ASSET PRICE MODELS

BLACK AND SCHOLES MODEL

BROWNIAN MOTION (= MOTO BROWNIANO)

↳ $\boxed{(W_t)_{t\geq 0}}$ is BROWNIAN MOTION on $(\Omega, \mathcal{F}, \mathbb{P})$ if:
- all TRAJECTORIES start from 0 ⇒ $W_0 = 0$
- has CONTINUOS PATH $t \longrightarrow W_t$, $\mathbb{R}_+ \longrightarrow \mathbb{R}^t$
- $s<t$ $\boxed{W_t - W_s = dW_t}$ = INCREMENT OF BROWNIAN MOTION is distributed $\boxed{N(0, t-s)}$
- $u<s<t$ $\left.\begin{matrix} W_t - W_s \\ W_s - W_u \end{matrix}\right\}$ are INDEPENDENT ⇒ NO OVERLAPPING

BACHELIER MODEL

$P_t - P_s = \mu(t-s)$

↳ $\boxed{P_t = P_s + \mu(t-s) + \sigma(W_t - W_s)}$ is N distributed: $\boxed{P_t \sim N\left(P_s + \mu(t-s);\ \sigma^2(t-s)\right)}$

CONSTANT, CONSTANT, CONSTANT

↳ P_t is CONTINUOS and DETERMINISTIC: only $W_t - W_s$ changes

⇒ RANDOM WALK

$X_i = \begin{cases} -\sqrt{\delta} & 1/2 \\ +\sqrt{\delta} & 1/2 \end{cases}$ $\quad t/\delta$, with CLT: $\dfrac{S_t - E[S_t]}{\sqrt{t}} = \dfrac{S_t}{\sqrt{t}} \sim N(0,1)$

⇒ $\boxed{S_t \sim N(0,t)}$

ARITMETIC BM: BACHELIER MODEL

$\boxed{dP_t = P_{t+dt} - P_t = \underbrace{\mu\, dt}_{\text{DETERMINISTIC PART}} + \underbrace{\sigma\, dW_t}_{\text{DRIFT PART}}}$ ⇒ if $\bar{t}$: $\boxed{P_{\bar{t}} = P_t + \mu(\bar{t} - t) + \sigma(W_{\bar{t}} - W_t)}$

↳ $P_{\bar{t}} \sim N\left(P_t + \mu(\bar{t}-t),\ \sigma^2(\bar{t}-t)\right)$

GEOMETRIC BM: BLACK AND SCHOLES (consider the PROPORTION to the PRICE)

$\boxed{dP_t = \mu \cdot P_t\, dt + \sigma\, P_t \cdot dW_t}$ ⇒ we call $Y_t = f(P_t)$ ⇒ $d[f(P_t)] = f(P_{t+dt}) - f(P_t) = Y_{t+dt} - Y_t$

⇒ ITÔ FORMULA

TAYLOR FORMULA: $f(x) \simeq f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$

↳ ITÔ: trovare una BUONA APPROSSIMAZIONE per il VALORE dell'ASSET X, considerando il valore oggi X_0.

$X_0 = P_t$, $X = P_{t+dt}$

$$f(P_{t+dt}) = f(P_t) + f'(P_t)(P_{t+dt} - P_t) + \frac{1}{2} f''(P_t)(P_{t+dt} - P_t)^2$$ ⇒ BLACK AND SCHOLES MODEL

$$f(P_{t+dt}) = f(P_t) + f'(P_t)(\mu \cdot P_t \cdot dt + \sigma \cdot P_t \cdot dW_t) + \frac{1}{2} f''(P_t)(\mu \cdot P_t \cdot dt + \sigma \cdot P_t \, dW_t)^2$$ ⇒ scelgo io l'ordine a cui troncare dt

↳ $dW_t \sim N(0, dt) \Rightarrow E[dW_t] = 0 \; ; \; E[(dW_t)^2] - E[(dW_t)]^2 = Var[dW_t] = dt$

↳ focus on $[\mu \cdot P_t \cdot dt + \sigma \cdot P_t \cdot dW_t]^2 = \cancel{(\mu P_t)^2 (dt)^2} + (\sigma P_t)^2 (dW_t)^2 + \cancel{2 \mu \sigma P_t^2 \, dt \, dW_t}$

dt^2 NON VA BENE

$dW_t = \sqrt{dt} \Rightarrow dt \cdot \sqrt{dt} = dt^{3/2}$ $3/2 > 1$ NON VA BENE

