_2]$ where

$$\begin{aligned}y_{1,2} &= e^{\alpha T} \pm \sqrt{e^{2\alpha T} - e^{\alpha T}\cdot e^{\delta T}} = e^{0.0344\cdot 4} \pm \sqrt{e^{2\cdot 0.0344\cdot 4} - e^{0.0344\cdot 4}\cdot e^{0.02\cdot 4}}\\ &= e^{0.0344\cdot 4} \pm \sqrt{7.3704\times 10^{-2}} = \begin{cases} 0.87603 \\ 1.419 \end{cases}\end{aligned}$$

(notice that the argument of the square root is positive if and only if $2\alpha > \alpha + \delta$, which is true since $\alpha = \sigma^2 + \delta$).

Therefore the *historical probability* $\mathbb{P}$ that the terminal value of this buy-and-hold strategy superreplicates the terminal payoff of the derivative at maturity is

$$\begin{aligned}
\mathbb{P}\left[V_{\vartheta^{BH}}(T) \geq (S(T))^2\right] &= \mathbb{P}\left[y_1 \leq S(T) \leq y_2\right] = \\
&= \mathbb{P}\left[y_1 \leq S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4+\sigma W_4} \leq y_2\right] = \\
&= \mathbb{P}\left[y_1 \leq S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4+\sigma\sqrt{4}Z} \leq y_2\right] = \\
&= \mathbb{P}\left[\ln\frac{y_1}{S(0)} \leq \left(\mu-\frac{\sigma^2}{2}\right)\cdot 4+\sigma\sqrt{4}Z \leq \ln\frac{y_2}{S(0)}\right] = \\
&= \mathbb{P}\left[\frac{1}{\sigma\sqrt{4}}\left(\ln\frac{y_1}{S(0)}-\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4\right) \leq Z \leq \frac{1}{\sigma\sqrt{4}}\left(\ln\frac{y_2}{S(0)}-\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4\right)\right] \\
&= \mathbb{P}\left[\frac{1}{0.12\sqrt{4}}\left(\ln\frac{0.87603}{1}-\left(0.05-\frac{0.12^2}{2}\right)\cdot 4\right) \leq Z \leq \frac{1}{0.12\sqrt{4}}\left(\ln\frac{1.419}{1}-\left(0.05-\frac{0.12^2}{2}\right)\cdot 4\right)\right] \\
&= \mathbb{P}\left[-1.2648 \leq Z \leq 0.7448\right] \\
&= N(0.7448) - N(-1.2648) = \\
&= 0.771803683 - 0.102971516 = 0.66883
\end{aligned}$$

(in the previous lines Z denotes a standard normal random variable with respect to $\mathbb{P}$).

Though not requested by the exam, we compute here also the risk-neutral probability of the same event

$$\begin{aligned}
&\mathbb{Q}\left[V_{\vartheta^{BH}}(T) \geq (S(T))^2\right] = \\
&= \mathbb{Q}\left[\frac{1}{0.12\sqrt{4}}\left(\ln\frac{0.87603}{1}-\left(0.02-\frac{0.12^2}{2}\right)\cdot 4\right) \leq Z^{\mathbb{Q}} \leq \frac{1}{0.12\sqrt{4}}\left(\ln\frac{1.419}{1}-\left(0.02-\frac{0.12^2}{2}\right)\cdot 4\right)\right] \\
&= \mathbb{Q}\left[-0.76481 \leq Z^{\mathbb{Q}} \leq 1.2448\right] \\
&= 0.893397362 - 0.222192346 = 0.67121
\end{aligned}$$

where $Z^{\mathbb{Q}}$ denotes a standard normal random variable with respect to $\mathbb{Q}$.

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Give a detailed justification to all your answers!

EXERCISE 1

A one period market is constituted by the riskless asset B that provides the riskless interest rate $r = 25\%$, and by the risky security S_1 whose prices at time $T = 1$ are

$$S_1(1)(\omega_1) = 0.5$$
$$S_1(1)(\omega_2) = 2.5$$
$$S_1(1)(\omega_3) = 1$$

1. Is the market complete?
2. Suppose that the risky security S_1 trades at $t = 0$ at the price
$$S_1(0) = S_0 = 1.0$$
Do state price vectors/risk neutral probabilities exist? If your answer is positive, find the whole set of state price vectors/ risk neutral probabilities, and discuss no-arbitrage in the market.
3. Suppose that a *put* option on S_1 with strike $K = 1$ and maturity $T = 1$ is introduced in the market. Write $p(1)$ the payoff of the option at maturity $T = 1$ in all ω_k for $k = 1, 2, 3$. Can the *put* option on S_1 be replicated with B and S_1? If your answer is negative, determine the interval of no-arbitrage prices at $t = 0$ for the security *put* option on S_1.
4. If the *put* option trades at $t = 0$ at the price
$$p(0) = 0.16$$
is the extended market B, S_1 and the put option free of arbitrage opportunities? If your answer is positive, find the set of risk neutral probabilities in the extended market.
5. A *passport option on* S is introduced in the extended market that includes B, S, and the put option as in point 4. A *passport option on* S is a European call on the profits of a trading account on S. The holder of this deritative chooses at the initial date a trading strategy $u \in [-1, 1]$ on the stock S. At maturity she receives the gain of such strategy if positive; in the case of (potential) losses the position is closed without any obligation for the holder. In our framework, the final payoff determined by the trading strategy $u \in [-1, 1]$ is
$$X_u = (u \cdot (S(1) - S(0)))^+ .$$
whose initial no-arbitrage value is then
$$\frac{E^Q [X_u]}{1 + r}.$$

