KINEMATICS

$$T_n^0(\mathbf{q}) = \begin{bmatrix} \mathbf{n}^0(\mathbf{q}) & \mathbf{s}^0(\mathbf{q}) & \mathbf{a}^0(\mathbf{q}) & \mathbf{p}^0(\mathbf{q}) \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_1^0(\mathbf{q}_1)A_2^1(\mathbf{q}_2)\cdots A_n^{n-1}(\mathbf{q}_n)$$

Operational space or task space:

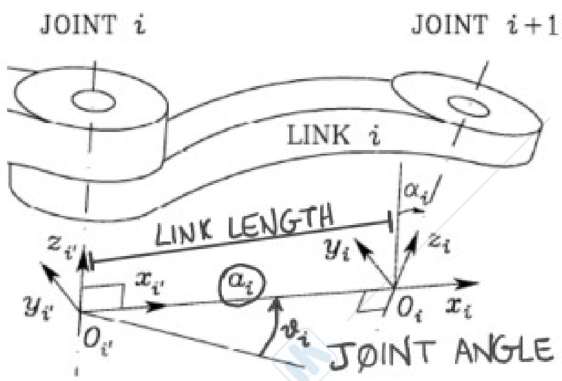
$$T_e^b = \begin{bmatrix} R & p \\ 0^T & 1 \end{bmatrix} \implies \mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \phi \end{bmatrix} = \begin{bmatrix} x(\mathbf{q}) \\ y(\mathbf{q}) \\ z(\mathbf{q}) \\ \zeta(\mathbf{q}) \\ \psi(\mathbf{q}) \end{bmatrix}$$

Post-multiplication:

$$A_i^{i-1}(q_i) = A_{i'}^{i-1}A_i^{i'} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $q_i = \theta_i$ for revolute joints;
- $q_i = d_i$ for prismatic joints.

Spherical Wrist $\quad \mathbf{W}(q_1, q_2, q_3) = \mathbf{p} - d_6\mathbf{a}$



- $d_i \to$ **link offset**: coordinata di $O_{i'}$ lungo z_{i-1}
- $\theta_i \to$ **joint angle**: angolo di rotazione da x_{i-1} a x_i attorno all'asse $z_{i'}$ (positivo quando la rotazione è anti-oraria)
- $a_i \to$ **link length**: distanza (con segno) fra O_i e $O_{i'}$
- $\alpha_i \to$ **link twist**: angolo di rotazione da z_{i-1} a z_i attorno all'asse x_i (positivo quando la rotazione è anti-oraria)

DIFFERENTIAL KINEMATICS

$$\boxed{\dot{p}_i = \dot{p}_{i-1} + v_{i-1,i} + \omega_{i-1} \times r_{i-1,i}} \qquad \boxed{\omega_i = \omega_{i-1} + \omega_{i-1,i}}$$

Geometric Jacobian:

$$\mathbf{v} = \begin{bmatrix} \dot{\mathbf{p}} \\ \omega \end{bmatrix} = J(\mathbf{q})\dot{\mathbf{q}} = \begin{bmatrix} J_P(\mathbf{q}) \\ J_O(\mathbf{q}) \end{bmatrix}\dot{\mathbf{q}}$$

$$J(\mathbf{q}) = \begin{bmatrix} z_0 \times (\mathbf{p} - p_0) & z_1 \times (\mathbf{p} - p_1) & z_2 \times (\mathbf{p} - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix}$$

$$J = \begin{bmatrix} J_{p,i} \\ J_{o,i} \end{bmatrix} = \begin{cases} \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix} & \text{for a prismatic joint} \\ \begin{bmatrix} z_{i-1} \times (p_e - p_{i-1}) \\ z_{i-1} \end{bmatrix} & \text{for a revolute joint} \end{cases}$$