ITÔ FORMULA: $$\boxed{f(P_{t+dt}) = f(P_t) + f'(P_t)(\mu P_t \cdot dt + \sigma \cdot P_t \cdot dW_t) + \frac{1}{2} f''(P_t) \cdot \sigma^2 \cdot P_t^2 \cdot dt}$$

OSS. $d[f(P_t)] = f(P_{t+dt}) - f(P_t)$

$$\begin{aligned} d[f(P_t)] = f(P_{t+dt}) - f(P_t) &= f'(P_t)(P_{t+dt} - P_t) + \frac{1}{2} f''(P_t)(P_{t+dt} - P_t)^2 = \\ &= f'(P_t)(\mu \cdot P_t \cdot dt + \sigma \cdot P_t \cdot dW_t) + \frac{1}{2} f''(P_t)(\sigma^2 \cdot P_t^2 \cdot dt) \\ &= \left[f'(P_t)(\mu \cdot P_t) + \frac{1}{2} f''(P_t)(\sigma^2 \cdot P_t^2)\right] dt + f'(P_t)(\sigma \cdot P_t \cdot dW_t) \end{aligned}$$

3 CASI PER ITÔ FORMULA

$Y_t = f(t, P_t)$

$$dY_t = d f(t, P_t) = \frac{\delta f}{\delta t} \cdot dt + \frac{\delta f}{\delta P_t} \cdot dP_t + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta t} \cdot (dt)^2 + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta P_t} \cdot (dP_t)^2 + 2\left(\frac{1}{2}\right) \frac{\delta f}{\delta t \cdot \delta P_t} \cdot (dt \cdot dP_t) =$$

$$= \frac{\delta f}{\delta t} \cdot dt + \frac{\delta f}{\delta P_t} \cdot \underbrace{dP_t}_{\mu P_t dt + \sigma P_t dW_t} + \cancel{\frac{1}{2} \frac{\delta^2 f}{\delta t} (dt)^2} + \frac{1}{2} \frac{\delta^2 f}{\delta P_t} (\sigma^2 P_t^2 \, dt) + \cancel{2 \cdot \left(\frac{1}{2}\right) \frac{\delta^2 f}{\delta t \cdot \delta P_t} (dt \, dP_t)} =$$

$$\boxed{dY_t = d f(t, P_t) = \left(\frac{\delta f}{\delta t} + \frac{\delta f}{\delta P_t} \cdot \mu P_t + \frac{1}{2} \frac{\delta^2 f}{\delta P_t} \sigma^2 P_t^2\right) dt + \frac{\delta f}{\delta P_t} \, \sigma P_t \cdot dW_t}$$

DISCOUNTED ASSET PRICE (= PREZZO DEL BENE ATTUALIZZATO)

$\tilde{P}_t = e^{-rt} \cdot P_t$ ⇒ remember $df(t, P_t) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial P_t} \mu P_t + \frac{1}{2} \frac{\partial^2 f}{\partial P_t^2} \cdot \sigma^2 P_t\right) dt + \frac{\partial f}{\partial P_t} \sigma P_t \, dW_t$

$$\Rightarrow d\tilde{P}_t = (-r e^{-rt} P_t + e^{-rt} \mu P_t + 0) dt + e^{-rt} \sigma P_t \, dW_t = e^{-rt} \cdot P_t \, (\mu - r) \, dt + e^{-rt} P_t \, \sigma \, dW_t =$$

$$d\tilde{P}_t = \tilde{P}_t (\mu - r) dt + \tilde{P}_t \cdot \sigma \, dW_t$$

$f(P_t^1, P_t^2)$

$$df(P^1,P^2) = \frac{\partial f}{\partial P_t^1} dP_t^1 + \frac{\partial f}{\partial P_t^2} dP_t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial P_t^{1}}(\sigma^1 P_t^1)^2 dt + \frac{1}{2}\frac{\partial^2 f}{\partial P_t^2}(\sigma^2 P_t^2) dt + 2\cdot\frac{1}{2}\left(\frac{\partial^2 f}{\partial P_t^1 \partial P_t^2}\right)\cdot dP_t^1 \cdot dP_t^2$$

$$\boxed{df(P^1,P^2) = \frac{\partial f}{\partial P_t^1}\cdot dP_t^1 + \frac{\partial f}{\partial P_t^2}\cdot dP_t^2 + \frac{1}{2}\frac{\partial^2 f}{\partial P_t^{1}}(\sigma^1 P_t^1)^2 dt + \frac{1}{2}\frac{\partial^2 f}{\partial P_t^2}(\sigma^2 P_t^2)^2 dt + 2\cdot\frac{1}{2}\frac{\partial^2 f}{\partial P_t^1 \partial P_t^2}\cdot \sigma^1\cdot\sigma^2\cdot P_t^1\cdot P_t^2 \cdot \rho_{1,2}\underbrace{(dW_t)^2}_{=dt}}$$