Find the trading strategy $u^* \in [-1, 1]$ that maximizes

$$\frac{E^Q [X_u]}{1+r}.$$

Compute the maximum

$$\frac{E^Q [X_{u^*}]}{1+r}.$$

This value is the *intial no-arbitrage price of the passport option*

$$S_X (0) = \frac{E^Q [X_{u^*}]}{1+r},$$

that the buyer of the option pays to the writer at $t = 0$.

6. Suppose the holder of the passport option enters in the *non-optimal* trading strategy $u_o = -\frac{u^*}{2}$. Compute the initial replicating cost of the final payoff X_{u_o}. Determine the difference that the writer of the option pockets at $t = 0$ due to the suboptimal choice of the trading strategy by the holder.

EXERCISE 2

Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 1$, $\delta = 4\%$, $\mu = 6\%$ and $\sigma = 14\%$.

1. Determine the *historical probability* $\mathbb{P}$ that a European *call* option on S with strike price $K = 0.5$ closes at maturity $T = 1$ *in the money.*

2. Consider the European path-dependent derivative on S whose payoff X at maturity $T = 1$ is

$$X = 100 \cdot \left(\ln \frac{S(1)}{S(0.5)} \right)^2$$

Compute its initial no-arbitrage price $S_X (0)$.

3. Show that the initial price of the derivative of the previous step $S_X (0)$ is increasing with respect to the volatility parameter σ.

4. Find the risk-neutral probability that the payoff X of point 2. exceeds the level $b = 1$ at maturity $T = 1$.

THEORY QUESTION

State Ito formula for a diffusion process.

Solution of the Exercises

Solution of EXERCISE 1

1. The market is incomplete, because the number of scenarios $K = 3 > 2$, the number of traded securities.

2. Since the market is incomplete, the risk-neutral measures (if any) cannot be unique. Denoting by $q_i = \mathbb{Q}(\omega_i) > 0$ for $i = 1, ..., 3$, we have that

$$\begin{aligned}
\frac{1}{1.25}\left(0.5q_1 + 2.5q_2 + 1\underbrace{(1 - q_1 - q_2)}_{q_3}\right) &= 1 \\
1.2q_2 - 0.4q_1 + 0.8 &= 1 \\
1.2q_2 - 0.4q_1 - 0.2 &= 0
\end{aligned}$$

that gives

$$\begin{cases}
q_1 \in \left(0; \frac{5}{8}\right) = (0; 0.625) \\
q_2 = \frac{0.4q_1 + 0.2}{1.2} = \frac{1}{3}q_1 + \frac{1}{6} = 0.333\,33q_1 + 0.166\,67 \\
q_3 = 1 - q_1 - \left(\frac{1}{3}q_1 + \frac{1}{6}\right) = -\frac{4}{3}q_1 + \frac{5}{6} = 0.833\,33 - 1.333\,3q_1
\end{cases}$$

The state price vectors are

$$\psi_i = \frac{\mathbb{Q}(\omega_i)}{1 + r}$$

so that

$$\begin{cases}
\psi_1 \in \left(\frac{0}{1.25}; \frac{5}{8} \cdot \frac{1}{1.25}\right) = (0; 0.5) \\
\psi_2 = \frac{1}{3}\psi_1 + \frac{1}{6} \cdot \frac{1}{1.25} = 0.333\,33\psi_1 + 0.133\,33 \\
\psi_3 = -\frac{4}{3}\psi_1 + \frac{5}{6} \cdot \frac{1}{1.25} = -1.333\,3\psi_1 + 0.666\,67
\end{cases}$$

3. The final payoff of a *put* option on S_1 with strike $K = 3$ and maturity $T = 1$ is

$$\begin{gathered}
p(1)(\omega_1) = (1 - 0.5)^+ = 0.5 \\
p(1)(\omega_2) = (1 - 2.5)^+ = 0 \\
p(1)(\omega_3) = (1 - 1)^+ = 0
\end{gathered}$$

and cannot be replicated because

$$\det \begin{bmatrix} 1.25 & 0.5 & 0.5 \\ 1.25 & 2.5 & 0 \\ 1.25 & 1 & 0 \end{bmatrix} = -0.937\,5 \neq 0$$

$$rank \begin{bmatrix} 1.25 & 0.5 & 0.5 \\ 1.25 & 2.5 & 0 \\ 1.25 & 1 & 0 \end{bmatrix} = 3,$$

i.e. the put payoff is independent from the terminal prices of B and S_1.