Analytical Jacobian:

$$x = k(\mathbf{q}) = \begin{bmatrix} p(\mathbf{q}) \\ \phi(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \theta \\ \varphi \\ \psi \end{bmatrix}$$

$$\dot{p} = \frac{\partial p}{\partial q}\dot{q} = J_P(q)\dot{q}$$

$$\dot{\varphi} = \frac{\partial \varphi}{\partial q}\dot{q} = J_\varphi(q)\dot{q}$$

$$\boxed{\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} = J_A(\mathbf{q})\dot{\mathbf{q}}}$$

$$\omega = \begin{bmatrix} 0 & -s_\phi & c_\phi s_\theta \\ 0 & c_\phi & s_\phi s_\theta \\ 1 & 0 & c_\theta \end{bmatrix}\dot{\varphi} = T(\varphi)\dot{\varphi}$$

$$J(\mathbf{q}) \leftrightarrow J_A(\mathbf{q})$$

$$\omega = T(\phi)\dot{\phi}$$

$$J(\mathbf{q}) = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{0} & T(\phi) \end{bmatrix} J_A(\mathbf{q})$$

$$v = J\dot{q} \implies \dot{q} = J^{-1}v \implies J^{-1} = \frac{1}{det(J)}adj(J)$$

$$det(J) \to 0 \implies J^{-1} \to \infty$$

$$\boxed{det(J(\mathbf{q}_s)) = 0 \implies \mathbf{q}_s \text{ is a singular configuration}}$$

Inverse Kinematics: $\dot{\mathbf{q}} = J^{-1}(\mathbf{q})v \qquad \dot{\mathbf{q}} = J^{+}v.$ M $\qquad g(\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T W\dot{\mathbf{q}} \qquad v = J(\mathbf{q})\dot{\mathbf{q}}$

$$JJ^+ = JJ^T(JJ^T)^{-1} = I_r$$

Lagrange: $g(\ddot{q}, \lambda) = \frac{1}{2}\dot{q}^T W\dot{q} + \lambda^T(v - J\dot{q})$ **=>** $\dot{q} = W^{-1}J^T(JW^{-1}J^T)^{-1}v$

$J^+J = J^T(JJ^T)^{-1}J \neq I_r$ => Right pseudo-inverse $\qquad \dot{q} = J^+\dot{p} + (I - J^+J)\dot{q}_d = J^+\dot{p}$

Damped least-square:

$$\boxed{J^* = J^T(JJ^T + k^2I)^{-1}}$$

STATICS $\tau = J_g^T(\mathbf{q})\mathbf{F}$

$$H(q) = \sqrt{\det(J(q)J^T(q))}$$

Manipulability Measure: $\boxed{w(\mathbf{q}) = \sqrt{\det(JJ^T)} = |\lambda_1\lambda_2\cdots\lambda_n| = |\det(J)|}$

$$\tau^T d\mathbf{q} = \mathbf{F}^T J(\mathbf{q})d\mathbf{q} \qquad J^{-1} = \frac{1}{det(J)}adj(J)$$

Velocity manipulability Ellipsoid: $\boxed{E_v = \{v \ : v^T(JJ^T)^{-1}v = 1\}}$

Force manipulability Ellipsoid: $\boxed{E_F = \{\mathbf{F} \ : \mathbf{F}^T(JJ^T)\mathbf{F} = 1\}}$

Elasticity: $\delta W_{TCP} = \delta W_g \quad \forall \mathbf{q}(t) \iff \tau^T\delta\mathbf{q} = \mathbf{F}^T\delta\mathbf{p}$

$$\Delta \boldsymbol{p} = \boldsymbol{J}\Delta\boldsymbol{q} = \boldsymbol{J}\boldsymbol{K}_q^{-1}\boldsymbol{\tau}_e = \boldsymbol{J}\boldsymbol{K}_q^{-1}\boldsymbol{J}^\top\boldsymbol{F} = \boldsymbol{C}_p\boldsymbol{F}$$

$$J \approx U\overline{\Sigma}V^T \qquad J^+ \approx J^* = V\overline{\Sigma}^+U^T$$

DYNAMICS

$$\dot{\mathbf{q}}^T\dot{\mathbf{q}} = 1 \qquad \tau^T\tau = 1$$

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \mathcal{F}_i \qquad i = 1, \ldots, n \qquad \mathcal{T} = \sum_{i=1}^n \mathcal{T}_{l_i} + \mathcal{T}_{m_i}$$

$$\boxed{\mathcal{T}_{l_i} = \frac{1}{2}m_{l_i}\dot{\mathbf{p}}_i^T\dot{\mathbf{p}}_i + \frac{1}{2}\omega_i^T\mathbf{R}_i^b\mathbf{I}_{l_i}(\mathbf{R}_i^b)^T\omega_i}$$

