Quantitative Finance and Derivatives - MSc FiRM
19/12/2018

1 Consider two assets with prices X_t and Y_t, following two independent Bachelier models. The drifts are μ_X, μ_Y, the volatilities are σ_X, σ_Y. The initial values are X_0 and Y_0. Determine the distribution, mean and variance at time t for the following portfolios:
i) $X_t + 3Y_t$, **(3)**
ii) $2X_t - 3Y_t$. **(3)**

Sol. Note that the asset price models are

$$X_t = X_0 + \mu_X t + \sigma_X W_t, \qquad Y_t = Y_0 + \mu_Y t + \sigma_Y Z_t,$$

with W_t and Z_t two independent Brownian motions. We know that

$$X_t \sim \mathcal{N}(X_0 + \mu_X t; \sigma_X^2 t), \qquad Y_t \sim \mathcal{N}(Y_0 + \mu_Y t; \sigma_Y^2 t).$$

i) By the fact that the sum of independent Gaussian r.v. is Gaussian we have that the distribution is

$$\mathcal{N}(X_0 + 3Y_0 + (\mu_X + 3\mu_Y)\, t; (\sigma_X^2 + 9\sigma_Y^2)\, t).$$

ii) By the fact that the sum of independent Gaussian r.v. is Gaussian we have that the distribution is

$$\mathcal{N}(2X_0 - 3Y_0 + (2\mu_X - 3\mu_Y)\, t; (4\sigma_X^2 + 9\sigma_Y^2)\, t).$$

2 The underlying asset price follows a Binomial model: the initial price is $S_0 = 10$ euro, in the next two semesters the price can increase by 25% or decrease by 20% with (historical) probability equal to 50% in each semester. The risk free interest rate is 4% per year.
Consider a American Put option written on this underlying with strike $K = 12$ and maturity $T = 1$ year.
i) Determine if it is optimal to exercise the American Put option before the maturity. **(2)**
ii) Compute the (initial) option price. **(3)**
iii) If the (historical) probability of increasing (resp. decreasing) is equal to 80% (resp. 20%), which is the option price? Justify your answer. **(2)**

Sol. i)+ii) See exercise 8.2 in the Exercise book.
iii) The option price remains the same, because the option price is computed using the risk neutral probability, not the historical one.

3 Consider an option whose payoff at maturity T is S_T^4, where S_t is the underlying asset price which follows a Black-Scholes model, with drift μ, volatility σ. The risk free rate is r.
(i) Using the risk neutral valuation compute the option price at time t. **(4)**
(ii) Verify that the option price obtained in (i) satisfies the Black-Scholes pricing PDE. **(3)**

Sol. Under the Black-Scholes model the option payoff is

$$S_T^4 = S_t^4 e^{4(r-\frac{1}{2}\sigma^2)(T-t)+4\sigma(W_T-W_t)} = S_t^4 e^{(4r-2\sigma^2)(T-t)+4\sigma(W_T-W_t)}.$$

i) Using the risk neutral valuation, denote $f(t, S_t)$ the the option price at time t

$$f(t,S_t) = e^{-r(T-t)} E[S_t^4 e^{(4r-2\sigma^2)(T-t)+4\sigma(W_T-W_t)} | \mathcal{F}_t]$$

$$= e^{3r(T-t)} S_t^4 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(4\sigma^2(T-t) - 8\sigma y\sqrt{T-t} + y^2)} dy$$

$$= e^{(3r+6\sigma^2)(T-t)} S_t^4 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(16\sigma^2(T-t) - 8\sigma y\sqrt{T-t} + y^2)} dy = e^{(3r+6\sigma^2)(T-t)} S_t^4 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-4\sigma\sqrt{T-t})^2} dy.$$

With the change of variable $z = y - 4\sigma\sqrt{T-t}$, it is equal to

$$e^{(3r+6\sigma^2)(T-t)} S_t^4 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.$$