Hence there is an interval of no-arbitrage prices at $t=0$ for the security *put* option on S_1. To retrieve such interval we compute

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[p\left(1\right)\right]=\frac{0.5\cdot q_1+0\cdot q_2+0\cdot q_3}{1.25}=\frac{0.5}{1.25}q_1=0.4q_1$$

therefore

$$\inf_{\mathbb{Q}}\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[p\left(1\right)\right]=0.4\cdot 0=0$$

and

$$\sup_{\mathbb{Q}}\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[p\left(1\right)\right]=0.4\cdot 0.625=0.25$$

Hence the no-arbitrage interval for the put option is $(0;0.25)$

4. If the *put* option trades at $t=0$ at the price

$$p(0)=0.16$$

the extended market B, S_1 and the put option is free of arbitrage opportunities, because $0.16\in(0;0.25)$. The set of risk neutral probabilities in the extended market is determined by imposing

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[p\left(1\right)\right]=0.4q_1=p(0)=0.16$$

that yields

$$q_1=\frac{0.16}{0.4}=0.4$$

and, consequentely,

$$\begin{cases} q_1=0.4 \\ q_2=\frac{1}{3}\cdot 0.4+\frac{1}{6}=0.3 \\ q_3=-\frac{4}{3}\cdot 0.4+\frac{5}{6}=0.3 \end{cases}$$

5. Let $\Delta S=S(1)-S(0)$. The final payoff of the passport option determined by the trading strategy $u\in[-1;1]$ is

$$X_u=\left(u[S(1)-S(0)]\right)^+=\left(u\Delta S\right)^+$$

We first compute ΔS in the different ω_k, for $k=1,2,3$:

$$\begin{gathered}\Delta S(\omega_1)=0.5-1=-0.5\\ \Delta S(\omega_2)=2.5-1=1.5\\ \Delta S(\omega_3)=1-1=0\end{gathered}$$

Therefore if the holder is *short* the stock, i.e. $u<0$,

$$\begin{gathered}\left(u\Delta S(\omega_1)\right)^+=\left(u\cdot(-0.5)\right)^+=(-u)\cdot 0.5>0\\ \left(u\Delta S(\omega_2)\right)^+=\left(u\cdot 1.5\right)^+=0\\ \left(u\Delta S(\omega_3)\right)^+=\left(u\cdot 0\right)^+=0\end{gathered}$$

and if the holder is *long* the stock, i.e. $u > 0$,

$$(u\Delta S(\omega_1))^+ = (u \cdot (-0.5))^+ = 0$$
$$(u\Delta S(\omega_2))^+ = (u \cdot 1.5)^+ = u \cdot 1.5 > 0$$
$$(u\Delta S(\omega_3))^+ = (u \cdot 0)^+ = 0$$

Therefore

$$\begin{aligned} E^Q\left[(u\Delta S)^+\right] &= \begin{cases} (-u) \cdot 0.5 \cdot \mathbb{Q}(\omega_1) + 0 + 0 & \text{if } u < 0 \\ 0 + u \cdot 1.5 \cdot \mathbb{Q}(\omega_2) + 0 & \text{if } u > 0 \end{cases} \\ &= \begin{cases} (-u) \cdot 0.5 \cdot 0.4 = 0.2 \cdot (-u) & \text{if } u < 0 \\ u \cdot 1.5 \cdot 0.3 = 0.45 \cdot u & \text{if } u > 0 \end{cases} \end{aligned}$$

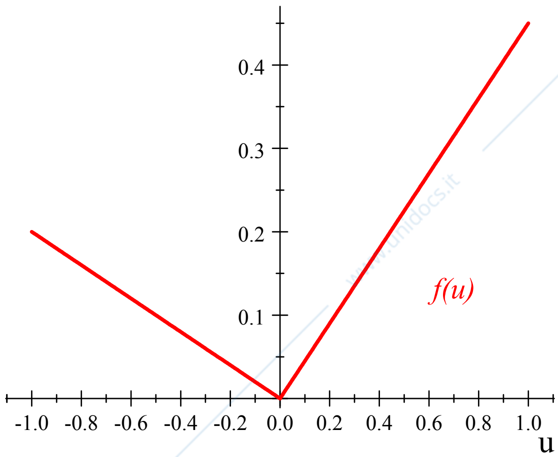
The trading strategy $u^* \in [-1, 1]$ that maximizes

$$\frac{E^Q[X_u]}{1+r}$$

is the trading strategy $u^* \in [-1, 1]$ that maximizes the piecewise linear function

$$f(u) = E^Q\left[(u\Delta S)^+\right] = \begin{cases} -0.2 \cdot u & \text{if } u \in [-1; 0] \\ 0.45 \cdot u & \text{if } u \in [0; 1] \end{cases}$$

on the domain $[-1; 1]$. As we can easily see, the function $f(u)$ reaches its maximum at $u^* = 1$.



