Exams-2013 - exam

Quantitative Finance And Derivatives - Module 1 (Università Commerciale Luigi Bocconi)

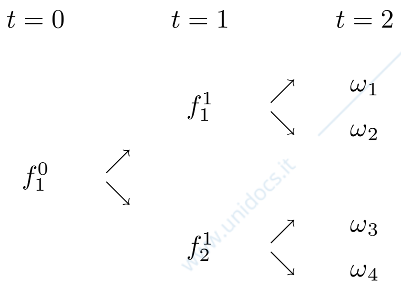
Quantitative Finance and Derivatives I
Finanza Quantitativa e Derivati I

code 20188
a.y. 2012/13, January 2013

POINTS WILL BE AWARDED ONLY IF THE ANSWER IS SUPPORTED BY A DETAILED LOGICAL GIUSTIFICATION

EXERCISE 1 (35 points out of 100).

Consider a multiperiod discrete market with $t = 0, 1, 2$ with the following information structure:



The *historical probability* is *uniform* on Ω, i.e. $\mathbb{P}(\omega_k) = 0.25$ for $k = 1, ..., 4$.

Two securities are traded in the market. The first is a *locally risk-free asset* B that provides the locally riskless interest rate

$$\begin{aligned} r(0) &= 5\% \\ r(1)(f_1^1) &= 6\% \\ r(1)(f_2^1) &= 0\%. \end{aligned}$$

The second security is a *risky asset* S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 11$$
$$S(1)(f_2^1) = 9$$

and with time 2 prices

$$S(2)(\omega_1) = 13.2$$
$$S(2)(\omega_2) = 8.8$$
$$S(2)(\omega_3) = 10.35$$
$$S(2)(\omega_4) = 8.55$$

1. **(5 points)** Is the market dynamically complete?
2. **(10 points)** Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
3. **(6 points)** Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *European digital option* on S with maturity $T = 2$ and strike $K = 10$. Recall that the final payoff of the digital option is

$$X(2) = \begin{cases} 1 & \text{if } S(2) > K \\ 0 & \text{otherwise} \end{cases}$$

4. **(14 points)** Is the final payoff $X(2)$ of the digital option of the previous point independent of $\mathcal{P}_1$ with respect to *the risk neutral probability* $\mathbb{Q}$? Is $X(2)$ independent of $\mathcal{P}_1$ with respect to the *historical probability* $\mathbb{P}$?

EXERCISE 2 (50 points out of 100).
Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 1$, $\delta = 3\%$, $\mu = 7\%$ and $\sigma = 14\%$.

1. **(5 points)** Determine the *historical probability* $\mathbb{P}$ that a European *put* option on S with strike price $K = 1$ closes *in the money* at the maturity $T = 4$ (Here *you only need to express the probability in terms of the distribution function* $N(\cdot)$ *of a standard Normal random variable*).

2. **(15 points)** Compute the stochastic differential of
$$Y(t) = (S(t))^{\frac{1}{3}}$$
with respect to *both the risk-neutral measure* $\mathbb{Q}$ and the *historical probability measure* $\mathbb{P}$. Is $Y(t)$ lognormal with respect to both probabilities? If your answer is positive, compute both the risk-neutral and the historical drift.

3. **(8 points)** Consider the European derivative on S with payoff X at maturity $T = 4$ given by
$$X = (S(T))^{\frac{1}{3}}.$$
Compute $S_X(0)$, its no-arbitrage price at time 0.

4. **(10 points)** Compute the buy-and-hold strategy with B and S such that the initial cost of this strategy is equal to the initial cost of replication, and whose terminal value at T replicates the final payoff X *on average with respect to the historical probability measure* $\mathbb{P}$.

5. **(12 points)** Let $(\vartheta_0, \vartheta_1) = (\vartheta_0(t), \vartheta_1(t))$ denote the *self-financing replicating portfolio* of the derivative of point 3. Compute $\vartheta_1(0)$. Are you long or short on S at the initial date?

QUESTION (15 points out of 100).
In a discrete multiperiod arbitrage-free market consider a redundant derivative security with cash-flow process $X = \{X(t)\}_{t=1}^{T}$. The payout $X(t)$ is paid at t to the holder of the derivative security. *State* the equivalent conditions that the *price process* of this redundant derivative security has to satisfy at any $t = 0, ..., T$ in order to preserve no-arbitrage in the extended market.

SOLUTIONS TO EXERCISES

Exercise 1

1. The prices of security B are
$$B(1)(f_1^1) = B(1)(f_2^1) = 1.05$$
and at the final date $T = 2$
$$\begin{array}{c} B(2)(\omega_1) = B(2)(\omega_2) = 1.05 \cdot 1.06 = 1.113 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.05 \end{array}$$
The market is dynamically complete, because each one-period submarket is complete (in your exam check explicitly that the rank of the *terminal* payoff matrix of each one-period submarket has rank 2).

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems
$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases} \quad (1)$$
for m_0,
$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases} \quad (2)$$
for $m_{1,1}$, and
$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases} \quad (3)$$
System (1) can be rewritten as
$$\begin{cases} 10 = \frac{1}{1.05} \left\{ 11 \cdot \mathbb{Q}[f_1^1] + 9 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$
and is solved by
$$\begin{array}{rcl} \mathbb{Q}[f_1^1] & = & 0.75 \\ \mathbb{Q}[f_2^1] & = & 0.25 \end{array}$$
System (2) can be rewritten as
$$\begin{cases} 11 = \frac{1}{1.06} \left\{ 13.2 \cdot \mathbb{Q}[\omega_1|f_1^1] + 8.8 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$
and is solved by
$$\begin{array}{rcl} \mathbb{Q}[\omega_1|f_1^1] & = & 0.65 \\ \mathbb{Q}[\omega_2|f_1^1] & = & 0.35 \end{array}$$

and System (3) can be rewritten as

$$\begin{cases} 9 = \frac{1}{1+r(1)(f_2^1)} \left\{ 10.35 \cdot \mathbb{Q}[\omega_3|f_2^1] + 8.55 \cdot \mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.25 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.75 \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.75 \cdot 0.65 = 0.487\,5 \\ \mathbb{Q}[\omega_2] &= 0.75 \cdot 0.35 = 0.262\,5 \\ \mathbb{Q}[\omega_3] &= 0.25 \cdot 0.25 = 0.062\,5 \\ \mathbb{Q}[\omega_4] &= 0.25 \cdot 0.75 = 0.187\,5 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

3. The terminal payoff of *European digital option* on S with maturity $T = 2$ and strike $K = 10$ is

$$\begin{aligned} X(2)(\omega_k) &= 1 \text{ for } k = 1, 3 \\ X(2)(\omega_k) &= 0 \text{ for } k = 2, 4 \end{aligned}$$

The no arbitrage prices of the *digital* option at $t = 0$ and in the nodes f_1^1 and f_2^1 at $t = 1$ are

$$\begin{aligned} S_X(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}} \left[\frac{X(2)}{1+r(1)} \middle| \mathcal{P}_1 \right] (f_1^1) \\ &= \frac{1 \cdot 0.65 + 0 \cdot 0.35}{1.06} = 0.613\,21 \\ S_X(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}} \left[\frac{X(2)}{1+r(1)} \middle| \mathcal{P}_1 \right] (f_2^1) \\ &= \frac{1 \cdot 0.25 + 0 \cdot 0.75}{1.0} = 0.25 \\ S_X(0) &= \mathbb{E}^{\mathbb{Q}} \left[\frac{S_X(1)}{1+r(0)} \right] \\ &= \frac{0.613\,21 \cdot 0.75 + 0.25 \cdot 0.25}{1.05} = 0.497\,53 \end{aligned}$$

4. The final payoff $X(2)$ of the digital option of the previous point is *not* independent of $\mathcal{P}_1$ with respect to the risk neutral probability $\mathbb{Q}$, because

$$\mathbb{Q}\left((X(2) = 1) \cap f_1^1 \right) = \mathbb{Q}(\omega_1) = 0.487\,5$$

is different from the product of the two probabilities

$$\begin{aligned} \mathbb{Q}(X(2) = 1) \cdot \mathbb{Q}\left(f_1^1\right) &= \mathbb{Q}(\omega_1 \cup \omega_3) \cdot \mathbb{Q}\left(f_1^1\right) \\ &= (0.487\,5 + 0.062\,5) \cdot 0.75 = 0.412\,5. \end{aligned}$$