Equation of motion

$$\sum_{j=1}^n b_{ij}(\boldsymbol{q})\ddot{q}_j + \underbrace{\sum_{j=1}^n\sum_{k=1}^n h_{ijk}(\boldsymbol{q})\dot{q}_k\dot{q}_j}_{\triangleq\, C(q,\dot{q})q} + g_i(\boldsymbol{q}) = \xi_i \qquad (h(q,\dot{q}))$$

$$\mathcal{U} = -\sum_{i=1}^n m_{l_i}\mathbf{g}_0^T\mathbf{p}_{l_i} + m_{m_i}\mathbf{g}_0^T\mathbf{p}_{m_i}$$

Total Kinetic Energy

$$\boxed{\mathcal{T} = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n b_{ij}(\mathbf{q})\dot{\mathbf{q}}_i\dot{\mathbf{q}}_j = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{B}(\mathbf{q})\dot{\mathbf{q}}}$$

Dynamical model in the joint space

$$\boxed{\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} + \mathbf{F}_v\dot{\mathbf{q}} + \mathbf{F}_s\text{sgn}(\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \tau - J^T(\mathbf{q})\mathbf{h}}$$

TRAJECTORY PLANNING

$$\begin{cases} \mathbf{q}(t_i) = a_3 t_i^3 + a_2 t_i^2 + a_1 t_i + a_0 \\ \mathbf{q}(t_f) = a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 \\ \dot{\mathbf{q}}(t_i) = 3a_3 t_i^2 + 2a_2 t_i + a_1 \\ \dot{\mathbf{q}}(t_f) = 3a_3 t_f^2 + 2a_2 t_f + a_1 \\ \mathbf{q}(t_i) = \mathbf{q}_i \\ \mathbf{q}(t_f) = \mathbf{q}_f \\ \dot{\mathbf{q}}(t_i) = \dot{\mathbf{q}}_i \\ \dot{\mathbf{q}}(t_f) = \dot{\mathbf{q}}_f \end{cases}$$

qi=0 =>

$$\begin{cases} \mathbf{q}_i = a_0 \\ \mathbf{q}_f = a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 \\ \dot{\mathbf{q}}_i = a_1 \\ \dot{\mathbf{q}}_f = 3a_3 t_f^2 + 2a_2 t_f + a_1 \end{cases}$$

Solution of the trapezoidal profile:

$$\mathbf{q}(t) = \begin{cases} \mathbf{q}_i + \frac{1}{2}\ddot{\mathbf{q}}_c t^2 & 0 \le t \le t_c \\ \mathbf{q}_i + \ddot{q}_c t_c (t - t_c/2) & t_c < t \le t_f - t_c \\ \mathbf{q}_i - \frac{1}{2}\ddot{\mathbf{q}}_c (t_f - t)^2 & t_f - t_c < t_f \end{cases}$$

$$t_c = \frac{t_f}{2} - \frac{1}{2}\sqrt{\frac{t_f^2 \ddot{\mathbf{q}} - 4(\mathbf{q}_f - \mathbf{q}_i)}{\ddot{\mathbf{q}}_c}} \qquad |\ddot{\mathbf{q}}| \ge \frac{4|\mathbf{q}_f - \mathbf{q}_i|}{t_f^2}$$

$$\mathrm{sgn}(\ddot{\boldsymbol{q}}) = \mathrm{sgn}(\boldsymbol{q}_f - \boldsymbol{q}_i)$$

Bang Bang: $t_c = \dfrac{t_f}{2}$

$$\dot{\mathbf{q}}_c = \frac{\mathbf{q}_m - \mathbf{q}_c}{t_m - t_c} = \frac{\text{rise}}{\text{run}} \qquad t_m = \frac{t_f}{2} \qquad \mathbf{q}_m = \frac{\mathbf{q}_f + \mathbf{q}_i}{2}$$