The maximum value $f(u^*)$ is equal to

$$\max_u E^Q\left[(u\Delta S)^+\right] = f(u^*) = 0.45 \cdot (1) = 0.45$$

The intial no-arbitrage price of the passport option is therefore

$$S_X(0) = \frac{E^Q[X_{u^*}]}{1+r} = \frac{0.45}{1.25} = 0.36$$

6. If the holder of the passport option enters in the *non-optimal* trading strategy

$$u_o = -\frac{u^*}{2} = -\frac{1}{2} = -0.5 < 0,$$

the final payoff is

$$X_{u_o} = \begin{cases} (u_o \Delta S(\omega_1))^+ = (-u_o \cdot 0.5)^+ = 0.5 \cdot 0.5 = 0.25 \\ (u_o \Delta S(\omega_2))^+ = (u_o \cdot 1.5)^+ = ((-0.5) \cdot 1.5)^+ = 0 \\ (u_o \Delta S(\omega_3))^+ = (u_o \cdot 0)^+ = 0 \end{cases}$$

with replicating cost equal to

$$S_{X_{u_o}}(0) = \frac{E^Q\left[(u_o \Delta S)^+\right]}{1+r} = \frac{-0.2 \cdot u_o}{1.25} = \frac{-0.2 \cdot (-0.5)}{1.25} = 0.08,$$

since *the risk-neutral expected value of the discounted terminal payoff equals its initial replication cost,* by no-arbitrage.

The difference that can be pocketed by the writer of the option due to the suboptimal choice of the trading strategy is

$$S_X(0) - S_{X_{u_o}}(0) = 0.36 - 0.08 = 0.28.$$

Solution of EXERCISE 2

1. The *historical probability* $\mathbb{P}$ that a European *call* option on S with strike price $K = 0.5$ closes at maturity $T = 1$ *in the money* is

$$\begin{aligned}
\mathbb{P}[S(1) > K] &= \mathbb{P}\left[e^{\left(\mu - \frac{\sigma^2}{2}\right)\cdot 1 + \sigma W_1} > \frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[\left(\mu - \frac{\sigma^2}{2}\right)\cdot 1 + \sigma W_1 > \ln \frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[Z > \frac{1}{\sigma\sqrt{1}}\left(\ln \frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 1\right)\right] = \\
&= N\left(-\frac{1}{\sigma\sqrt{1}}\left(\ln \frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 1\right)\right) \\
&= N\left(-\frac{1}{0.14\sqrt{1}}\left(\ln \frac{0.5}{1} - \left(0.06 - \frac{0.14^2}{2}\right)\cdot 1\right)\right) = \\
&= N(5.3096) \approx 1
\end{aligned}$$

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2. The initial no-arbitrage price of the European derivative on S whose payoff X at maturity $T = 1$ is

$$X = 100 \cdot \left(\ln \frac{S(1)}{S(0.5)} \right)^2$$

is given by

$$S_X(0) = 100 \cdot \mathbb{E}^{\mathbb{Q}} \left[\left(\left(\ln \frac{S(1)}{S(0.5)} \right)^2 \right) e^{-\delta \cdot 1} \right] = 100 \cdot \mathbb{E}^{\mathbb{Q}} \left[\left(\left(\delta - \frac{\sigma^2}{2} \right) \cdot 0.5 + \sigma \left(W^*(1) - W^*(0.5) \right) \right)^2 \right] e^{-\delta} =$$

$$= 100 \cdot \mathbb{E}^{\mathbb{Q}} \left[\left(\left(\delta - \frac{\sigma^2}{2} \right) \cdot 0.5 + \sigma \left(\sqrt{0.5} \cdot Z \right) \right)^2 \right] e^{-\delta}, \quad \text{because } W^*(1) - W^*(0.5) \overset{\mathbb{Q}}{\sim} \sqrt{0.5} \cdot Z \text{ with } Z \overset{\mathbb{Q}}{\sim} \mathcal{N}(0$$

$$= 100 \cdot e^{-\delta} \left[\left(\delta - \frac{\sigma^2}{2} \right)^2 0.5^2 + \sigma^2 \cdot 0.5 \right], \quad \text{since } \mathbb{E}^{\mathbb{Q}} \left[\sigma \left(\sqrt{0.5} \cdot Z \right) \right] = 0 \text{ and } \mathbb{E}^{\mathbb{Q}} \left[\left(\sigma \left(\sqrt{0.5} \cdot Z \right) \right)^2 \right] = \sigma^2 (0.5$$

$$= 100 \cdot e^{-0.04} \left(\left(0.04 - \frac{0.14^2}{2} \right)^2 (0.5)^2 + 0.14^2 \cdot 0.5 \right) = 0.963\,48$$

3. To show that $S_X(0)$ is increasing w.r.t. σ we compute

$$\begin{aligned} \frac{\partial S_X(0)}{\partial \sigma} &= \frac{\partial}{\partial \sigma} \left(100 \cdot e^{-\delta} \left[\left(\delta - \frac{\sigma^2}{2} \right)^2 0.5^2 + \sigma^2 \cdot 0.5 \right] \right) \\ &= \left(100 \cdot e^{-\delta} \left[2 \left(\delta - \frac{\sigma^2}{2} \right) 0.5^2 (-\sigma) + \sigma \right] \right). \end{aligned}$$

Such derivative is positive if and only if

$$-\sigma 2 \left(\delta - \frac{\sigma^2}{2} \right) 0.5^2 + \sigma > 0$$
$$\sigma \left(0.25\sigma^2 - 0.5\delta + 1 \right) > 0$$
$$0.25\sigma^2 - 0.5\delta + 1 > 0 \quad \text{since } \sigma > 0$$
$$\sigma^2 > \frac{0.5\delta - 1}{0.25} = 2\delta - 4 = 2 \cdot 0.04 - 4 = -3.92$$

which is always satisfied, since $\sigma^2 > 0$. Therefore $S_X(0)$ is increasing with respect to σ, for any $\sigma \in \Re$.