On the contrary, with respect to the uniform historical probability, the probability of the following intersections

$$\begin{aligned}\mathbb{P}\left((X(2)=1)\cap f_1^1\right) &= \mathbb{P}(\omega_1)=0.25\\ \mathbb{P}\left((X(2)=1)\cap f_2^1\right) &= \mathbb{P}(\omega_3)=0.25\\ \mathbb{P}\left((X(2)=0)\cap f_1^1\right) &= \mathbb{P}(\omega_2)=0.25\\ \mathbb{P}\left((X(2)=0)\cap f_2^1\right) &= \mathbb{P}(\omega_4)=0.25\end{aligned}$$

do concide (resp.) with the product of the probabilities:

$$\begin{aligned}\mathbb{P}(X(2)=1)\cdot\mathbb{P}\left(f_1^1\right) &= \mathbb{P}(\omega_1\cup\omega_3)\cdot\mathbb{P}\left(f_1^1\right)=(0.25+0.25)\cdot 0.5=0.25\\ \mathbb{P}(X(2)=1)\cdot\mathbb{P}\left(f_2^1\right) &= \mathbb{P}(\omega_1\cup\omega_3)\cdot\mathbb{P}\left(f_2^1\right)=(0.25+0.25)\cdot 0.5=0.25\\ \mathbb{P}(X(2)=0)\cdot\mathbb{P}\left(f_1^1\right) &= \mathbb{P}(\omega_2\cup\omega_4)\cdot\mathbb{P}\left(f_1^1\right)=(0.25+0.25)\cdot 0.5=0.25\\ \mathbb{P}(X(2)=0)\cdot\mathbb{P}\left(f_2^1\right) &= \mathbb{P}(\omega_2\cup\omega_4)\cdot\mathbb{P}\left(f_2^1\right)=(0.25+0.25)\cdot 0.5=0.25\end{aligned}$$

Thus the payoff $X(2)$ is independent of $\mathcal{P}_1$ with respect to historical ptobability $\mathbb{P}$.

Exercise 2.

1. The *historical probability* $\mathbb{P}$ that a European *put* option on S with strike price $K=1$ closes at maturity $T=4$ *in the money is*

$$\begin{aligned}\mathbb{P}[S(4)<K] &= \mathbb{P}\left[e^{\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4+\sigma W_4}<\frac{K}{S(0)}\right]=\\ &= \mathbb{P}\left[\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4+\sigma W_4<\ln\frac{K}{S(0)}\right]=\\ &= \mathbb{P}\left[Z<\frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K}{S(0)}-\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4\right)\right]=\\ &= N\left(\frac{1}{\sigma\sqrt{4}}\left(\ln\frac{K}{S(0)}-\left(\mu-\frac{\sigma^2}{2}\right)\cdot 4\right)\right)\\ &= N\left(\frac{1}{0.14\sqrt{4}}\left(\ln\frac{1}{1}-\left(0.07-\frac{0.14^2}{2}\right)\cdot 4\right)\right)=\\ &= N(-0.86)=0.194\,89\end{aligned}$$

   where Z denotes a standard normal random variable with respect to the *historical probability* $\mathbb{P}$.

2. To compute the stochastic differential of

$$Y(t)=(S(t))^{\frac{1}{3}}$$

   with respect to the risk-neutral measure we consider the function

$$f(t,S)=(S)^{\frac{1}{3}}$$

   and its derivatives

$$\tfrac{\partial f(t,S)}{\partial t}=0;\qquad \tfrac{\partial f(t,S)}{\partial S}=\tfrac{1}{3}S^{-\frac{2}{3}};\qquad \tfrac{\partial^2 f(t,S)}{\partial S^2}=-\tfrac{2}{9}S^{-\frac{5}{3}}$$

Applying Ito formula we get

$$\begin{aligned} dY(t) &= 0\cdot dt+\frac{1}{3}S(t)^{-\frac{2}{3}}\cdot dS(t)+\frac{1}{2}\cdot\left(-\frac{2}{9}S(t)^{-\frac{5}{3}}\right)\cdot S^{2}(t)\,\sigma^{2}dt \\ &= (S)^{\frac{1}{3}}\left(\frac{1}{3}\left(\delta dt+\sigma dW^{*}(t)\right)-\frac{1}{9}\sigma^{2}dt\right) \\ &= Y(t)\left(\left(\frac{1}{3}\delta+-\frac{1}{9}\sigma^{2}\right)dt+\frac{1}{3}\sigma dW^{*}(t)\right) \end{aligned}$$

Hence $Y(t)$ is lognormal with respect to the risk-neutral measure and its risk-neutral drift.is

$$\mu_Y^{\mathbb{Q}}=\frac{1}{3}\delta-\frac{1}{9}\sigma^{2}=\frac{1}{3}\cdot 0.03-\frac{1}{9}0.14^{2}=7.822\,2\times 10^{-3}$$

With similar steps, Y is lognormal also with respect to the *historical probability measure* $\mathbb{P}$, since

$$dY(t)=Y(t)\left(\left(\frac{1}{3}\mu+-\frac{1}{9}\sigma^{2}\right)dt+\frac{1}{3}\sigma dW(t)\right)$$

and its historical drift is

$$\mu_Y^{\mathbb{P}}=\frac{1}{3}\mu-\frac{1}{9}\sigma^{2}=\frac{1}{3}\cdot 0.07-\frac{1}{9}0.14^{2}=2.115\,6\times 10^{-2}$$

3. The initial price of the European derivative on S whose payoff X at maturity $T=4$ is

$$X=(S(T))^{\frac{1}{3}}.$$

is

$$\begin{aligned} S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\delta T}\cdot(S(T))^{\frac{1}{3}}\right]=e^{-\delta T}\cdot\mathbb{E}^{\mathbb{Q}}\left[(S(T))^{\frac{1}{3}}\right] \\ &= e^{-\delta T}\cdot Y(0)\,e^{\mu_Y^{\mathbb{Q}}\cdot T} \end{aligned}$$

because Y is lognormal, and hence $\mathbb{E}^{\mathbb{Q}}\left[(S(T))^{\frac{1}{3}}\right]=\mathbb{E}^{\mathbb{Q}}[Y(T)]=Y(0)\,e^{\mu_Y^{\mathbb{Q}}\cdot T}$. Therefore

$$\begin{aligned} S_X(0) &= e^{-\delta T}\cdot Y(0)\,e^{\left(\frac{1}{3}\delta-\frac{1}{9}\sigma^{2}\right)\cdot T} \\ &= Y(0)\cdot e^{\left(-\frac{2}{3}\delta-\frac{1}{9}\sigma^{2}\right)\cdot T}=1\cdot e^{\left(-\frac{2}{3}\cdot 0.03-\frac{1}{9}0.14^{2}\right)\cdot 4}=0.915\,11 \end{aligned}$$

4. Denote with h_0,h_1 the number of units of B and S that constitute the buy-and-hold strategy. We have to determine h_0,h_1 such that the cost of the strategy

$$h_0\cdot 1+h_1\cdot S(0)=S_X(0)$$

and such that at maturity

$$\mathbb{E}^{\mathbb{P}}\left[(S(T))^{\frac{1}{3}}-\left(h_0\cdot e^{\delta T}+h_1\cdot S(T)\right)\right]=0.$$

This average equation can be rewritten as

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[(S(T))^{\frac{1}{3}}\right]-\mathbb{E}^{\mathbb{P}}\left[\left(h_0\cdot e^{\delta T}+h_1\cdot S(T)\right)\right] &= 0 \\ (S(0))^{\frac{1}{3}}\,e^{\mu_Y^{\mathbb{P}}\cdot T}-h_0\cdot e^{\delta T}-h_1 S(0)\,e^{\mu T} &= 0 \text{ because } (S(t))^{\frac{1}{3}} \text{ and } S(t) \text{ are lognormal} \end{aligned}$$