Cruise velocity assignment:

$$\frac{|\mathbf{q}_f - \mathbf{q}_i|}{t_f} < |\dot{\mathbf{q}}_c| \le 2\frac{|\mathbf{q}_f - \mathbf{q}_i|}{t_f} \qquad \dot{\mathbf{q}}_c = \ddot{\mathbf{q}}_c t_c \qquad \ddot{\mathbf{q}}_c t_c^2 - \ddot{\mathbf{q}}_c t_f t_c + \mathbf{q}_f - \mathbf{q}_i = 0$$

$$t_c = \frac{\mathbf{q}_i - \mathbf{q}_f + \dot{\mathbf{q}}_c t_f}{\dot{\mathbf{q}}_c} \qquad \ddot{\mathbf{q}}_c = \frac{\dot{\mathbf{q}}_c^2}{\mathbf{q}_i - \mathbf{q}_f + \dot{\mathbf{q}}_c t_f}$$

CONTROL

Dynamical model manipulator : $B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F_v\dot{q} + g(q) = \tau$

$$q_m = K_r q \qquad \tau_m = K_r^{-1}\tau$$

$$B(q) = \overline{B} + \Delta B(q)$$

$$\begin{cases} K_r^{-1}\tau = K_t i_a \\ i_a = G_i v_c \end{cases} \qquad f_\infty = \begin{bmatrix} \frac{k_{Px}k_x}{k_{Px}+k_x}(x_d - x_e) \\ 0 \end{bmatrix} \qquad \exists V(t,x) : \forall x \in N \begin{cases} V(t,x) > 0 \\ \dot{V}(t,x) \le 0 \end{cases}$$

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = u - J^T(q)h \qquad \text{PI control: } C(s) = K_c\frac{1 + sT_c}{s}$$

$$V(\dot{q},\tilde{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q} + \frac{1}{2}\tilde{q}^T K_P \tilde{q} > 0 \quad \forall \dot{q},\tilde{q} \longrightarrow \dot{V} = \dot{q}^T B(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{B}(q)\dot{q} - \dot{q}^T K_P \tilde{q}$$

MOBILE ROBOTS

$$\dot{q} = \sum_{j=1}^{n-k} g_j(q) u_j = G(q)u \qquad \dot{q} \in \mathcal{N}(A^T(q))$$

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0$$

$$\dot{x}\sin(\theta + \phi) - \dot{y}\cos(\theta + \phi) - L\dot{\theta}\cos\phi = 0$$

Bicycle

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \cos\theta\cos\phi & 0 \\ \sin\theta\cos\phi & 0 \\ \sin\phi/L & 0 \\ 0 & 1 \end{bmatrix} \qquad v = \frac{r(\omega_R + \omega_L)}{2} \qquad \omega = \frac{r(\omega_R - \omega_L)}{d}$$

$$bel_k \approx \sum_{i=1}^{n} \mu_k^{[i]} \delta(q_k - \hat{q}_k^{[i]})$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \boldsymbol{G}_1(\boldsymbol{q})u_1 + \boldsymbol{G}_2(\boldsymbol{q})u_2 \quad (u_2 \equiv \omega) \quad \text{if front drive: } u_1 = v \text{, if back d.: } u_1 = \frac{v}{\cos\phi}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos(\alpha); \quad \cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha)$$

$$\sin\left(\frac{\pi}{2} + \alpha\right) = \cos(\alpha); \quad \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin(\alpha)$$

$$\sin(\pi - \alpha) = \sin(\alpha); \quad \cos(\pi - \alpha) = -\cos(\alpha)$$

$$\sin(\pi + \alpha) = -\sin(\alpha); \quad \cos(\pi + \alpha) = -\cos(\alpha)$$

$$\sin\left(\frac{3}{2}\pi - \alpha\right) = -\cos(\alpha); \quad \cos\left(\frac{3}{2}\pi - \alpha\right) = -\sin(\alpha)$$

$$\sin\left(\frac{3}{2}\pi + \alpha\right) = -\cos(\alpha); \quad \cos\left(\frac{3}{2}\pi + \alpha\right) = \sin(\alpha)$$

$$\sin(-\alpha) = -\sin(\alpha); \quad \cos(-\alpha) = \cos(\alpha)$$