4. The risk-neutral probability that X at maturity $T = 1$ exceeds the level $b = 1$ is

$$
\begin{aligned}
\mathbb{Q}[X > b] &= \mathbb{Q}\left[100 \cdot \left(\left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5 + \sigma\left(W^*(1) - W^*(0.5)\right)\right)^2 > b\right] \\
&= \mathbb{Q}\left[100 \cdot \left(\left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5 + \sigma\left(\sqrt{0.5} \cdot Z\right)\right)^2 > b\right] \\
&= \mathbb{Q}\left[\left\{\left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5 + \sigma\sqrt{0.5}Z > \sqrt{\frac{b}{100}}\right\} \cup \left\{\left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5 + \sigma\sqrt{0.5}Z < -\sqrt{\frac{b}{100}}\right\}\right] \\
&= \mathbb{Q}\left[\left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5 + \sigma\sqrt{0.5}Z > \sqrt{\frac{b}{100}}\right] + \mathbb{Q}\left[\left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5 + \sigma\sqrt{0.5}Z < -\sqrt{\frac{b}{100}}\right] \\
&\qquad \text{because the two events are disjoint} \\
&= \mathbb{Q}\left[Z > \left(\sqrt{\frac{b}{100}} - \left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5\right)\frac{1}{\sigma\sqrt{0.5}}\right] + \mathbb{Q}\left[Z < \left(-\sqrt{\frac{b}{100}} - \left(\delta - \frac{\sigma^2}{2}\right) \cdot 0.5\right)\frac{1}{\sigma\sqrt{0.5}}\right] \\
&= \mathbb{Q}\left[Z > \left(\sqrt{0.01} - \left(0.04 - \frac{0.14^2}{2}\right) \cdot 0.5\right)\frac{1}{0.14\sqrt{0.5}}\right] + \mathbb{Q}\left[Z < \left(-\sqrt{0.01} - \left(0.04 - \frac{0.14^2}{2}\right) \cdot 0.5\right)\frac{1}{0.14\sqrt{0.5}}\right] \\
&= \mathbb{Q}[Z > 0.85762] + \mathbb{Q}[Z < -1.1627] \\
&= \mathbb{Q}[Z < -0.85762] + \mathbb{Q}[Z < -1.1627], \quad \text{by the properties of } Z \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1) \\
&= N(-0.85762) + N(-1.1627) \\
&= 0.195551164 + 0.122475621 = 0.31803
\end{aligned}
$$

Quantitative Finance and Derivatives I
Finanza Quantitativa e Derivati I

code 20188

a.y. 2011/12, September 2012

Give a detailed justification to all your answers!

EXERCISE 1

Consider a multiperiod discrete market with $t = 0, 1, 2$ with the following information structure:

$$
\begin{array}{ccccc}
t=0 & & t=1 & & t=2 \\
 & & & \nearrow & \omega_1 \\
 & & f_1^1 & & \\
 & \nearrow & & \searrow & \omega_2 \\
f_1^0 & & & & \\
 & \searrow & & \nearrow & \omega_3 \\
 & & f_2^1 & & \\
 & & & \searrow & \omega_4
\end{array}
$$

Two securities are traded in the market. The first is the *locally risk-free asset* B that provides the locally riskless interest rate

$$
\begin{aligned}
r(0) &= 10\% \\
r(1)(f_1^1) &= 0\% \\
r(1)(f_2^1) &= 5\%.
\end{aligned}
$$

The second security is the *risky asset* S, whose price at time 0 is

$$S(0) = 10,$$

whereas at $t = 1$ its prices are

$$
\begin{aligned}
S(1)(f_1^1) &= 12 \\
S(1)(f_2^1) &= 9
\end{aligned}
$$

and at the final date $T = 2$

$$
\begin{aligned}
S(2)(\omega_1) &= 14.4 \\
S(2)(\omega_2) &= 10.8 \\
S(2)(\omega_3) &= 10.8 \\
S(2)(\omega_4) &= 8.1
\end{aligned}
$$

1. Write the prices of the locally riskless asset B in each node of the information structure for $t = 1, 2$.

2. Is the market dynamically complete?

3. Determine the set of the risk neutral probabilities $\mathbf{Q}$ for the market, specifying $\mathbf{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?

4. Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *European call option* on S with maturity $T = 2$ and strike $K = 9.8$.

5. Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *European knock-in call option* on S with maturity $T = 2$ and strike $K = 9.8$ and barrier $b = 9.5$. The call option is activated only if the stock S falls below the barrier $b = 9.5$ during the life of the option. The terminal payoff is therefore
$$X(2) = \begin{cases} (S(2) - K)^+ & \text{if } \min_{t=0,1,2} S(t) \leq b \\ 0 & \text{otherwise} \end{cases}$$

6. Consider a *forward contract* on S and maturity $T = 2$. The holder of a long position on the forward contract buys the stock S at maturity $T = 2$ at the price F. The *forward price* F is settled so that the no-arbitrage price of the contract at date 0 is null, that is
$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(2) - F}{B(2)}\right] = 0.$$
Compute F.