From the initial cost equation we get

$$h_0 = S_X(0) - h_1 \cdot S(0),$$

and substituting into the average equation we obtain

$$\begin{aligned}(S(0))^{\frac{1}{3}} e^{\mu_Y^{\mathbb{P}} \cdot T} - (S_X(0) - h_1 \cdot S(0)) \cdot e^{\delta T} - h_1 S(0) e^{\mu T} &= 0 \\ h_1 S(0) \left(e^{\delta T} - e^{\mu T}\right) &= S_X(0) e^{\delta T} - (S(0))^{\frac{1}{3}} e^{\mu_Y^{\mathbb{P}} \cdot T} \\ h_1 &= \frac{S_X(0) e^{\delta T} - (S(0))^{\frac{1}{3}} e^{\mu_Y^{\mathbb{P}} \cdot T}}{S(0)(e^{\delta T} - e^{\mu T})} \\ &= \frac{0.915\,11 \cdot e^{0.03 \cdot 4} - (1)^{\frac{1}{3}} e^{2.115\,6 \times 10^{-2} \cdot 4}}{1 \cdot (e^{0.03 \cdot 4} - e^{0.07 \cdot 4})} = 0.288\,93\end{aligned}$$

Therefore

$$h_0 = 0.915\,11 - 0.288\,93 \cdot 1 = 0.626\,18.$$

5. The initial no-arbitrage price of the derivative is

$$\begin{aligned}S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\delta T} \cdot (S(T))^{\frac{1}{3}}\right] = e^{-\delta T} \cdot \mathbb{E}^{\mathbb{Q}}\left[(S(T))^{\frac{1}{3}}\right] \\ &= e^{-\delta T} \cdot (S(0))^{\frac{1}{3}} e^{\mu_Y^{\mathbb{Q}} \cdot T} \\ &= (S(0))^{\frac{1}{3}} e^{\left(-\frac{2}{3}\delta - \frac{1}{9}\sigma^2\right) \cdot T} \text{ since } \mu_Y^{\mathbb{Q}} = \frac{1}{3}\delta - \frac{1}{9}\sigma^2\end{aligned}$$

The initial number of units of S of the self-financing replicating strategy is

$$\vartheta_1(0) = \frac{\partial S_X(0)}{\partial S(0)} = \frac{1}{3}(S(0))^{\frac{-2}{3}} \cdot e^{\left(-\frac{2}{3}\delta - \frac{1}{9}\sigma^2\right) \cdot T} > 0$$

Hence the replicating strategy is initially long on S.

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POINTS WILL BE AWARDED ONLY IF THE ANSWER IS SUPPORTED BY A DETAILED LOGICAL JUSTIFICATION

EXERCISE 1 (50 points out of 100).
Consider a one period market with the riskless asset B yielding a risk-free rate $r = 4\%$, and a risky security S_1 whose prices at time $T = 1$ are

$$S_1(1)(\omega_1) = 9$$
$$S_1(1)(\omega_2) = 10$$
$$S_1(1)(\omega_3) = 8$$

Assume that the *historical probability* $\mathbb{P}$ on Ω is

$$\mathbb{P}(\omega_1) = 0.4$$
$$\mathbb{P}(\omega_2) = 0.5$$
$$\mathbb{P}(\omega_3) = 0.1.$$

1. **(3 points)** Is the market complete?
2. **(7 points)** Suppose that the risky security S_1 trades at $t = 0$ at the price
$$S_1(0) = S_0 = 8.75$$
Do state price vectors/risk neutral probabilities exist? If your answer is positive, find both the set of state price vectors and that of risk neutral probabilities, and discuss no-arbitrage in the market.
3. **(10 points)** Suppose that a *call* option on S_1 with strike $K = 9.22$ and maturity $T = 1$ is introduced in the market. Determine $c(1)(\omega_k)$, the payoff of the option at maturity $T = 1$ for $k = 1, 2, 3$. Is this *call* option replicable with B and S_1? If your answer is negative, determine the interval of no-arbitrage prices at $t = 0$ for this *call* option on S_1.
4. **(10 points)** Compute the strategy $\vartheta^{INF} = \left(\vartheta_0^{INF}, \vartheta_1^{INF}\right)$ that minimizes in the *original* market the initial cost of super-replicating the maturity payoff of the call option introduced in point 3.
5. **(15 points)** Compute the strategy $\vartheta^{MV} = \left(\vartheta_0^{MV}, \vartheta_1^{MV}\right)$ that allows you to *mean-variance-hedge* in the *original* market the maturity payoff of the call option introduced in point 3. To determine the strategy ϑ^{MV} you need to impose the following two conditions on $V_{\vartheta^{MV}}(1)$, its value at time $T = 1$:
   (a) $V_{\vartheta^{MV}}(1)$ must *replicate* the maturity payoff of the call option $c(1)$ *on average with respect to the historical probability measure* $\mathbb{P}$;

(b) $V_{\vartheta^{MV}}(1)$ must *minimize the quadratic error of replication with respect to the hystorical probability measure* $\mathbb{P}$, that is $\mathbb{E}^{\mathbb{P}}\left[(c(1) - V_{\vartheta}(1))^2\right]$ must reach its minimum at $\vartheta = \vartheta^{MV}$.

6. **(5 points)** Compare the initial costs of the strategies ϑ^{INF} and ϑ^{MV} computed above, and give a financial intrerpretation to your findings.

EXERCISE 2 (30 points out of 100)
Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 1$, $\delta = 2\%$, $\mu = 7\%$ and $\sigma = 10\%$.

1. **(5 points)** Determine the *historical probability* $\mathbb{P}$ that a European *put* option on S with strike price $K = 0.5$ closes *out of the money* at the maturity $T = 1$.
2. **(15 points)** Given the parameter $\alpha > 0$, apply Ito Formula to compute the stochastic differential of
$$Y(t) = e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot (S(t))^{\alpha},$$
with respect to the *risk-neutral probability* $\mathbb{Q}$.
3. **(10 points)** Consider the European derivative on S whose payoff at maturity $T = 1$ is
$$Y(1) = e^{-\frac{1}{2}\sigma^2\alpha^2} \cdot (S(1))^{\alpha}$$
Determine the value of the parameter α consistent with the fact that the initial no-arbitrage price of the derivative is
$$S_Y(0) = 1.5.$$

THEORY QUESTION (20 points out of 100)
State and derive the Black-Scholes partial differential equation. Write the terminal condition at T for a European call option on S.

Solution of the Exercises

Solution of EXERCISE 1

1. The market is incomplete, because the number of scenarios $K = 3 > 2$, the number of traded securities.

2. Since the market is incomplete, the risk-neutral measures (if any) cannot be unique. Denoting by $q_i = \mathbb{Q}(\omega_i) > 0$ for $i = 1, ..., 3$, we have that

$$\frac{1}{1.04}\left(9q_1 + 10q_2 + 8\underbrace{(1 - q_1 - q_2)}_{q_3}\right) = 8.75$$

$$q_1 + 2q_2 + 8 = 9.1$$

$$2q_2 = 1.1 - q_1$$

$$q_2 = 0.55 - 0.5q_1$$

that gives

$$\begin{cases} q_1 \in \left(0; \frac{5}{8}\right) = (0; 0.9) \\ q_2 = 0.55 - 0.5q_1 \\ q_3 = 1 - q_1 - (0.55 - 0.5q_1) = 0.45 - 0.5q_1 \end{cases}$$

the state price vectors are

$$\psi_i = \frac{\mathbb{Q}(\omega_i)}{1 + r}$$

so that

$$\begin{cases} \psi_1 \in \left(\frac{0}{1.04}; 0.9 \cdot \frac{1}{1.04}\right) = (0; 0.865\,38) \\ \psi_2 = 0.55 \cdot \frac{1}{1.04} - 0.5\psi_1 = 0.528\,85 - 0.5\psi_1 \\ \psi_3 = 0.45 \cdot \frac{1}{1.04} - 0.5\psi_1 = 0.432\,69 - 0.5\psi_1 \end{cases}$$

3. The final payoff of a *call* option on S_1 with strike $K = 9.22$ and maturity $T = 1$ is

$$c(1)(\omega_1) = (9 - 9.22)^+ = 0$$

$$c(1)(\omega_2) = (10 - 9.22)^+ = 0.78$$

$$c(1)(\omega_3) = (8 - 9.22)^+ = 0$$

and cannot be replicated because

$$\det\begin{bmatrix} 1.04 & 9 & 0 \\ 1.04 & 10 & 0.78 \\ 1.04 & 8 & 0 \end{bmatrix} = 0.811\,2 \neq 0$$

$$rank\begin{bmatrix} 1.04 & 9 & 0 \\ 1.04 & 10 & 0.78 \\ 1.04 & 8 & 0 \end{bmatrix} = 3,$$

i.e. the call payoff is linearly independent from the terminal prices of B and S_1.