Finally this is

$$e^{(3r+6\sigma^2)(T-t)} S_t^4.$$

ii) Remember that the pricing PDE is

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r S_t + \frac{1}{2} \frac{\partial^2 f}{\partial S_t^2} \sigma^2 S_t^2 = rf.$$

So compute:

$$\frac{\partial f}{\partial t} = -(3r+6\sigma^2) e^{(3r+6\sigma^2)(T-t)} S_t^4, \quad \frac{\partial f}{\partial S_t} = 4S_t^3 e^{(3r+6\sigma^2)(T-t)}, \quad \frac{\partial^2 f}{\partial S_t^2} = 12 S_t^2 e^{(3r+6\sigma^2)(T-t)}.$$

Finally substitute these derivatives into the PDE .... and see that the price $f(t, S_T)$ obtained in i) satisfies also ii).

4 The underlying asset price follows a Black-Scholes model with initial price $S_0 = 10$ euro, volatility $\sigma = 20\%$ per year, risk free rate $r = 2\%$ per year. You sell an option that pays 100 if $S_1^2 \geq 100$ and 0 otherwise. (Note that $T = 1$ year). Show how to render Δ-neutral your position. **(5)**

Sol. Notice that $S_1^2 \geq 100$ iff $S_1 \geq 10$, therefore we have a cash-or-nothing option with strike 10, with the payoff

$$D_1 = \begin{cases} 100 & \text{if } S_1 \geq 10 \\ 0 & \text{otherwise.} \end{cases}$$

We compute then the initial price, which is

$$D_0 = 100\, e^{-rT} \mathcal{N}(d_2),$$

where $d_2 := \frac{\ln(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)}{\sigma T} = 0$.

Compute now the Δ as the partial derivative of D_0 with respect to S_0 and we have

$$\Delta = 100\, e^{-rT} \frac{n(d_2)}{\sigma S_0 \sqrt{T}} = 100\, e^{-0.02} \frac{1}{2} \frac{1}{\sqrt{2\pi}} = 19.55.$$

Finally, we have to buy 19.55 units of the underlying in order to render the position Δ-neutral.

5 Suppose that the spot interest rate process $(r_t)_{t \geq 0}$ follows the Vasicek model, with the following parameters under the historical probability: speed a, level $\bar{b}$ and volatility σ. Assume a constant market price of risk λ.

(i) Write the model of r_t under the (forward) risk neutral probability. **(2)**

(ii) Does the long run mean and variance are the same in the two models? Justify your answer and give their expressions. **(2)**

(iii) Suppose now that the parameters under the (forward) risk neutral probability are: the speed $a = 0.4$, the level $b = 0.01$ and the volatility $\sigma = 0.2$. The initial value is $r_0 = 4\%$ per year. Compute the price of a zero coupon bond with nominal value $N = 40$ Euro and maturity two years. (Recall the formulae $B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}$, $A(t,T) = [B(t,T) - (T-t)](b - \frac{\sigma^2}{2a^2}) - \frac{\sigma^2}{4a} B^2(t,T)$.) **(3)**

Sol. i) The spot interest rate process under the historical probability is

$$dr_t = a(\bar{b} - r_t)dt + \sigma dW_t.$$

From the PDE for the term structure of interest rates we know that if $\mu(t, r_t)$ is the drift under the historical probability, then the drift under the (forward) risk neutral probability is $\mu(t, r_t) - \lambda\sigma$. In our case $\mu(t, r_t) = a\bar{b} - ar_t$, therefore the drift becomes $a\bar{b} - ar_t - \lambda\sigma = a(\bar{b} - \frac{\lambda\sigma}{a} - r_t)$ and

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

where $b = \bar{b} - \frac{\lambda\sigma}{a}$.

(ii) The long run variance is the same $\frac{\sigma^2}{2a}$, not the long run mean which becomes $b = \bar{b} - \frac{\lambda\sigma}{a}$.

(iii) It is enough to substitute the numerical values, for the formula of the affine term structure see the exercise book.