7. Suppose that the historical probability is uniform on the set $\Omega = \{\omega_1, ..., \omega_4\}$, that is
$$\mathbb{P}(\omega_k) = 0.25 \quad \text{for } k = 1, ..., 4.$$
Compute the historical probability that the terminal payoff of a long position on the forward contract of the previous point is positive, that is compute
$$\mathbb{P}[S(2) > F]$$

EXERCISE 2

Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 1$, $\delta = 3\%$, $\mu = 6\%$ and $\sigma = 10\%$.

1. Determine the *risk neutral probability* $\mathbb{Q}$ that a European *put* option on S with strike price $K = 1$ closes at maturity $T = 4$ *in the money.*

2. Find the strike price K' such that the *historical probability* $\mathbb{P}$ that the European *put* option on S with strike price K' closes at maturity $T = 4$ strictly *in the money* is equal to 50%.

3. Compute the differential of
$$Y(t) = \sqrt{S(t)}$$
with respect to the *risk-neutral probability* $\mathbb{Q}$. Is the process Y lognormal? Justify your answer. In case of a positive answer, compute the *risk-neutral drift* of the process Y.

4. Consider the European derivative whose payoff at maturity $T = 4$ is $Y(4)$. Compute its initial no-arbitrage price.

(For question 1 *you only need to determine the formula for this probability, expressed in terms of the distribution function* $N(\cdot)$ *of a standard Normal random variable*).

QUESTION

In a discrete multiperiod arbitrage-free market consider a redundant derivative security with cash-flow process $X = \{X(t)\}_{t=1}^{T}$. The payout $X(t)$ is paid at t to the holder of the derivative security. *State* the equivalent conditions that the *price process* of this redundant derivative security has to satisfy at any $t = 0, ..., T$ in order to preserve no-arbitrage in the extended market.

SOLUTIONS TO EXERCISES

Exercise 1

1. The prices of security B are
$$B(1)(f_1^1) = B(1)(f_2^1) = 1.1$$
and at the final date $T = 2$
$$B(2)(\omega_1) = B(2)(\omega_2) = 1.1$$
$$B(2)(\omega_3) = B(2)(\omega_4) = 1.1 \cdot 1.05 = 1.155$$

2. The market is dynamically complete, because each one-period submarket is complete (in your exam check explicitly that the rank of the *terminal* payoff matrix of each one-period submarket has rank 2).

3. We look for risk neutral probabilities $\mathbf{Q}$ for the market. We have to solve the systems
$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbf{Q}[f_1^1] + S(1)(f_2^1)\mathbf{Q}[f_2^1] \right\} \\ \mathbf{Q}[f_1^1] + \mathbf{Q}[f_2^1] = 1 \\ \mathbf{Q}[f_1^1], \mathbf{Q}[f_2^1] > 0 \end{cases} \tag{1}$$
for m_0,
$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbf{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbf{Q}[\omega_2|f_1^1] \right\} \\ \mathbf{Q}[\omega_1|f_1^1] + \mathbf{Q}[\omega_2|f_1^1] = 1 \\ \mathbf{Q}[\omega_1|f_1^1], \mathbf{Q}[\omega_2|f_1^1] > 0 \end{cases} \tag{2}$$
for $m_{1,1}$, and
$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbf{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbf{Q}[\omega_4|f_2^1] \right\} \\ \mathbf{Q}[\omega_3|f_2^1] + \mathbf{Q}[\omega_4|f_2^1] = 1 \\ \mathbf{Q}[\omega_3|f_2^1], \mathbf{Q}[\omega_4|f_2^1] > 0 \end{cases} \tag{3}$$

System (1) can be rewritten as
$$\begin{cases} 10 = \frac{1}{1.1} \left\{ 12 \cdot \mathbf{Q}[f_1^1] + 9 \cdot \mathbf{Q}[f_2^1] \right\} \\ \mathbf{Q}[f_1^1] + \mathbf{Q}[f_2^1] = 1 \\ \mathbf{Q}[f_1^1], \mathbf{Q}[f_2^1] > 0 \end{cases}$$
and is solved by
$$\begin{aligned} \mathbf{Q}[f_1^1] &= 0.666666667 \\ \mathbf{Q}[f_2^1] &= 1 - 0.666666667 = 0.333\,33 \end{aligned}$$