BLACK AND SCHOLES MARKET

(MODELLO DI VALUTAZIONE delle OPZIONI EUROPEE)

È un MODELLO di NON ARBITRAGGIO, ossia calcola il prezzo di EQUILIBRIO delle OPZIONI assumendo che nel MERCATO NON esistano opportunità di ARBITRAGGIO ⇒ parte dalla COSTRUZIONE di un portafoglio PRIVO DI RISCHIO composto da OPZIONI e ATTIVITÀ sottostante e ne calcola il VALORE ATTUALE ipotizzando che il suo RENDIMENTO debba necessariamente essere uguale al TASSO RISK-FREE.

2 ASSET
- RISKY ASSET: $dS_t = \mu S_t dt + \sigma S_t dW_t \Rightarrow f(\ln S_t) \xRightarrow{\text{ITÔ}} \boxed{S_t = S_0 \cdot e^{(\mu - \frac{1}{2}\sigma^2)t - \sigma dW_t}}$
- RISK-LESS ASSET: $dB_t = r B_t dt \Rightarrow \boxed{B_t = e^{rt}}$

} NO ARBITRAGE ASSUMPTION

⇒ PORTFOLIO: VALUE of PORTFOLIO at time t $= \pi_t =$
- -1 OPTION ⇒ $-\Phi_t = -f(t,S_t)$
- $\frac{\partial f(t,S_t)}{\partial S_t}$ STOCK

$$d\pi_t = \pi_{t+dt} - \pi_t = -\underbrace{f(t,S_t)}_{\text{PRICE of OPTION}} + \underbrace{\frac{\partial f(t,S_t)}{\partial S_t}}_{\#\text{ of STOCK}} \cdot \underbrace{S_t}_{\text{PRICE of STOCK}}$$

PRICE of OPTION ↳ ITÔ FORMULA; PRICE of STOCK ↳ BS MODEL

⇒ we want a PORTFOLIO totally RISKFREE

$$d\pi_t = -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2\right)dt \xRightarrow{\text{NO ARBITRAGE}} d\pi_t = r\cdot\pi_t\cdot dt$$

$$-\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2\right)\cancel{dt} = r\overbrace{\left(-f(t,S_t) + \frac{\partial f}{\partial S_t}S_t\right)}^{\pi_t}\cancel{dt}$$

$$\boxed{r f(t,S_t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S_t} r\cdot S_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\cdot\sigma^2\cdot S_t^2}$$

⟹ ma voglio trovare il VALORE all'INIZIO $f(0,S_0)$; io conosco $f(T,S_T) =$ PAYOFF

BLACK AND SCHOLES: ANOTHER APPROACH

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dB_t = r B_t dt \end{cases}$$

ρ = DIVIDEND RATE = è un TASSO che esprime la porzione di DIVIDENDI

$d\pi_t = \pi_{t+dt} - \pi_t \Rightarrow$ PORTFOLIO
- -1 OPTION $\Rightarrow f(t, S_t) =$ PRICE OF OPTION
- $\frac{\partial f}{\partial S_t}$ STOCK $\Rightarrow$ c'è anche una parte di DIVIDENDI che dipende dal TEMPO (dt) e dal PREZZO DELLE AZIONI (S_t)

$$d\pi_t = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial^2 S_t^2}\cdot\sigma^2 \cdot S_t^2 + \rho S_t \frac{\partial f}{\partial S_t}\right)\cdot dt \xRightarrow{\text{NO ARBITRAGE}} d\pi_t = r\,\pi_t\,dt$$

$$\boxed{r f(t, S_t) = \frac{\partial f}{\partial t} + (r-\rho) S_t \frac{\partial f}{\partial S_t} + \frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2 \quad \Bigg|\ \text{PAYOFF} = f(T, S_T)}$$