Hence there is an interval of no-arbitrage prices at $t = 0$ for the security *call* option on S_1. To retrieve such interval we compute

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[c\left(1\right)\right] = \frac{0 \cdot q_1 + 0.78 \cdot q_2 + 0 \cdot q_3}{1.04} = \frac{0.78}{1.04} \cdot (0.55 - 0.5q_1) = 0.412\,5 - 0.375\,q_1$$

therefore

$$\inf_{\mathbb{Q}} \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[c\left(1\right)\right] = 0.412\,5 - 0.375 \cdot 0.9 = 0.075$$

and

$$\sup_{\mathbb{Q}} \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[c\left(1\right)\right] = 0.412\,5 - 0.375 \cdot 0 = 0.412\,5$$

Hence the no-arbitrage interval for the call option is $(0.075; 0.4125)$

4. The strategy $\vartheta^{INF} = \left(\vartheta_0^{INF}, \vartheta_1^{INF}\right)$ that minimizes in the *original* market the initial cost of superreplicating the payoff of the option introduced in point 3 at time $T = 1$ has initial cost $V_{\vartheta^{INF}}(0) = 0.4125$.

   The strategy ϑ^{INF} belongs to the set of super-replicating strategies, whose final payoff $V_{\vartheta}(1) \geq c(1)$. More precisely, the set of superreplicating strategies ϑ is defined by the system of linear inequalities:

$$\begin{cases} \vartheta_0 \cdot 1.04 + \vartheta_1 \cdot 9 \geq 0 \\ \vartheta_0 \cdot 1.04 + \vartheta_1 \cdot 10 \geq 0.78 \\ \vartheta_0 \cdot 1.04 + \vartheta_1 \cdot 8 \geq 0 \end{cases}$$

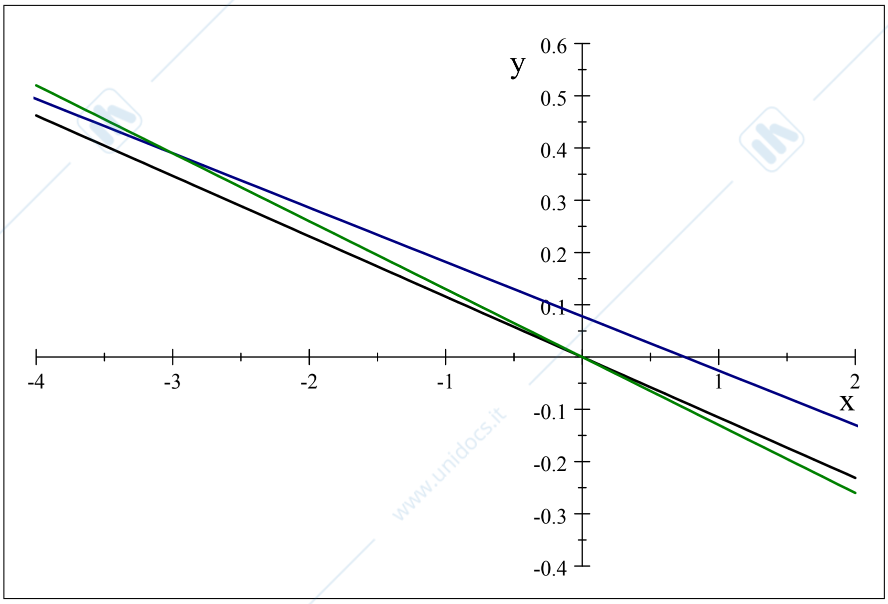
   or, equivalently,

$$\begin{cases} \vartheta_1 \geq -\frac{1.04}{9}\vartheta_0 = -0.115\,56\vartheta_0 \\ \vartheta_1 \geq -\frac{1.04}{10}\vartheta_0 + 0.078 = -0.104\vartheta_0 + 0.078 \\ \vartheta_1 \geq -\frac{1.04}{8}\vartheta_0 = -0.13\vartheta_0 \end{cases}$$

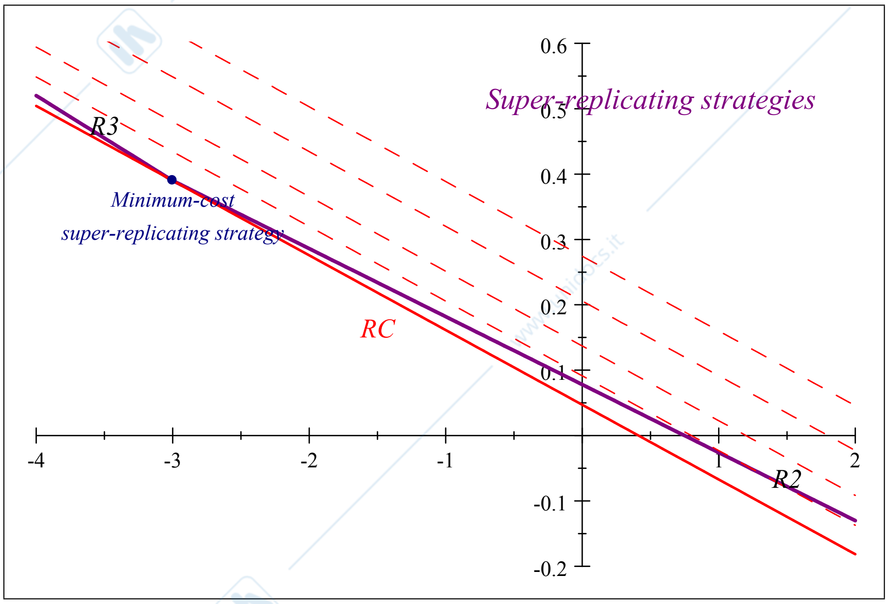
   Denoting by $R1$, $R2$ and $R3$ the three lines defining the boundaries of the system of linear inequalities,

$$R1: \vartheta_1 = -0.115\,56\vartheta_0 \quad \text{(black)}$$
$$R2: \vartheta_1 = -0.104\vartheta_0 + 0.078 \quad \text{(blue)}$$
$$R3: \vartheta_1 = -0.13\vartheta_0 \quad \text{(green)},$$

   their graph in the $(x, y) = (\vartheta_0, \vartheta_1)$ plane is



Thus, the super-replicating strategies $(\vartheta_0, \vartheta_1)$ in the $(x, y) = (\vartheta_0, \vartheta_1)$ plane define the region above the maximum between the two lines $R2$ (blue) *and* $R3$ (green). Such maximum has the purple graph in the plot below.



The strategy whose initial cost is c lies on the line

$$\vartheta_0 + \vartheta_1 \cdot 8.75 = c,$$

defining the equation of RC

$$RC : \vartheta_1 = -0.114\,29 \cdot \vartheta_0 + -\frac{c}{8.75}$$

As c varies, the graph of RC defines the set of red-dashed parallel lines in the above plot. The strategy in the super-replicating region with minimum c is therefore $\left(\vartheta_0^{INF}, \vartheta_1^{INF}\right) = R2 \cap R3$. Hence

$$\begin{cases} \vartheta_1 = -0.104\vartheta_0 + 0.078 \\ \vartheta_1 = -0.13\vartheta_0 \end{cases}$$

delivers

$$\vartheta_1^{INF} = 0.39 \text{ and } \vartheta_0^{INF} = -3$$

whose initial cost is $V_{\vartheta^{INF}}(0) = \vartheta_0^{INF} + \vartheta_1^{INF} \cdot 8.75 = -3 + 0.39 \cdot 8.75 = 0.412\,5$, as we know.