System (2) can be rewritten as
$$\begin{cases} 12 = \left\{ 14.4 \cdot \mathbf{Q}[\omega_1|f_1^1] + 10.8 \cdot \mathbf{Q}[\omega_2|f_1^1] \right\} \\ \mathbf{Q}[\omega_1|f_1^1] + \mathbf{Q}[\omega_2|f_1^1] = 1 \\ \mathbf{Q}[\omega_1|f_1^1], \mathbf{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned}\mathbf{Q}[\omega_1|f_1^1] &= 0.333\,33\\ \mathbf{Q}[\omega_2|f_1^1] &= 0.666666667\end{aligned}$$

and System (3) can be rewritten as

$$\begin{cases} 9 = \frac{1}{1.05}\left\{10.8\cdot\mathbf{Q}[\omega_3|f_2^1] + 8.1\cdot\mathbf{Q}[\omega_4|f_2^1]\right\}\\ \mathbf{Q}[\omega_3|f_2^1] + \mathbf{Q}[\omega_4|f_2^1] = 1\\ \mathbf{Q}[\omega_3|f_2^1], \mathbf{Q}[\omega_4|f_2^1] > 0\end{cases}$$

and is solved by

$$\begin{aligned}\mathbf{Q}[\omega_3|f_2^1] &= 0.5\\ \mathbf{Q}[\omega_4|f_2^1] &= 0.5\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{Q}[\omega_1] &= 0.666666667\cdot 0.333\,33 = 0.222\,22\\ \mathbf{Q}[\omega_2] &= 0.666666667\cdot 0.666666667 = 0.444\,44\\ \mathbf{Q}[\omega_3] &= 0.333\,33\cdot 0.5 = 0.166\,67\\ \mathbf{Q}[\omega_4] &= 0.333\,33\cdot 0.5 =: 0.166\,67\end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

4. The terminal payoff of *European call option* on S with maturity $T = 2$ and strike $K = 9.8$ is in the nodes ω_k for $k = 1, ..., 4$

$$c\,(2) = \begin{cases} (S(2)(\omega_1) - 9.8)^+ = (14.4 - 9.8)^+ = 4.6\\ (S(2)(\omega_2) - 9.8)^+ = (10.8 - 9.8)^+ = 1\\ (S(2)(\omega_3) - 9.8)^+ = (10.8 - 9.8)^+ = 1\\ (S(2)(\omega_4) - 9.8)^+ = (8.1 - 9.8)^+ = 0\end{cases}$$

The no arbitrage prices of the *call* option at $t = 0$ and in the nodes f_1^1 and f_2^1 at $t = 1$ are

$$\begin{aligned}c(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(2)}{1+r\,(1)}\middle|\mathcal{P}_1\right](f_1^1)\\ &= \frac{4.6\cdot 0.333\,33 + 1\cdot 0.666666667}{1.0} = 2.2\\ c(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(2)}{1+r\,(1)}\middle|\mathcal{P}_1\right](f_2^1)\\ &= \frac{1\cdot 0.5 + 0\cdot 0.5}{1.05} = 0.476\,19\\ c(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1+r\,(0)}\right]\\ &= \frac{2.2\cdot 0.666666667 + 0.476\,19\cdot 0.333\,33}{1.1} = 1.\,477\,6\end{aligned}$$

5. A *European knock-in call option* on S with maturity $T = 2$ and strike $K = 9.8$ and barrier $b = 9.5.$ is activated only if the stock S falls below the barrier $b = 9.5$ during the life of the option. The terminal payoff is therefore

$$X(2) = \begin{cases} (K)^+ & \text{if } \min_{t=0,1,2} S(t) \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} X(2)(\omega_1) &= X(2)(\omega_2) = 0 \text{ since the barrier is not activated} \\ X(2)(\omega_3) &= (S(2)(\omega_3) - 9.8)^+ = (10.8 - 9.8)^+ = 1 \\ X(2)(\omega_4) &= (S(2)(\omega_4) - 9.8)^+ = (8.1 - 9.8)^+ = 0 \end{aligned}$$

The no-arbitrage price at $t = 0$ and at $t = 1$ of the *European knock-in call option* on S are

$$\begin{aligned} S_X(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right| \mathcal{P}_1\right](f_1^1) = 0 \\ S_X(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right| \mathcal{P}_1\right](f_2^1) \\ &= \frac{1 \cdot 0.5 + 0 \cdot 0.5}{1.05} = c(1)(f_2^1) = 0.476\,19 \\ S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1)}{1+r(0)}\right] \\ &= \frac{0 \cdot 0.666666667 + 0.476\,19 \cdot 0.333\,33}{1.1} = 0.144\,30 \end{aligned}$$

6. The *forward price* F for stock S at date 2 is such that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(2) - F}{B(2)}\right] = 0$$

hence

$$F \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S(2)}{B(2)}\right] = S(0)$$

where the last equality follows by the martingality property of the discounted risky asset with respect to the risk-neutral measure. We compute

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right] = \frac{1}{1.1} \cdot 0.666666667 + \frac{1}{1.155} \cdot 0.333\,33 = 0.894\,66$$

We get therefore

$$F = \frac{S(0)}{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right]} = \frac{10}{0.894\,66} = 11.177$$