EUROPEAN CALL OPTION

$$r f(t, S_t) = \frac{\partial f}{\partial t} + r S_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}\sigma^2 S_t^2 \quad \Bigg|\ \text{PAYOFF} = f(T, S_T) = (S_T - K)^+$$

$$dS_t = r S_t dt + \sigma S_t dW_t \xRightarrow[\text{RISK NEUTRAL}]{S_t \text{ non dipende da } \mu:} dS_t = r S_t dt + \sigma S_t dW_t + \left(\frac{\mu - r}{\sigma}\right)\cdot \sigma S_t dt \xRightarrow{\text{define}} \widetilde{W}_t = W_t + \left(\frac{\mu - r}{\sigma}\right)t$$

$dS_t = r S_t dt + \sigma S_t d\widetilde{W}_t \Rightarrow \widetilde{W}_t$ is NOT a BROWNIAN MOTION, cause of $\widetilde{W}_t - \widetilde{W}_s \sim N\left(\frac{\mu - r}{\sigma}(t-s)\ ; \dots\right)$

$\Rightarrow$ TH: $(W_t)_{t\geq 0}$ is a BM on $(\Omega, \mathcal{F}, \mathbb{P})$, but $\exists$ an EQUIVALENT PROBABILITY MEASURE $\widetilde{\mathbb{P}}$ such that $\widetilde{W}_t$ is a BM on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$
  ↳ $dS_t = r S_t dt + \sigma S_t d\widetilde{W}_t$ is the same STOCK PRICE on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$

$\Rightarrow$ EUROPEAN CALL OPTION PAYOFF: $(S_T - K)^+$

$C_t = E^{\widetilde{\mathbb{P}}}\left(e^{-r(T-t)}(S_T - K)^+ \mid \mathcal{F}_t\right) \rightarrow$ ma il MERCATO è EFFICIENTE: i PREZZI riflettono quello che è successo prima: S_t;

$C_t = E^{\widetilde{\mathbb{P}}}\left(e^{-r(T-t)}(S_T - K)^+ \mid S_t\right) \rightarrow S_T = S_0\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma \widetilde{W}_T}$, ma si parte da oggi: $S_T = S_t \cdot e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{W}_T - \widetilde{W}_t)}$;

$C_t = E^{\widetilde{\mathbb{P}}}\left[e^{-r(T-t)}\cdot\left(S_t\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(\widetilde{W}_T - \widetilde{W}_t)} - K\right)^+\right]$
- $\rightarrow \widetilde{W}_T - \widetilde{W}_t = Y\sqrt{T-t}$
- $\rightarrow$ if X RV. $f(x)$, $Y = \Phi(X) \Rightarrow E(Y) = \int_{-\infty}^{+\infty}\Phi(x)\cdot f_X(x)\,dx$

$$\boxed{C_t = \int_{-\infty}^{+\infty} e^{-r(T-t)}\left(S_t \cdot e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma Y\sqrt{T-t}} - K\right)^+ \cdot \frac{1}{\sqrt{2\pi}}\, e^{-Y^2/2}\, dy}$$

QUANDO è POSITIVO?

$S_t\, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma Y\sqrt{T-t}} - K \geq 0 \qquad e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma Y\sqrt{T-t}} \geq \frac{K}{S_t}$

$(r - \frac{1}{2}\sigma^2)(T-t) + \sigma Y\sqrt{T-t} \geq \ln\left(\frac{K}{S_t}\right) \qquad Y \geq \frac{(r-\frac{1}{2}\sigma^2)(T-t) - \ln\left(\frac{K}{S_t}\right)}{\sigma\sqrt{T-t}} \doteq -d_2 \rightarrow$ quando $y \geq -d_2$ posso integrare

$$C_t = \underbrace{\int_{-d_2}^{+\infty} e^{-r(T-t)}\left(S_t \cdot e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma y\sqrt{T-t}}\right)\frac{1}{\sqrt{2\pi}}\cdot e^{-y^2/2}\,dy}_{\int_{-d_2}^{+\infty} \frac{S_t\cdot e^{-\frac{1}{2}\sigma^2(T-t) + \sigma y\sqrt{T-t} - y^2/2}}{\sqrt{2\pi}} = \int_{-\infty}^{+d_2}\dots} - \underbrace{\int_{-d_2}^{+\infty} K\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-y^2/2}\,dy}_{K e^{-r(T-t)}\int_{-d_2}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy}$$

$K e^{-r(T-t)} \cdot N(d_2)$

$S_t \cdot \int_{-d_2}^{+\infty} \frac{e^{-1/2(y + \sigma\sqrt{T-t})^2}}{\sqrt{2\pi}}$: $Z = y + \sigma\sqrt{T-t}$, e per $-\infty < y \leq d_2$, allora $-\infty < Z \leq d_2 + \sigma\sqrt{T-t} \doteq d_1$

$$S_t \int_{-\infty}^{+d_1} \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2} \, dz = S_t \cdot N(d_1)$$