5. A strategy $\vartheta = (\vartheta_0, \vartheta_1)$ in the *original* market has a terminal value that replicates the final payoff of the call option $c(1)$ *on average with respect to the historical probability measure* $\mathbb{P}$ if

$$\mathbb{E}^{\mathbb{P}}\left[V_{\vartheta}(1)\right] = \mathbb{E}^{\mathbb{P}}\left[c(1)\right].$$

Since $\mathbb{P}(\omega_1) = 0.4$, $\mathbb{P}(\omega_2) = 0.5$, and $\mathbb{P}(\omega_3) = 0.1$, we have that

$$\mathbb{E}^{\mathbb{P}}\left[S_1(1)\right] = 9 \cdot 0.4 + 10 \cdot 0.5 + 8 \cdot 0.1 = 9.4$$

and

$$\mathbb{E}^{\mathbb{P}}\left[c(1)\right] = 0 \cdot 0.4 + 0.78 \cdot 0.5 + 0 \cdot 0.1 = 0.39$$

These computations yield

$$\mathbb{E}^{\mathbb{P}}\left[V_{\vartheta}(1)\right] = \vartheta_0\left(1 + r\right) + \vartheta_1 \mathbb{E}^{\mathbb{P}}\left[S_1(1)\right] = 1.04 \cdot \vartheta_0 + 9.4 \cdot \vartheta_1 = \mathbb{E}^{\mathbb{P}}\left[c(1)\right] = 0.39$$

and therefore

$$\begin{aligned} 1.04 \cdot \vartheta_0 + 9.4 \cdot \vartheta_1 &= 0.39 \\ 1.04 \cdot \vartheta_0 &= 0.39 - 9.4 \cdot \vartheta_1 \end{aligned}$$

The *mean-variance-hedging strategy* for the call option $\vartheta^{MV} = \left(\vartheta_0^{MV}, \vartheta_1^{MV}\right)$ replicates the final payoff of the call option $c(1)$ *on average with respect to the historical probability measure* $\mathbb{P}$, and hence

$$1.04 \cdot \vartheta_0^{MV} = 0.39 - 9.4 \cdot \vartheta_1^{MV}.$$

Its terminal value $V_{\vartheta^{MV}}(1)$ is therefore

$$V_{\vartheta^{MV}}(1) = \vartheta_0^{MV}(1 + r) + \vartheta_1^{MV} S_1(1) = 0.39 - 9.4 \cdot \vartheta_1^{MV} + \vartheta_1^{MV} S_1(1)$$

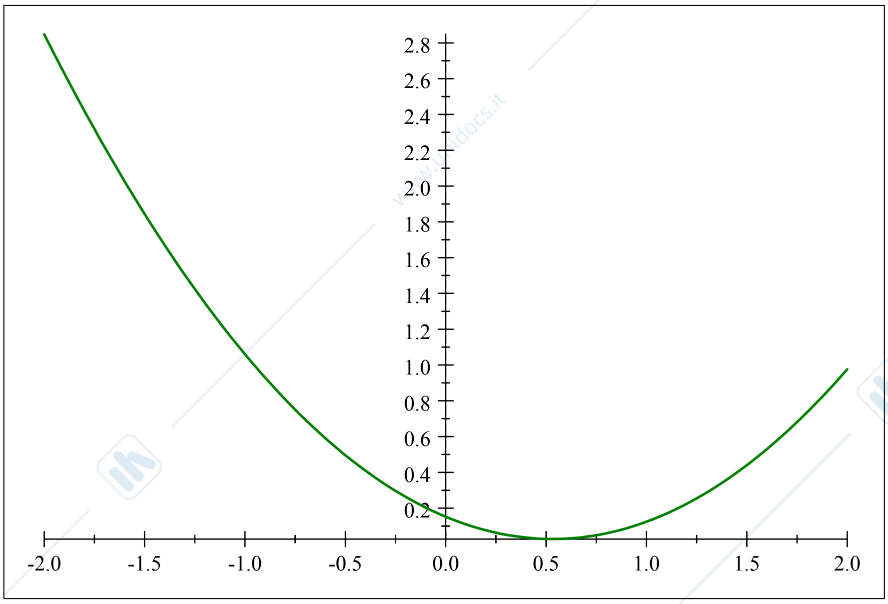
leading to

$$\begin{aligned} V_{\vartheta^{MV}}(1)(\omega_1) &= 0.39 - 9.4 \cdot \vartheta_1^{MV} + \vartheta_1^{MV} \cdot 9 = 0.39 - 0.4\vartheta_1^{MV} \\ V_{\vartheta^{MV}}(1)(\omega_2) &= 0.39 - 9.4 \cdot \vartheta_1^{MV} + \vartheta_1^{MV} \cdot 10 = 0.6\vartheta_1^{MV} + 0.39 \\ V_{\vartheta^{MV}}(1)(\omega_3) &= 0.39 - 9.4 \cdot \vartheta_1^{MV} + \vartheta_1^{MV} \cdot 8 = 0.39 - 1.4\vartheta_1^{MV}. \end{aligned}$$

The quadratic error of replication with respect to the hystorical probability measure $\mathbb{P}$ is.

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left(V_{\vartheta^{MV}}(1)-c(1)\right)^2\right] &= \left(0.39-0.4\vartheta_1^{MV}-0\right)^2\cdot 0.4+\left(0.6\vartheta_1^{MV}+0.39-0.78\right)^2\cdot 0.5+\\ &\quad+\left(0.39-1.4\vartheta_1^{MV}-0\right)^2\cdot 0.1\\ &= 0.44\left(\vartheta_1^{MV}\right)^2-0.468\,\vartheta_1^{MV}+0.152\,1.\end{aligned}$$

The quadratic expression in ϑ_1^{MV} is a parabola



whose minimum is reached at

$$\vartheta_1^{MV}=-\frac{-0.468}{2\cdot 0.44}=0.531\,82.$$

Therefore, the *mean-variance-hedging strategy* for the call option is ϑ^{MV} with

$$\begin{aligned}\vartheta_0^{MV} &= 0.375\,-9.038\,5\cdot 0.531\,82=-4.431\,9\\ \vartheta_1^{MV} &= 0.531\,82\end{aligned}$$

Its terminal values are

$$\begin{gathered}V_{\vartheta^{MV}}(1)(\omega_1)=0.39-0.4\cdot 0.531\,82=0.177\,27>0=c(1)(\omega_1)\\ V_{\vartheta^{MV}}(1)(\omega_2)=0.6\cdot 0.531\,82+0.39=0.709\,09<0.78=c(1)(\omega_2)\\ V_{\vartheta^{MV}}(1)(\omega_3)=0.39-1.4\cdot 0.531\,82=-0.354\,55<0=c(1)(\omega_3)\end{gathered}$$

and its initial cost is

$$V_{\vartheta^{MV}}(0)=-4.431\,9+0.531\,82\cdot 8.75=0.221\,53$$

6. The strategy ϑ^{MV}, whose initial cost is $0.221\,53$ does not super-replicate the option $c\,(1)$. On the contrary, the more expensive strategy ϑ^{INF} replicates the option $c\,(1)$ in ω_2 and ω_3 (because $\vartheta^{INF} = R2 \cap R3$), and strictly super-replicates $c\,(1)$ in ω_1, where

$$\begin{aligned} V_{\vartheta^{INF}}\,(1)\,(\omega_1) &= \vartheta_0^{INF}\,(1+r) + \vartheta_1^{INF} S_1(1)(\omega_1) \\ &= -3 \cdot 1.04 + 0.39 \cdot 9 \\ &= 0.39 > 0 = c\,(1)\,(\omega_1) \end{aligned}$$

Solution of EXERCISE 2

1. The *historical probability* $\mathbb{P}$ that a European *put* option on S with strike price $K = 0.5$ closes at maturity $T = 1$ *out of the money* is

$$\begin{aligned} \mathbb{P}\,[S(1) \geq K] &= \mathbb{P}\left[e^{\left(\mu - \frac{\sigma^2}{2}\right)\cdot 1 + \sigma W_1} \geq \frac{K}{S(0)}\right] = \\ &= \mathbb{P}\left[\left(\mu - \frac{\sigma^2}{2}\right)\cdot 1 + \sigma W_1 \geq \ln\frac{K}{S(0)}\right] = \\ &= \mathbb{P}\left[Z \geq \frac{1}{\sigma\sqrt{1}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 1\right)\right] = \\ &= N\left(-\frac{1}{\sigma\sqrt{1}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot 1\right)\right) \\ &= N\left(-\frac{1}{0.10\sqrt{1}}\left(\ln\frac{0.5}{1} - \left(0.07 - \frac{0.10^2}{2}\right)\cdot 1\right)\right) = \\ &= N\,(7.\,581\,5) \approx 1 \end{aligned}$$

2. To compute the stochastic differential of

$$Y\,(t) = e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot (S\,(t))^{\alpha}\,,$$

with $\alpha > 0$, with respect to the *risk-neutral probability* $\mathbb{Q}$, we consider the function

$$f\,(t, S) = e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot (S)^{\alpha}$$

and its derivatives

$\frac{\partial f(t,S)}{\partial t} = -\frac{1}{2}\sigma^2\alpha^2 \cdot e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot (S\,(t))^{\alpha}\,;$ $\quad \frac{\partial f(t,S)}{\partial S} = e^{-\frac{1}{2}\sigma^2\alpha^2 t}\alpha S^{\alpha-1};$ $\quad \frac{\partial^2 f(t,S)}{\partial S^2} = e^{-\frac{1}{2}\sigma^2\alpha^2 t}\alpha\,(\alpha-1)\,S^{\alpha-}$