7. The historical probability that the terminal payoff of a long position on the forward contract of the previous point is positive is

$$\mathbb{P}[S(2) > F]) = \mathbb{P}(\omega_1) = 0.25,$$

because

$$S(2)(\omega_k) > F \text{ for } k = 1 \text{ only.}$$

Exercise 2

1. The *risk neutral probability* $\mathbb{Q}$ that a European *put* option on S with strike price $K = 1$ closes at maturity $T = 4$ *in the money* is

$$\begin{aligned}
\mathbb{P}[S(4) < K] &= \mathbb{P}\left[e^{\left(\delta - \frac{\sigma^2}{2}\right)\cdot 4 + \sigma W_4} < \frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[\left(\delta - \frac{\sigma^2}{2}\right)\cdot 4 + \sigma W_4 < \ln\frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[Z < \frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K}{S(0)} - \left(\delta - \frac{\sigma^2}{2}\right)\cdot 4\right)\right] = \\
&= N\left(\frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K}{S(0)} - \left(\delta - \frac{\sigma^2}{2}\right)\cdot 4\right)\right) \\
&= N\left(\frac{1}{0.1\sqrt{4}}\left(\ln\frac{1}{1} - \left(0.03 - \frac{0.10^2}{2}\right)\cdot 4\right)\right) = \\
&= N(-0.5)
\end{aligned}$$

   where Z denotes a standard normal random variable with respect to the *historical probability* $\mathbb{P}$.

2. The strike price K' such that the *historical probability* $\mathbb{P}$ that the European *put* option on S with strike price K' closes at maturity $T = 4$ strictly *in the money* is equal to 50% is determined by

$$\mathbb{P}\left[S(4) < K^{'}\right] = 0.5$$

   that is

$$\begin{aligned}
\mathbb{P}\left[S(4) < K^{'}\right] &= \mathbb{P}\left[e^{\left(\mu - \frac{\sigma^2}{2}\right)\cdot 4 + \sigma W_4} < \frac{K^{'}}{S(0)}\right] = \\
&= \mathbb{P}\left[\left(\mu - \frac{\sigma^2}{2}\right)\cdot 4 + \sigma W_4 < \ln\frac{K^{'}}{S(0)}\right] = \\
&= \mathbb{P}\left[Z < \frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K^{'}}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 4\right)\right] = \\
&= N\left(\frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K^{'}}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 4\right)\right) = 0.5
\end{aligned}$$

   that is true for

$$\frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K^{'}}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 4\right) = 0,$$

because of the properties of the standard normal distribution. The last equality implies

$$\begin{aligned} \ln \frac{K'}{S(0)} &= \left(\mu - \frac{\sigma^2}{2}\right) \cdot 4 \\ K' &= S(0) e^{\left(\mu - \frac{\sigma^2}{2}\right) \cdot 4} \\ &= 1 \cdot e^{\left(0.06 - \frac{0.1^2}{2}\right) \cdot 4} = 1.2461 \end{aligned}$$

3. To compute the differential of

$$Y(t) = \sqrt{S(t)}$$

with respect to the *risk-neutral probability* $\mathbb{Q}$ we first calculate the derivatives of

$$f(t, S) = \sqrt{S}.$$

These derivatives are

$$\tfrac{\partial f(t,S)}{\partial t} = 0; \qquad \tfrac{\partial f(t,S)}{\partial S} = \tfrac{1}{2} S^{-\frac{1}{2}}; \qquad \tfrac{\partial^2 f(t,S)}{\partial S^2} = -\tfrac{1}{4} S^{-\frac{3}{2}}$$

The differential by Ito formula is

$$\begin{aligned} dY(t) &= \frac{\partial f(t, S(t))}{\partial t} dt + \frac{\partial f(t, S(t))}{\partial S} dS(t) + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 f(t, S(t))}{\partial S^2} dt \\ &= \frac{1}{2} S(t)^{-\frac{1}{2}} dS(t) + \frac{1}{2}\sigma^2 S(t)^2 \left(-\frac{1}{4} S(t)^{-\frac{3}{2}}\right) dt \\ &= \sqrt{S(t)} \left(\frac{1}{2}(\delta dt + \sigma dW^*(t)) - \frac{1}{8}\sigma^2 dt\right) \\ &= Y(t) \left(\left(\frac{1}{2}\delta - \frac{1}{8}\sigma^2\right) dt + \frac{1}{2}\sigma dW^*(t)\right) \end{aligned}$$

where W^* is a standard Brownian motion under the *risk-neutral probability* $\mathbb{Q}$. Therefore Y is lognormal with respect to the *risk-neutral probability* $\mathbb{Q}$, and the *risk-neutral drift is*

$$\begin{aligned} \mu_Y^{\mathbb{Q}} &= \frac{1}{2}\delta - \frac{1}{8}\sigma^2 = \\ &= \frac{1}{2} 0.03 - \frac{1}{8} 0.1^2 = 0.01375 \end{aligned}$$

4. The initial no-arbitrage price of the European derivative whose payoff at maturity $T = 4$ is $Y(4)$ is

$$\mathbb{E}^{\mathbb{Q}}\left[Y(4) e^{-\delta 4}\right] = e^{-\delta 4} \mathbb{E}^{\mathbb{Q}}[Y(4)] = e^{-\delta 4} Y(0) e^{\mu_Y^{\mathbb{Q}} \cdot 4},$$

because Y is lognormal with respect to the *risk-neutral probability* $\mathbb{Q}$.

Therefore

$$\mathbb{E}^{\mathbb{Q}}\left[Y(4) e^{-\delta 4}\right] = e^{-0.03 \cdot 4} \cdot 1 \cdot e^{0.01375 \cdot 4} = 0.93707$$