$\Rightarrow$ PRICE FOR EUROPEAN CALL OPTION:
$$\boxed{C_t = S_t N(d_1) - K \cdot e^{-r(T-t)} N(d_2)}$$
$$\boxed{C_t = E^{\tilde{P}}\left[e^{-r(T-t)} (S_T - K)^+ \mid \mathcal{F}_t\right]}$$

$$\begin{cases} d_1 = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 = d_1 - \sigma\sqrt{T-t} = \\ = \dfrac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \end{cases}$$

EUROPEAN PUT OPTION

$$\boxed{P_t = E^{\tilde{P}}\left[e^{-r(T-t)} (K - S_T)^+ \mid \mathcal{F}_t\right]}$$

→ VOGLIO TROVARE P_t USANDO LA EUROPEAN PUT-CALL PARITY
- Ⓐ 1 EUROPEAN CALL + Ke^{-rT}
- Ⓑ 1 EUROPEAN PUT + 1 STOCK

→ ALLA MATURITÀ T
- Ⓐ $(S_T - K)^+ + Ke^{-rT} \cdot e^{rT} = (S_T - K)^+ + K$ $\quad \max(S_T, K)$
- Ⓑ $(K - S_T)^+ + S_T$ $\quad \max(S_T, K)$

$\Rightarrow \boxed{C_0 + Ke^{-rT} = P_0 + S_0}$

$$P_t = C_t + K \cdot e^{-r(T-t)} - S_t \qquad P_t = S_t \cdot N(d_1) - Ke^{-r(T-t)} \cdot N(d_2) + Ke^{-r(T-t)} - S_t$$

$\Rightarrow$ PRICE FOR EUROPEAN PUT OPTION:
$$\boxed{P_t = -S_t \cdot N(-d_1) + Ke^{-r(T-t)} \cdot N(-d_2)}$$
$$\boxed{P_t = E^{\tilde{P}}\left[e^{-r(T-t)} (K - S_T)^+ \mid \mathcal{F}_t\right]}$$

DIGITAL OPTION CALL

① CASH OR NOTHING $\quad f(S_T) = \begin{cases} H & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases} = H \cdot 1_{(S_T > K)} + 0$

$$D_t = E^{\tilde{P}}\left[e^{-r(T-t)} H 1_{(S_t > K)} \mid S_t\right] \qquad D_t = e^{-r(T-t)} H \cdot E^{\tilde{P}}\left[1_{(S_t > K)} \mid S_t\right]$$

→ fare il valore atteso dell'indicatore di una funzione è come calcolare la P dell'evento

→ $\tilde{P}\left(S_t \cdot e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)} > K\right)$

$$D_t = e^{-r(T-t)} H \cdot \tilde{P}\left(Y > -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$\to 1 - P(Y \le -d_2) = 1 - N(-d_2) = 1 - (1 - N(d_2)) = N(d_2)$

$$\boxed{D_t = He^{-r(T-t)} \cdot N(d_2)}$$

② ASSET OR NOTHING $\quad f(S_T) = \begin{cases} S_T & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases} = S_T \cdot 1_{(S_T > K)} + 0$

$$\bar{D}_t = E^{\tilde{P}}\left[e^{-r(T-t)} S_T \cdot 1_{(S_T > K)} \mid S_t\right] \qquad \bar{D}_t = e^{-r(T-t)} \cdot E^{\tilde{P}}\left[S_T \cdot 1_{(S_T > K)} \mid S_t\right]$$

$$\bar{D}_t = e^{-r(T-t)} \cdot E^{\tilde{P}}\left[S_t e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)} \cdot 1_{(S_t \cdot e^{\dots} > K)} \mid S_t\right]$$

→ VALORE ATTESO di una DISTRIBUZIONE GAUSSIANA: $E^{\tilde{P}}[\dots] = \int_{-\infty}^{+\infty}$

$$\bar{D}_t = e^{-r(T-t)} \int_{-d_2}^{+\infty} S_t \cdot e^{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma Y\sqrt{T-t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-Y^2/2} \, dY$$

$$\bar{D}_t = \int_{-d_2}^{+\infty} S_t \cdot \frac{e^{-\frac{1}{2}\sigma^2(T-t) + \sigma Y\sqrt{T-t} - Y^2/2}}{\sqrt{2\pi}} \, dY$$