Applying Ito formula we get

$$\begin{aligned} dY\,(t) &= -\frac{1}{2}\sigma^2\alpha^2 \cdot e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot (S\,(t))^{\alpha} \cdot dt + \alpha e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot S\,(t)^{\alpha-1} \cdot dS\,(t) + \\ &\quad + \frac{1}{2} \cdot \alpha\,(\alpha-1)\,e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot S\,(t)^{\alpha-2} \cdot S^2\,(t)\,\sigma^2 dt \\ &= e^{-\frac{1}{2}\sigma^2\alpha^2 t} \cdot (S\,(t))^{\alpha}\left(-\frac{1}{2}\sigma^2\alpha^2 dt + \delta\alpha dt + \sigma\alpha dW^*\,(t) + \frac{1}{2}\cdot\alpha\,(\alpha-1)\,\sigma^2 dt\right) \\ &= Y\,(t)\left(\left(\delta\alpha - \frac{1}{2}\cdot\alpha\sigma^2\right)dt + \sigma\alpha dW^*\,(t)\right) \end{aligned}$$

Hence $Y(t)$ is lognormal with respect to the risk-neutral measure and its risk-neutral drift.is

$$\begin{aligned}\mu_Y^{\mathbb{Q}} &= \alpha\left(\delta-\frac{1}{2}\cdot\sigma^2\right)\\ &= \alpha\left(0.02-\frac{1}{2}\cdot 0.1^2\right)\\ &= 0.015\,\alpha\end{aligned}$$

3. The initial no-arbitrage price of the European derivative on S whose payoff at maturity $T=1$ is

$$Y(1)=e^{-\frac{1}{2}\sigma^2\alpha^2}(S(1))^{\alpha}$$

is given by

$$\begin{aligned}S_Y(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\delta}\cdot Y(1)\right]\\ &= e^{-\delta}\cdot Y(0)\,e^{\mu_Y^{\mathbb{Q}}\cdot 1} \quad \text{since } Y \text{ is lognormal}\\ &= e^{-0.02}\cdot e^{(0.015\,\alpha)}\\ &= e^{-0.02+0.015\,\alpha}\end{aligned}$$

Since

$$S_Y(0)=1.5,$$

we obtain

$$\begin{aligned}e^{-0.02+0.015\,\alpha} &= 1.5\\ -0.02+0.015\,\alpha &= \ln 1.5\\ \alpha &= \frac{\ln 1.5+0.02}{0.015}=28.364\end{aligned}$$

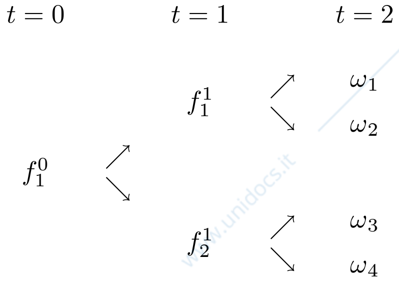
Quantitative Finance and Derivatives I
Finanza Quantitativa e Derivati I

code 20188
a.y. 2012/13, September 2013

POINTS WILL BE AWARDED ONLY IF THE ANSWER IS SUPPORTED BY A DETAILED LOGICAL JUSTIFICATION

EXERCISE 1 (35 points out of 100).

Consider a multiperiod discrete market with $t = 0, 1, 2$ with the following information structure:



Two securities are traded in the market. The first is a *locally risk-free asset* B that provides the locally riskless interest rate

$$\begin{aligned} r(0) &= 1\% \\ r(1)(f_1^1) &= 1\% \\ r(1)(f_2^1) &= 0\%. \end{aligned}$$

The second security is a *risky asset* S, with time 0 price

$$S(0) = 100,$$

with time 1 prices

$$S(1)(f_1^1) = 105$$
$$S(1)(f_2^1) = 95$$

and with time 2 prices

$$S(2)(\omega_1) = 110.25$$
$$S(2)(\omega_2) = 99.75$$
$$S(2)(\omega_3) = 104.5$$
$$S(2)(\omega_4) = 85.5$$

1. **(5 points)** Is the market dynamically complete?
2. **(10 points)** Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
3. **(5 points)** Consider the *European derivative* X with maturity $T = 2$, whose terminal payoff at $T = 2$ is

$$X(2) = \begin{cases} S(2) & \text{if } S(2) < 100 \\ 100 & \text{otherwise} \end{cases}$$

Compute the no-arbitrage prices at $t = 0$ and at $t = 1$ of the derivative X.

4. **(4 points)** Consider the derivative Y whose *cashflow* is
$$Y(t) = \begin{cases} S(t) & \text{if } S(t) < 100 \\ 100 & \text{otherwise} \end{cases}$$
for $t = 1$ *and* $T = 2$. Compute the no-arbitrage prices at $t = 0$ and at $t = 1$ of the derivative Y.

5. (**5 points**) Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *zero-coupon bond* with maturity $T = 2$, whose terminal payoff is 1 in any state of the world.

6. **(6 points)** Determine a *buy-and-hold strategy* ϑ that replicates a *European put option on* S with maturity $T = 2$ and strike 100 using only **two** financial securities in the market extended to all the derivatives of previous points, that is in the market constituted by B, S, the derivatives X of point 3.,Y of point 4., and the zero coupon bond of point 5.

EXERCISE 2 (40 points out of 100).
Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 1$, $\delta = 5\%$, $\mu = 8\%$, $\sigma = 16\%$, and $T = 4$.

1. **(5 points)** Determine the *historical probability* $\mathbb{P}$ that a European *call* option on S with strike price $K = 1$ closes *in the money* at the maturity T (Here *you only need to express the probability in terms of the distribution function* $N(\cdot)$ *of a standard Normal random variable*).

2. **(5 points)** Find the strike price K' such that the *risk neutral probability* $\mathbb{Q}$ that a European *call* option on S with strike price K' closes at the maturity T *in the money* is equal to the probability computed at the previous point.

3. **(12 points)** Let W^* be the standard Brownian motion under the *risk neutral probability* $\mathbb{Q}$. Compute
$$\mathbb{E}^{\mathbb{Q}}\left[\left(W^*(T) - W^*(t)\right)^2 \middle| \mathcal{F}_t\right],$$
$$\mathbb{E}^{\mathbb{Q}}\left[\left(W^*(t)\right)^2 \middle| \mathcal{F}_t\right],$$
$$\mathbb{E}^{\mathbb{Q}}\left[\left(W^*(T) - W^*(t)\right) \cdot W^*(t) \middle| \mathcal{F}_t\right],$$
and
$$\mathbb{E}^{\mathbb{Q}}\left[\left(W^*(T)\right)^2 \middle| \mathcal{F}_t\right]$$
for $t \in (0, T)$ (explain carefully the steps of your computations).

4. **(8 points)** Consider the European derivative whose final payoff is
$$Y = (W(T))^2$$
where W is the standard Brownian motion under the *historical probability* $\mathbb{P}$. Determine its no arbitrage price for any $t \in [0, T]$.

5. **(10 points)** Consider the European derivative whose terminal payoff X at maturity T is given by
$$X = (W(T) - E)^2,$$
where W is the standard Brownian motion under the *historical probability* $\mathbb{P}$, and E is fixed in such a way that $S_X(0)$, the initial no-arbitrage price of the derivative X, is equal to $5 \cdot E$. Find E such that $S_X(0) < 10$.

QUESTION (25 points out of 100).

In the Black and Scholes market model state the Partial Differential Equation for a European derivative whose terminal payoff is a deterministic function of the terminal underlying value.

Write the terminal condition for a digital European option that pays at maturity 1 if $S(T) > 100$, and 0 otherwise.

SOLUTIONS TO EXERCISES

Exercise 1

1. The prices of security B are

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.01$$

and at the final date $T = 2$

$$\begin{aligned} B(2)(\omega_1) = B(2)(\omega_2) = 1.01 \cdot 1.01 = 1.020\,1 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.01. \end{aligned}$$

The market is dynamically complete, because each one-period submarket is complete (in your exam check explicitly that the rank of the *terminal* payoff matrix of each one-period submarket has rank 2).

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases} \tag{1}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases} \tag{2}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases} \tag{3}$$

System (1) can be rewritten as

$$\begin{cases} 100 = \frac{1}{1.01} \left\{ 105 \cdot \mathbb{Q}[f_1^1] + 95 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.6 \\ \mathbb{Q}[f_2^1] &= 0.4 \end{aligned}$$

System (2) can be rewritten as

$$\begin{cases} 105 = \frac{1}{1.01} \left\{ 110.25 \cdot \mathbb{Q}[\omega_1|f_1^1] + 99.75 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.6 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.4 \end{aligned}$$

and System (3) can be rewritten as

$$\begin{cases} 95 = \frac{1}{1+r(1)(f_2^1)}\left\{104.5 \cdot \mathbb{Q}[\omega_3|f_2^1] + 85.5 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.5 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.5 \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.6 \cdot 0.6 = 0.36 \\ \mathbb{Q}[\omega_2] &= 0.6 \cdot 0.4 = 0.24 \\ \mathbb{Q}[\omega_3] &= 0.4 \cdot 0.5 = 0.2 \\ \mathbb{Q}[\omega_4] &= 0.4 \cdot 0.5 = 0.2 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

3. The terminal payoff

$$X(2) = \begin{cases} S(2) \text{ if } S(2) < 100 \\ 100 \text{ otherwise} \end{cases}$$

is equal to

$$\begin{gathered} X(2)(\omega_1) = 100 \\ X(2)(\omega_2) = 99.75 \\ X(2)(\omega_3) = 100 \\ X(2)(\omega_4) = 85.5 \end{gathered}$$

The no-arbitrage prices at $t = 1$ of this derivative are

$$\begin{aligned} S_X(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_1^1) \\ &= \frac{100 \cdot 0.6 + 99.75 \cdot 0.4}{1.01} = 98.911 \\ S_X(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_2^1) \\ &= \frac{100 \cdot 0.5 + 85.5 \cdot 0.5}{1.0} = 92.75 \end{aligned}$$

$$\begin{aligned} S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1)}{1+r(0)}\right] \\ &= \frac{98.911 \cdot 0.6 + 92.75 \cdot 0.4}{1.01} = 95.492. \end{aligned}$$

4. The terminal payoff of the derivative Y coincides with the terminal payoff of the derivative X. Therefore from $Y(2) = X(2)$ we get that their no-arbitrage prices at $t = 1$ coincide too, namely

$$\begin{aligned} S_Y(1)(f_1^1) &= S_X(1)(f_1^1) = 98.911 \\ S_Y(1)(f_2^1) &= S_X(1)(f_2^1) = 92.75. \end{aligned}$$

At $t=1$ the cashflow of the derivative Y is

$$\begin{array}{c} Y(1)(f_1^1)=100 \\ Y(1)(f_2^1)=95. \end{array}$$

The initial no-arbitrage price of the derivative Y is given by

$$\begin{aligned} S_Y(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{Y(1)+S_Y(1)}{1+r(0)}\right] \\ &= \frac{(100+98.911)\cdot 0.6+(95+92.75)\cdot 0.4}{1.01}=192.52. \end{aligned}$$

5. The no-arbitrage price at $t=0$ and at $t=1$ of a *zero-coupon bond* with maturity $T=2$, whose terminal payoff is 1 in any state of the world is at date $t=1$

$$\begin{aligned} ZCB(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{1+r(1)}\right|\mathcal{P}_1\right](f_1^1) \\ &= \frac{1}{1.01}=0.990\,10 \\ ZCB(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{1+r(1)}\right|\mathcal{P}_1\right](f_2^1) \\ &= \frac{1}{1.0}=1 \end{aligned}$$

At $t=0$ the price is

$$ZCB(0)=\mathbb{E}^{\mathbb{Q}}\left[\frac{ZCB(1)}{1+r(0)}\right]=\frac{0.990\,10\cdot 0.6+1\cdot 0.4}{1.01}=0.984\,22.$$

6. The terminal payoff of a *European put option on* S with maturity $T=2$ and strike 100 is

$$put\,(2)=(100-S(2))^{+}=\begin{cases} 100-S(2) & \text{if } S(2)<100 \\ 0 \text{ otherwise} \end{cases}$$

This payoff is equal to the constant amount 100 plus the final payoff of a short position on X, because

$$-X(2)=\begin{cases} -S(2) & \text{if } S(2)<100 \\ -100 \text{ otherwise} \end{cases}$$

and

$$100-X(2)=\begin{cases} 100-S(2) & \text{if } S(2)<100 \\ 100-100=0 \text{ otherwise} \end{cases}$$

In the extended market this final payoff $100-X(2)$ is obtained by buying at the initial date 100 units of the zero coupon bond of point 5. and by selling 1 unit of the derivative X. More formally consider the *buy-and-hold strategy* in the extended market

$$\begin{aligned} \vartheta_0(t) &= \vartheta_0=0 \quad \text{units of } B \\ \vartheta_1(t) &= \vartheta_1=0 \quad \text{units of } S \\ \vartheta_X(t) &= \vartheta_X=-1 \quad \text{units of } X \\ \vartheta_Y(t) &= \vartheta_Y=0 \quad \text{units of } Y \\ \vartheta_{ZCB}(t) &= \vartheta_{ZCB}=100 \quad \text{units of } ZCB \end{aligned}$$

for $t=0,1$. Then

$$\begin{aligned} C_\vartheta(2) &= V_\vartheta(2)=-1\cdot X(2)+100\cdot ZCB(2)=-1\cdot X(2)+100\cdot 1 \\ &= \begin{cases} 100-S(2) & \text{if } S(2)<100 \\ 100-100=0 \text{ otherwise} \end{cases} =(100-S(2))^{+}=put\,(2). \end{aligned}$$

At $t = 1$ the cashflow of the strategy $C_\vartheta(1) = 0$, because ϑ is buy-and-hold. Therefore, the cashflow process of ϑ coincides with the cashflow process of the European put option: hence ϑ replicates the put option.

Exercise 2.

1. The *historical probability* $\mathbb{P}$ that a European *call* option on S with strike price $K = 1$ closes at maturity $T = 4$ *in the money is*

$$\begin{aligned}
\mathbb{P}[S(T) > K] &= \mathbb{P}\left[e^{\left(\mu - \frac{\sigma^2}{2}\right)\cdot T + \sigma W_T} > \frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[\left(\mu - \frac{\sigma^2}{2}\right)\cdot T + \sigma W_T > \ln\frac{K}{S(0)}\right] = \\
&= \mathbb{P}\left[Z > \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot T\right)\right] = \\
&= \mathbb{P}\left[Z < -\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot T\right)\right] = \\
&= N\left(-\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot T\right)\right) = \qquad (4) \\
&= N\left(-\frac{1}{0.16\sqrt{4}}\left(\ln\frac{1}{1} - \left(0.08 - \frac{0.16^2}{2}\right)\cdot 4\right)\right) = \\
&= N(0.84) = 0.799\,55
\end{aligned}$$

where Z denotes a standard normal random variable with respect to the *historical probability* $\mathbb{P}$.

2. The *risk neutral probability* $\mathbb{Q}$ that a European *call* option on S with strike price K' closes at the maturity $T = 4$ *in the money* is

$$\begin{aligned}
\mathbb{Q}[S(T) > K'] &= \mathbb{Q}\left[e^{\left(\delta - \frac{\sigma^2}{2}\right)\cdot T + \sigma W_T^*} > \frac{K'}{S(0)}\right] = \\
&= \mathbb{Q}\left[\left(\delta - \frac{\sigma^2}{2}\right)\cdot T + \sigma W_T^* > \ln\frac{K'}{S(0)}\right] = \\
&= \mathbb{Q}\left[Z' > \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K'}{S(0)} - \left(\delta - \frac{\sigma^2}{2}\right)\cdot T\right)\right] = \\
&= \mathbb{Q}\left[Z' < -\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K'}{S(0)} - \left(\delta - \frac{\sigma^2}{2}\right)\cdot T\right)\right] = \qquad (5) \\
&= N\left(-\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K'}{S(0)} - \left(\delta - \frac{\sigma^2}{2}\right)\cdot T\right)\right)
\end{aligned}$$

where Z' is a standard normal random variable with respect to the *risk neutral probability* $\mathbb{Q}$. This probability is equal to the one computed at the previous point if the arguments of the normal cumulative distribution function in (4) and (5) coincide, that is if

$$-\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K'}{S(0)} - \left(\delta - \frac{\sigma^2}{2}\right)\cdot T\right) = -\frac{1}{\sigma\sqrt{T}}\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)\cdot T\right)$$

that is

$$\begin{gathered}
\ln\frac{K'}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)\cdot T=\ln\frac{K}{S(0)}-\left(\mu-\frac{\sigma^2}{2}\right)\cdot T\\
\ln K'-\ln S(0)=\left(\delta-\frac{\sigma^2}{2}\right)\cdot T+\ln K-\ln S(0)-\left(\mu-\frac{\sigma^2}{2}\right)\cdot T\\
\ln K'=\delta\cdot T+\ln K-\mu\cdot T\\
\ln K'=(\delta-\mu)\cdot 4+\ln 1=-0.03\cdot 4=-0.12\\
K'=\exp(-0.12)=0.886\,92
\end{gathered}$$