$Y \to -Y \qquad -\infty \le -Y \le d_2$

$$\bar{D}_t = \int_{-\infty}^{d_2} \frac{S_t \cdot e^{-\frac{1}{2}(y+\sigma\sqrt{T-t})^2}}{\sqrt{2\pi}}$$

$\to z = y + \sigma\sqrt{T-t}$, quindi $-\infty \le y \le d_2$ $\quad -\infty \le z \le \underbrace{d_2 + \sigma\sqrt{T-t}}_{d_1}$

$$\to \bar{D}_t = \int_{-\infty}^{d_1} \frac{S_t}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}} \, dz = S_t \cdot N(d_1)$$

$$\boxed{\bar{D}_t = S_t \cdot N(d_1)}$$

COX - ROSS - RUBISTEIN

$\Rightarrow$ NO ARBITRAGE CONDITION : $\boxed{d < 1+r < u}$

$\uparrow 25\%$ $\quad \downarrow 25\%$ $\quad p = 1/2$ $\quad 1-p = 1/2$ $\quad S_0 = 20$ € $\quad r = 4\%$

I) Consider two EUROPEAN CALL OPTIONS, the first one has strike 18 € and MATURITY one year, while the second has same strike, but MATURITY two years. Determine which one has higher PROBABILITY to be exercised.

20 → (1/2) 25 → (1/2) 31.25 $\Rightarrow 31.25 - 18 = 13.25$

25 → (1/2) 18.75 $\Rightarrow 18.75 - 18 = 0.75$

20 → (1/2) 15 → (1/2) 18.75

15 → (1/2) 11.25 $\Rightarrow 11.25 - 18 = 0$

Confronto $\to$ $\mathbb{P}(S_u > 18) = \mathbb{P}(S_u = 25) = 1/2$

$\to$ $\mathbb{P}(S_2 > 18) = \mathbb{P}(S_{ud} = 18.75) + \mathbb{P}(S_{uu} = 31.25) = 2p(1-p) + p^2 = 3/4$

II) Compute the INITIAL PRICES of two CALL OPTIONS.

$$B_0 = \frac{1}{(1+0.04)^2}\left[p^2(13.25) + 2(p)(1-p)(0.75) + (1-p)^2(0)\right] = 3.41$$

$$A_0 = \frac{1}{(1+0.04)}\left[p(7) + (1-p)(0)\right] = 3.37$$

$$(1-q^*) = \frac{u-(1+r)}{u-d} = \frac{1.25 - (1.04)}{0.5} = 0.42$$

$$q^* = \frac{(1+r)-d}{u-d} =$$ RISK NEUTRAL PROBABILITY

ASIATICHE : NON considero S_0 e faccio media;
LOOK-BACK : Considero S_0 e tolgo il VALORE più piccolo della TRAIETTORIA;
AMERICAN : faccio il CONFRONTO.

HEDGING STRATEGY

$$\delta_t^c = \frac{\partial C(t,S_t)}{\partial S_t} = N(d_1)$$

$$\delta_t^p = \delta_t^c - 1 = N(d_1) - 1 = N(-d_1)$$

→ per la PUT-CALL PARITY
→ maggiore di d_1 è come dire minore di $-d_1$

⇒ quando l'HEDGING ERROR è grande

ESERCIZIO

1) Sell 1 EUROPEAN CALL : $S_0 = 8$ $K^{(1)} = 8$
$\sigma = 40\%$ $r = 4\%$
$T = 1$ year

Determine # of STOCKS to buy/sell in order to obtain $\delta^\pi = 0$

⇒ Faccio il δ dell'OPZIONE : $\delta^{(1)} = \left(\frac{\partial C^{(1)}}{\partial S_t}\right) = N(d_1) = N\left(\frac{\ln\left(\frac{8}{8}\right) + \left(0.04 + \frac{1}{2}(0.4)^2\right)(1-0)}{0.4\sqrt{1-0}}\right) =$

$= N(0.3) = 0.618$

$\delta^\pi = -1\,(\text{OPTION}) + (x\ \text{stock})(S_t)$

$\Rightarrow \delta^\pi = -1 \cdot \delta^{(1)} + x \cdot \delta^S$

derivata di S_t rispetto a S_t è 1

$\Rightarrow \delta^\pi = 0$ per la NEUTRALITÀ : $x = \delta^{(1)} = 0.618$ (BUY: è positivo)