3. Let W^* be a standard Brownian motion under the *risk neutral probability* $\mathbb{Q}$. The required conditional expectations are

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\left.(W^*(T)-W^*(t))^2\right|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[(W^*(T)-W^*(t))^2\right] \quad \text{because } W^*(T)-W^*(t) \text{ is independent of}\\
&= T-t \quad \text{because } W^*(T)-W^*(t)\overset{\mathbb{Q}}{\sim}\mathcal{N}(0,T-t)
\end{aligned}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\left.(W^*(t))^2\right|\mathcal{F}_t\right]=(W^*(t))^2 \quad \text{because } W^*(t) \text{ is measurable w.r.t. } \mathcal{F}_t$$

$$\begin{gathered}
\mathbb{E}^{\mathbb{Q}}\left[(W^*(T)-W^*(t))\cdot W^*(t)|\,\mathcal{F}_t\right]=W^*(t)\cdot\mathbb{E}^{\mathbb{Q}}\left[(W^*(T)-W^*(t))|\,\mathcal{F}_t\right] \quad \text{because } W^*(t) \text{ is } \mathcal{F}_t\text{-meas.}\\
=W^*(t)\cdot\mathbb{E}^{\mathbb{Q}}\left[(W^*(T)-W^*(t))\right] \quad \text{because } W^*(T)-W^*(t) \text{ is } \mathcal{F}_t\text{-indep.}\\
=W^*(t)\cdot 0=0 \quad \text{because } W^*(T)-W^*(t)\overset{\mathbb{Q}}{\sim}\mathcal{N}(0,T-t)
\end{gathered}$$

and

$$\begin{gathered}
\mathbb{E}^{\mathbb{Q}}\left[\left.(W^*(T))^2\right|\mathcal{F}_t\right]=\mathbb{E}^{\mathbb{Q}}\left[\left.((W^*(T)-W^*(t))+W^*(t))^2\right|\mathcal{F}_t\right]=\\
=\mathbb{E}^{\mathbb{Q}}\left[\left.(W^*(T)-W^*(t))^2\right|\mathcal{F}_t\right]+\mathbb{E}^{\mathbb{Q}}\left[\left.(W^*(t))^2\right|\mathcal{F}_t\right]+2\mathbb{E}^{\mathbb{Q}}\left[((W^*(T)-W^*(t))\cdot W^*(t))|\,\mathcal{F}_t\right]=\\
=T-t+(W^*(t))^2+2\cdot 0 \quad \text{because of the previous computations.}
\end{gathered}$$

4. The no-arbtrage price of the European derivative whose terminal payoff is

$$Y=(W(T))^2,$$

where W is the standard Brownian motion under the *historical probability* $\mathbb{P}$, is given by

$$\begin{aligned}
S_Y(t) &= e^{-\delta(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left.(W(T))^2\right|\mathcal{F}_t\right]=e^{-\delta(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left.\left(W^*(T)-\frac{\mu-\delta}{\sigma}T\right)^2\right|\mathcal{F}_t\right]=\\
&= e^{-\delta(T-t)}\left\{\mathbb{E}^{\mathbb{Q}}\left[\left.(W^*(T))^2\right|\mathcal{F}_t\right]+\mathbb{E}^{\mathbb{Q}}\left[\left.\left(\frac{\mu-\delta}{\sigma}T\right)^2\right|\mathcal{F}_t\right]-2\frac{\mu-\delta}{\sigma}T\mathbb{E}^{\mathbb{Q}}\left[(W^*(T))|\,\mathcal{F}_t\right]\right\}=\\
&= e^{-\delta(T-t)}\left\{T-t+(W^*(t))^2+\left(\frac{\mu-\delta}{\sigma}T\right)^2-2\frac{\mu-\delta}{\sigma}TW^*(t)\right\},
\end{aligned}$$

because of the computations of the previous point, and since $\mathbb{E}^{\mathbb{Q}}\left[(W^*(T))|\,\mathcal{F}_t\right]=W^*(t)$, because W^* is a $\mathbb{Q}-$martingale. Therefore

$$\begin{aligned}
S_Y(t) &= e^{-\delta(T-t)}\left\{4-t+(W^*(t))^2+\left(\frac{0.08-0.05}{0.16}4\right)^2-2\frac{0.08-0.05}{0.16}4W^*(t)\right\}=\\
&= e^{-0.05(4-t)}\left\{4-t+(W^*(t))^2+0.562\,5-1.5W^*(t)\right\}.
\end{aligned}$$

5. The initial no-arbitrage price of the European derivative whose payoff X at maturity $T = 4$ is

$$X = (W(T) - E)^2,$$

where W is a standard Brownian motion under the *historical probability,* is given by

$$S_X(0) = e^{-\delta T}\ \mathbb{E}^{\mathbb{Q}}\left[(W(T) - E)^2\right] = e^{-\delta T}\ \mathbb{E}^{\mathbb{Q}}\left[\left(W^*(T) - \frac{\mu - \delta}{\sigma}T - E\right)^2\right],$$

where W^* is a standard Brownian motion under the risk neutral probability measure. Then

$$\begin{aligned}
S_X(0) &= e^{-\delta T}\ \mathbb{E}^{\mathbb{Q}}\left[\left(W^*(T) - \frac{\mu - \delta}{\sigma}T - E\right)^2\right] \\
&= e^{-\delta T}\mathbb{E}^{\mathbb{Q}}\left[(W^*(T))^2 + \left(-\frac{\mu - \delta}{\sigma}T - E\right)^2 + 2W^*(T)\left(-\frac{\mu - \delta}{\sigma}T - E\right)\right] \\
&= e^{-\delta T}\left(T + \left(\frac{\mu - \delta}{\sigma}T + E\right)^2 + 0\right) \\
&= \exp(-0.05 \cdot 4) \cdot \left(4 + \left(\frac{0.08 - 0.05}{0.16}4 + E\right)^2\right) \\
&= \exp(-0.05 \cdot 4) \cdot \left(4 + E^2 + 0.5625 + 1.5E\right) \\
&= \exp(-0.05 \cdot 4) \cdot \left(E^2 + 1.5E + 4.5625\right)
\end{aligned}$$

Since E is fixed in such a way that the initial no-arbitrage price of the derivative X is equal to $5 \cdot E$, we have

$$\exp(-0.05 \cdot 4) \cdot \left(E^2 + 1.5E + 4.5625\right) = 5 \cdot E$$

that is

$$\begin{aligned}
E^2 + 1.5E + 4.5625 &= 5 \cdot E \exp(0.05 \cdot 4) \\
E^2 + 1.5E + 4.5625 &= 5 \cdot 1.2214 \cdot E \\
E^2 - 4.607E + 4.5625 &= 0
\end{aligned}$$

which is solved by $E = 1.4412$ and $E = 3.1658$.

For $E = 1.4412$ we get $S_X(0) = \exp(-0.05 \cdot 4) \cdot \left(1.4412^2 + 1.5 \cdot 1.4412 + 4.5625\right) = 7.206 = 5 \cdot 1.4412 < 10$.

For $E = 3.1658$ we get $S_X(0) = \exp(-0.05 \cdot 4) \cdot \left(3.1658^2 + 1.5 \cdot 3.1658 + 4.5625\right) = 15.829 = 5 \cdot 3.1658 > 10$.

Therefore the solution is $E = 1.4412$, because $E = 1.4412$ does meet the constraint $S_X(0) < 10$, whereas $E = 3.1658$ does not.